FORCED CONVECTION IN DUCTS WITH PERMEABLE WALLS

Carlos Henrique Alexandrino[1]
Laboratory of Theoretical and Applied Mechanics - Mech. Engng. Dept. - Universidade Federal Fluminense
Rua Passo da Pátria, 156 -- 24210-240 Niterói/RJ, Brazil
chalex@bol.com.br

Maria Laura Martins-Costa[2]
Laboratory of Theoretical and Applied Mechanics - Mech. Engng. Dept. - Universidade Federal Fluminense
Rua Passo da Pátria, 156 -- 24210-240 Niterói/RJ, Brazil
laura@mec.uff.br

Abstract. A mixture theory model is employed in a local description of the energy transfer in a duct with permeable wall which is simulated by considering two distinct flow regions, one consisting of a Newtonian incompressible fluid and the other represented by a binary (solid-fluid) mixture. Compatibility conditions at the interface (pure fluid-mixture) for momentum and energy transfer are considered. The simulations are carried out by using a finite difference approach with an upwind strategy for the convective term discretization.

Keywords. Forced Convection in Porous Medium, Mixture Theory, Two Distinct Flow Regions.

1. Introduction

Ducts with permeable walls are present in many relevant engineering situations such as flow of perforation mud in oil wells and porous bearing lubrication. In the present work the momentum and energy transfer process in a permeable duct is modeled based on the continuum theory of mixtures, a generalization of the continuum mechanics specially developed to describe multiphase phenomena. The model considers two distinct flow regions – the former representing the region inside the duct where a Newtonian incompressible fluid flows, in which the classical continuum mechanics equations are recovered for a single constituent mixture. The latter one represents the duct's permeable walls – a fluid-saturated porous medium, from now on referred to as mixture region. In this region both fluid and porous matrix are treated as overlapping continuous constituents of a binary mixture, each of them occupying its whole volume. The fluid constituent and the fluid are assumed Newtonian and incompressible while the solid constituent, representing the porous matrix, is assumed rigid, homogeneous, isotropic and at rest.

Most of the works dealing with transport in porous media employ a local volume-averaging technique, discussed in detail by Whitaker (1969), to describe quantities such as temperature, pressure, concentration and the velocity components; allowing the use of the classical continuum mechanics approach. Vafai and Kim (1990) have used this approach and Darcy's law - with the addition of Brinkmann and Forchheimer terms to account for inertia and viscous effects and satisfy the no-slip condition - as the balance of linear momentum to analyze convection in two distinct flow regions (fluid and fluid-saturated porous medium). Huang and Vafai (1993) have studied the forced convection over a complex geometry consisting of multiple porous blocks attached to an impermeable wall where the fluid and the fluid flowing through the porous blocks represent two distinct flow regions. Vafai and Sozen (1990) have considered thermal nonequilibrium (a natural assumption in a mixture theory approach) in a continuum mechanics approach associated with a local volume-averaging technique, together with Erguns' correlation as the vapor phase momentum equation, in order to analyze the forced convective flow of a gas through a packed-bed of solid particles. Alazmi and Vafai (2000) have performed a complete analysis of the models for transport processes through porous media using a volume averaging technique. They have considered four categories of processes: constant porosity models, variable porosity models, models including thermal dispersion (combined with variable porosity or not) and models allowing for local thermal nonequilibrium.

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[1] This author is a graduate student at Mechanical Engineering Program (PGMEC)/UFF.
[2] All article correspondence should be addressed to this author.

A distinct approach is used in this work: the Continuum Theory of Mixtures (Truesdell, 1957, Truesdell and Toupin, 1960, Atkin and Craine, 1976, Bedford and Drumheller, 1983, Rajagopal and Tao, 1995) – supported by a local theory with thermodynamic consistency - which generalizes the classical continuum mechanics (Gurtin, 1981, Germain and Muller, 1986). It is employed to model flows through porous media in which the fluid and the porous matrix are treated as superimposed continuous constituents of a binary mixture - each of them occupying its whole volume. The mixture theory leads to an apparent thermomechanical independence allowing the existence of n distinct velocity fields and n distinct temperature fields (if thermal equilibrium is not assumed), simultaneously, at each spatial point, whenever a n-constituent mixture is considered. In order to provide dynamical and thermal interactions, additional terms, absent in a Continuum Mechanics description - playing the role of momentum and energy sources - are required to account for the thermomechanical coupling among the constituents in the balance equations. Constitutive relations for these sources, satisfying the material objectivity and the Second Law of Thermodynamics, are used (Martins-Costa et al., 1992; Costa Mattos et al., 1995).

Compatibility conditions at the interface (pure fluid-mixture) for momentum and energy transfer must be imposed in order to allow the solution of the problem. These conditions have been considered supposing no flow across the interface.

Starting from known velocity fields, the forced convection heating of a fluid flowing through the permeable duct is simulated by using a finite difference approach. A central finite difference discretization is used for the diffusive terms while an upwind strategy (Euvrard, 1987) is employed for the convective terms.

2. Mechanical model

2.1. Balance equations

Representing by the open set Ω_1, with boundary $\partial\Omega_1$, the region occupied by the pure fluid and by the open set Ω_2, with boundary $\partial\Omega_2$, the region occupied by the mixture and assuming absence of mass production by all the constituents in the latter region, the mass balance may be written as (Gurtin, 1981; Rajagopal and Tao, 1995; Atkin and Craine, 1976)

$$
\begin{aligned}
&\frac{\partial \boldsymbol{r}}{\partial t} + \nabla \cdot (\boldsymbol{r}\mathbf{v}) = 0 && \text{in } \Omega_1 \\
&\frac{\partial \boldsymbol{r}_i}{\partial t} + \nabla \cdot (\boldsymbol{r}_i \mathbf{v}_i) = 0 && \text{in } \Omega_2 , \quad i = 1, n
\end{aligned}
\tag{1}
$$

in which the actual fluid parameters – defined in Ω_1 – are $\boldsymbol{r}$ standing for its mass density and $\mathbf{v}$ for its velocity. In the mixture region Ω_2, n stands for the number of constituents and the fluid constituent parameters are $\boldsymbol{r}_i$, its mass density – representing locally the ratio between the i-th constituent mass and the respective volume of mixture and $\mathbf{v}_i$ standing for its velocity.

The balance of linear momentum is given by (Gurtin, 1981; Rajagopal and Tao, 1995; Atkin and Craine, 1976)

$$
\begin{aligned}
&\boldsymbol{r}\left[\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}\right] = \nabla \cdot \mathbf{T} + \boldsymbol{r}\mathbf{g} && \text{in } \Omega_1 \\
&\boldsymbol{r}_i\left[\frac{\partial \mathbf{v}_i}{\partial t} + (\nabla \mathbf{v}_i)\mathbf{v}_i\right] = \nabla \cdot \mathbf{T}_i + \mathbf{m}_i + \boldsymbol{r}_i\mathbf{g} && \text{in } \Omega_2 , \quad i = 1, n
\end{aligned}
\tag{2}
$$

where in the region Ω_1 the stress is given by Cauchy stress tensor $\mathbf{T}$ while in the mixture region Ω_2 $\mathbf{T}_i$ represents the partial stress tensor associated with the i-th constituent. The body force is represented by $\mathbf{g}$ everywhere while $\mathbf{m}_i$ is the momentum supply due to the interaction between the i-th constituent and the other constituents, being defined in the mixture region Ω_2 and representing an interactive body force. This momentum source arises from the possible existence of n distinct velocity fields in an n-constituents mixture at each spatial point so that the net momentum supply to the mixture due to all the constituents is zero:

$$
\sum_{i=1}^{n} \mathbf{m}_i = 0 \qquad \text{in } \Omega_2
\tag{3}
$$

The conservation of energy in both regions Ω_1 and Ω_2 is expressed as (Gurtin, 1981; Rajagopal and Tao, 1995; Atkin and Craine, 1976; Martins-Costa et al., 1993)

$$rc\left[\frac{\partial T}{\partial t}+(\nabla T)\mathbf{v}\right]=-\nabla\cdot\mathbf{q}+\mathbf{T}\cdot\mathbf{D}+q^{''} \qquad\qquad \text{in } \Omega_1$$

$$r_i c_i\left[\frac{\partial T_i}{\partial t}+(\nabla T_i)\mathbf{v}_i\right]=-\nabla\cdot\mathbf{q}_i+\mathbf{T}_i\cdot\mathbf{D}_i+q_i^{''}+\mathbf{y}_i \qquad \text{in } \Omega_2\,, \qquad i=1,n$$

(4)

where the parameters defined in the region Ω_1 are T, representing the fluid temperature, c, its specific heat, $\mathbf{D}$, the symmetrical part of $\nabla\mathbf{v}$ (the fluid velocity gradient) in such a way that the viscous dissipation is given by the inner product $\mathbf{T}\cdot\mathbf{D}$, the fluid heat flux vector is given by $\mathbf{q}$ and, finally, the external heat supply to the fluid is $q^{''}$. In the mixture region Ω_2, the i-th constituent temperature is given by T_i, the partial heat flux associated with the i-th constituent by $\mathbf{q}_i$, the symmetrical part of $\nabla\mathbf{v}_i$ by $\mathbf{D}_i$ and the external heat supply to the i-th constituent is $q_i^{''}$. Besides, c_i represents the i-th constituent specific heat, measured in a continuum mechanics context and $\mathbf{y}_i$ an internal heat source, both defined in Ω_2. The field $\mathbf{y}_i$ arises from the possible existence of n distinct temperature fields in an n-constituents mixture at each spatial point and represents the i-th constituent thermal interaction with the remaining constituents of the mixture in such a way that the following equation must hold (Martins-Costa et al., 1992; 1993):

$$\sum_{i=1}^{n}\mathbf{y}_i=0 \qquad\qquad \text{in } \Omega_2$$

(5)

2.2. Constitutive equations

Constitutive relations are required to solve the hydrodynamic problem considering a Newtonian fluid in a domain $\Omega_1\cup\Omega_2$. More specifically, for the stress tensor in Ω_1 and for the partial stress tensor and momentum source in Ω_2. Williams (1978) has proposed a constitutive relation for the partial stress tensor, by making a clear analogy to an incompressible Newtonian fluid as

$$\mathbf{T}_F=-p_F\mathbf{I}+2h\mathbf{D}_F\,,$$

with p_F representing the pressure acting on the fluid constituent and h a parameter related to its viscosity. Sampaio and Williams (1977) have analyzed mixtures of fluids and concluded that h should be represented by the product of j^2 and m, the base fluid viscosity (measured considering a continuum mechanics viewpoint). Williams (1978) extended that relation for solid-fluid mixtures proposing a scalar multiplicative factor l to account for the porous matrix presence - arguing that the presence of a porous matrix might enhance the drag or "present less window for the fluid to react on itself". The pressure acting on the fluid constituent is related to the pressure p acting on the mixture by $p_F=j\,p$.

In saturated flows the momentum source $\mathbf{m}_F$ accounts for the drag forces effect. Williams (1978) has proposed a constitutive relation for $\mathbf{m}_F=\hat{\mathbf{m}}_F(\mathbf{v}_F)$, by considering a limit case in which Darcy's law assumptions are valid - an infinite cross section (rigid, homogeneous and isotropic) porous medium saturated by a steady-state fluid flow having only the gradient of pressure as driving force.

So, the following constitutive relations will be employed in $\Omega_1\cup\Omega_2$:

$$\mathbf{T}=-p\mathbf{I}+2m\mathbf{D} \qquad\qquad \text{in } \Omega_1$$

$$\mathbf{T}_F=-pj\,\mathbf{I}+2lj^2 m\mathbf{D}_F \qquad \text{in } \Omega_2$$

$$\mathbf{m}_F=-\frac{j^2 m}{K} \qquad\qquad \text{in } \Omega_2$$

(6)

where m is the actual fluid viscosity, j and K are the porous matrix porosity and specific permeability, l is a scalar parameter depending on the porous matrix microstructure and the index F refers to the fluid constituent.

In order to build a mechanical model for the channel with permeable boundaries, constitutive relations are yet required for the heat flux $\mathbf{q}$ in Ω_1 and the partial heat fluxes $\mathbf{q}_S$ and $\mathbf{q}_F$ and the internal supplies $\mathbf{y}_F$ and $\mathbf{y}_S$ in Ω_2.

A constitutive model for the partial heat flux has been proposed by Saldanha da Gama (1989) considering a mixture composed by n rigid solids, in an analogy with Fourier's law $\mathbf{q}_i = \hat{\mathbf{q}}_i(\nabla T_i)$. To account for solid-fluid mixtures, the proportionality factor may depend on the i-th constituent thermal conductivity, on the n-constituents' temperatures and on the mixture internal structure and kinematics. The following constitutive relation will be considered

$$\mathbf{q}_i = -\Lambda k_i \boldsymbol{j}_i(\nabla T_i)$$

where k_i is the thermal conductivity of the material that is represented by the i-th constituent (the usual thermal conductivity – measured in a continuum mechanics viewpoint), T_i the i-th constituent temperature and, finally, Λ is a factor that may depend on the n-constituents' temperatures and on the mixture internal structure and kinematics. A linear dependence on $\boldsymbol{j}_a$ – the volume fraction of the $\boldsymbol{a}$-th constituent in the mixture has been assumed, making an analogy with the average techniques (Whitaker, 1969) broadly used to describe flows through porous media from a continuum mechanics viewpoint in which the fluid actually occupies a fraction $\boldsymbol{j}$ of the mixture volume.

The Λ factor must be a positive-valued parameter in order to satisfy the entropy inequality restriction, which requires that

$$\sum_{i=1}^{n} \frac{-\mathbf{q}_i \cdot \nabla T_i}{T_i} \geq 0$$

Considering the following definition for the heat flux

$$\mathbf{q} = \sum_{i=1}^{n} \mathbf{q}_i$$

combined with a particular example of an ideal binary mixture, Saldanha da Gama (1989) obtained a physical interpretation for Λ, which may be expressed as

$$\Lambda = \frac{1}{R_T \sum_{i=1}^{n} k_i \boldsymbol{j}_i}$$

where R_T is a thermal resistance. The specific values of Λ may be obtained by means of experimental observation.

The energy generation function, $\boldsymbol{y}_i$, which is an internal contribution, represents the energy supply to a given constituent, arising from its thermal interaction with the remaining constituents of the mixture. The $\boldsymbol{y}_i$ function is zero at a given point only if all the constituents are at the same temperature at this point. Any constituent i receives energy from its interaction with the j-constituents at a higher temperature and provides energy to those at a lower temperature, according to the following constitutive relation (Saldanha da Gama, 1989; Martins-Costa et al., 1992; Martins-Costa and Saldanha da Gama, 1996)

$$\boldsymbol{y}_i = \sum_{j=1}^{n} \hat{R}_{ij}(T_j - T_i).$$

The coefficient $\hat{R}_{ij}$ is a positive valued parameter depending on the thermal properties of j-constituents, on their velocities (accounting for convective heat transfer) and on the mixture internal structure. In a binary mixture $-\boldsymbol{y}_S = \boldsymbol{y}_F$ (Martins-Costa et al., 1993) and $R_{FS} = R_{SF}$.

The following constitutive relations will be employed for the heat flux in Ω_1 and for the partial heat fluxes and the internal supplies in Ω_2 (Martins-Costa et al., 1992; Costa Mattos et al., 1995):

$$\begin{aligned}
\mathbf{q} &= -k_f \nabla T && \text{in } \Omega_1 \\
\mathbf{q}_F &= -\Lambda k_f \boldsymbol{j} \nabla T_F && \text{in } \Omega_2 \\
\mathbf{q}_S &= -\Lambda k_s (1-\boldsymbol{j}) \nabla T_S && \text{in } \Omega_2 \\
\boldsymbol{y}_F &= -\boldsymbol{y}_S = R_{FS}(T_S - T_F) && \text{in } \Omega_2
\end{aligned} \qquad (7)$$

in which k_f is the Newtonian fluid thermal conductivity and k_s is the porous matrix thermal conductivity, Λ represents a scalar positive-valued parameter which may depend on both the internal structure and the kinematics of the

mixture and R_{FS} is a positive-valued factor which depends not only on spatial position and on both constituents' thermal properties but also on their velocities, accounting for the convective heat transfer.

2.3. Compatibility conditions

The interface between the regions Ω_1 and Ω_2 is defined by the set $\partial\Omega_I \equiv \bar{\Omega}_1 \cap \bar{\Omega}_2$. At this interface some compatibility conditions must be imposed in order to allow the solution of the problem. According to Williams (1978), since there is no flow across the interface, the following relations must hold

$$\mathbf{v} = \boldsymbol{j}\,\mathbf{v}_F$$
$$\boldsymbol{j}\,\mathbf{Tn}\cdot\mathbf{t} = \mathbf{T}_F\mathbf{n}\cdot\mathbf{t} \tag{8}$$

in which $\mathbf{n}$ is a unit outward normal to $\partial\Omega_I$ and $\mathbf{t}$ is any tangent to $\partial\Omega_I$. The compatibility equations (8) simulate the experimental condition proposed by Beavers and Joseph (1967) and confirmed and generalized by several authors (Nield and Bejan, 1992). The idealized model for a porous medium, employed by Taylor (1971) and Richardson (1971) gives theoretical support to Beavers and Joseph's condition. Equations (8), obtained from the solution of thermodynamically consistent equations in both regions, do not suffer from the difficulty of matching the porous medium flow equations with the Navier-Stokes equation, discussed by Nield and Bejan (1992). Williams (1978), based on a no-slip condition, concluded that the velocity should be zero (porous matrix velocity) on the solid parts of the boundary matching the fluid diffusing velocity on the fluid parts of the boundary. Besides both solid and fluid receive shearing stress from the fluid stream at the pure fluid region. It should be noticed that the pure fluid velocity and the fluid constituent velocity (in the mixture region) have distinct values at the interface.

In order to assure continuity of the temperature field and continuity and adequate distribution of the normal heat flux throughout the interface, the following compatibility conditions are imposed at $\partial\Omega_I$ (Martins-Costa et al., 1994):

$$T = \boldsymbol{j}\,T_F + \left(1-\boldsymbol{j}\right)T_S \qquad \text{at } \partial\Omega_I$$
$$\boldsymbol{j}\,\mathbf{q}\cdot\mathbf{n} = \mathbf{q}_F\cdot\mathbf{n} \qquad \text{at } \partial\Omega_I \tag{9}$$
$$\left(1-\boldsymbol{j}\right)\mathbf{q}\cdot\mathbf{n} = \mathbf{q}_S\cdot\mathbf{n} \qquad \text{at } \partial\Omega_I$$

where the first equation in (9) represents the continuity in the temperature field (the temperature of the fluid must be equal to the temperature of the mixture at the interface), while the latter ones represent the distribution of the heat flux from/to the mixture, at the interface. In Martins-Costa et al. (1994) - where two distinct flow regions are considered with the "pure fluid" limited by an impermeable surface with prescribed velocity - these compatibility conditions are adequately discussed.

Equations (9) impose continuity of temperature and heat flux fields at the interface. Vafai and Kim (1990) have already used similar conditions. It should be mentioned that these authors' compatibility conditions might be considered analogous to equation (9), once thermal nonequilibrium has been allowed.

3. Convection in permeable ducts

3.1. Hydrodynamic problem

Considering the porous matrix rigid and at rest it suffices to solve mass and momentum balances for the fluid constituent in the mixture region, while both constituents in this region must satisfy the energy balance. At this point the regions composing the permeable duct considered in this work are defined as

$$\Omega_1 \equiv \left\{(r,z) \text{ such that } 0 < r < R_1, \, 0 < z < L\right\}$$
$$\Omega_2 \equiv \left\{(r,z) \text{ such that } R_1 < r < R_2, \, 0 < z < L\right\} \tag{10}$$
$$\partial\Omega_I \equiv \left\{(r,z) \text{ such that } r = R_1, \, 0 < z < L\right\}$$

Supposing that the flow is not affected by the thermal problem, the velocity fields in Ω_1 and Ω_2 may be analytically obtained, provided that some hypotheses are considered. The former comes naturally from the problem geometry – the flow is axisymmetric, rendering a cylindrical coordinates system the most convenient one. The latter is that the flow is assumed to be fully developed – resulting in being not only a steady-state one, but also independent of the z-variable.
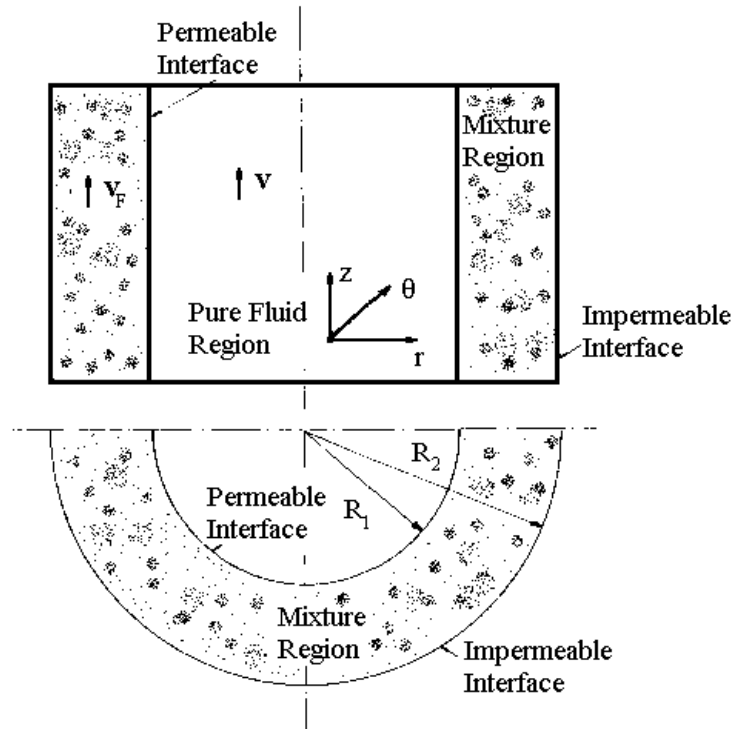
Figure 1. Problem scheme.

Denoting by v and v_F the z-components of $\mathbf{v}$ and $\mathbf{v}_F$ and taking into account that $\mathbf{r}_F = \boldsymbol{j}\,\mathbf{r}_f$ the hydrodynamic problem is represented by

$$
\begin{aligned}
&\frac{\partial \boldsymbol{f}}{\partial z} + \boldsymbol{m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = 0 && 0 \leq r < R_1 && 0 < z < L \\[2mm]
&\boldsymbol{j}\,\frac{\partial \boldsymbol{f}}{\partial z} + \boldsymbol{m}\boldsymbol{j}^2\boldsymbol{l}\,\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_F}{\partial r}\right) - \frac{\boldsymbol{j}^2\boldsymbol{m}}{K}v_F = 0 && R_1 < r < R_2 && 0 < z < L \\[2mm]
&\boldsymbol{f} \equiv -p + \boldsymbol{r}g_z z \qquad\Rightarrow\qquad \frac{\partial \boldsymbol{f}}{\partial z} = -\frac{\partial p}{\partial z} + \boldsymbol{r}g_z \\[2mm]
&v \text{ finite} && \forall r && 0 < z < L \\[2mm]
&v_F = 0 && r = R_2 && 0 < z < L \\[2mm]
&\boldsymbol{j}\,v_F = v && r = R_1 && 0 < z < L \\[2mm]
&\boldsymbol{l}\boldsymbol{j}\,\frac{\partial v_F}{\partial r} = \frac{\partial v}{\partial r} && r = R_1 && 0 < z < L
\end{aligned}
\tag{11}
$$

In equation (11) the boundary conditions are represented by the fourth and fifth equations – the former being employed to assure that the velocity must remain finite at $r = 0$ and the latter representing the classical no-slip condition at the impermeable interface located at $r = R_2$, while the compatibility of velocity and momentum transport across the interface located at $r = R_1$ is represented in the last two equations. The solution of equations (11) is given by (Saldanha da Gama and Sampaio, 1983):

$$v = \frac{K}{\boldsymbol{m}} \frac{\partial f}{\partial z} \left\{ \frac{\boldsymbol{l}}{4} \left(\boldsymbol{c}_1^2 - \boldsymbol{c}^2 \right) + \left(1 - \boldsymbol{y}_{I0} \right) - \frac{\boldsymbol{y}_{KI}}{\boldsymbol{y}_{IK}} \left(\boldsymbol{y}_I - \frac{\boldsymbol{c}_1}{2} \right) \right\}$$

$$v_F = \frac{K}{\boldsymbol{mj}} \frac{\partial f}{\partial z} \left\{ \left(1 - \frac{\mathrm{I}_0 \left(\boldsymbol{c} \right)}{\mathrm{I}_0 \left(\boldsymbol{c}_2 \right)} \right) + \left(\frac{\boldsymbol{c}_1}{2} - \boldsymbol{y}_I \right) \left(\frac{\mathrm{K}_0 \left(\boldsymbol{c} \right) \mathrm{I}_0 \left(\boldsymbol{c}_2 \right) - \mathrm{K}_0 \left(\boldsymbol{c}_2 \right) \mathrm{I}_0 \left(\boldsymbol{c} \right)}{\boldsymbol{y}_{IK}} \right) \right\}$$

$$\boldsymbol{c} = \frac{r}{\sqrt{K\boldsymbol{l}}} \qquad \boldsymbol{c}_1 = \frac{R_1}{\sqrt{K\boldsymbol{l}}} \qquad \boldsymbol{c}_2 = \frac{R_2}{\sqrt{K\boldsymbol{l}}} \qquad \boldsymbol{y}_{I0} = \frac{\mathrm{I}_0 \left(\boldsymbol{c}_1 \right)}{\mathrm{I}_0 \left(\boldsymbol{c}_2 \right)} \qquad \boldsymbol{y}_I = \frac{\mathrm{I}_1 \left(\boldsymbol{c}_1 \right)}{\mathrm{I}_0 \left(\boldsymbol{c}_2 \right)} \tag{12}$$

$$\boldsymbol{y}_{KI} = \mathrm{K}_0 \left(\boldsymbol{c}_1 \right) \mathrm{I}_0 \left(\boldsymbol{c}_2 \right) - \mathrm{K}_0 \left(\boldsymbol{c}_2 \right) \mathrm{I}_0 \left(\boldsymbol{c}_1 \right)$$

$$\boldsymbol{y}_{IK} = \mathrm{I}_0 \left(\boldsymbol{c}_2 \right) \mathrm{K}_1 \left(\boldsymbol{c}_1 \right) + \mathrm{I}_1 \left(\boldsymbol{c}_1 \right) \mathrm{K}_0 \left(\boldsymbol{c}_2 \right)$$

where v – the fluid velocity, is defined for $0 < r < R_1$ (the pure fluid region) while the fluid constituent velocity v_F is defined for $R_1 < r < R_2$ (the mixture region) and I_0, I_1, K_0 and K_1 are modified Bessel functions.

It is important to notice that for the limiting case in which the permeability $K \to 0$ a classical Hagen-Poiseuille flow is recovered for the fluid velocity v, since $\boldsymbol{c}^2 K\boldsymbol{l} = r^2$.

3.2. Thermal problem

Assuming the external sources $q^{''}$ and $q_i^{''}$ equals to zero and neglecting $\mathbf{T} \cdot \mathbf{D}$ and $\mathbf{T}_F \cdot \mathbf{D}_F$, the energy equations reduce to

$$\boldsymbol{a} v \frac{\partial T}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \qquad\qquad 0 < r < R_1 \quad\quad 0 < z < L$$

$$\boldsymbol{a}_F v_F \frac{\partial T_F}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_F}{\partial r} \right) + \frac{\partial^2 T_F}{\partial z^2} + \boldsymbol{b}_F \left(T_S - T_F \right) \qquad R_1 < r < R_2 \quad\quad 0 < z < L$$

$$0 = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T_S}{\partial r} \right) + \frac{\partial^2 T_S}{\partial z^2} + \boldsymbol{b}_S \left(T_F - T_S \right) \qquad R_1 < r < R_2 \quad\quad 0 < z < L \tag{13}$$

$$\boldsymbol{a} = \frac{\boldsymbol{r}_f c_f}{k_f} \qquad \boldsymbol{a}_F = \frac{\boldsymbol{r}_f c_f}{\Lambda k_f} \qquad \boldsymbol{b}_F = \frac{R_{FS}}{\Lambda k_f \boldsymbol{j}} \qquad \boldsymbol{b}_S = \frac{R_{FS}}{\Lambda k_s \left(1 - \boldsymbol{j} \right)}$$

subject to the following boundary conditions

$$
\begin{array}{llll}
T = T_0 & 0 < r < R_1 & z = 0 \\[4pt]
T_F = T_{F0} & R_1 < r < R_2 & z = 0 \\[4pt]
\dfrac{\partial T_S}{\partial z} = 0 & R_1 < r < R_2 & z = 0 \\[4pt]
T \quad \text{finite} & \forall r & 0 < z < L \\[4pt]
\dfrac{\partial T_F}{\partial r} = 0 & r = R_2 & 0 < z < L \\[4pt]
\dfrac{\partial T_S}{\partial r} = 0 & r = R_2 & 0 < z < L \\[4pt]
\dfrac{\partial T}{\partial z} = 0 & 0 < r < R_1 & z = L \\[4pt]
\dfrac{\partial T_F}{\partial z} = 0 & R_1 < r < R_2 & z = L \\[4pt]
\dfrac{\partial T_S}{\partial z} = 0 & R_1 < r < R_2 & z = L
\end{array}
\tag{14}
$$

and to the following compatibility conditions at the interface

$$T = \boldsymbol{j}\, T_F + \left(1 - \boldsymbol{j}\right) T_S \qquad\qquad r = R_1 \qquad 0 < z < L$$

$$\frac{\partial T}{\partial r} = \Lambda \frac{\partial T_F}{\partial r} \qquad\qquad r = R_1 \qquad 0 < z < L \qquad\qquad (15)$$

$$\frac{\partial T}{\partial r} = \Lambda \frac{k_s}{k_f} \frac{\partial T_S}{\partial r} \qquad\qquad r = R_1 \qquad 0 < z < L$$
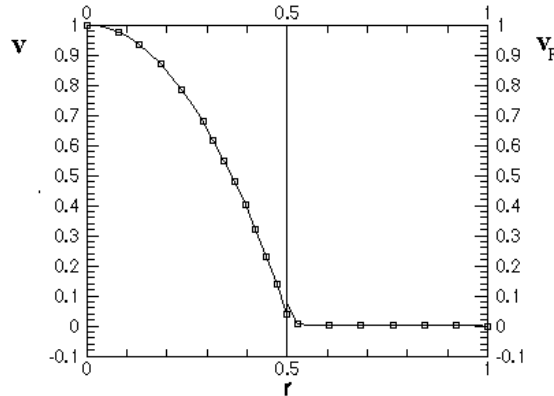
Figure 2. Fluid and fluid constituent velocity profiles.
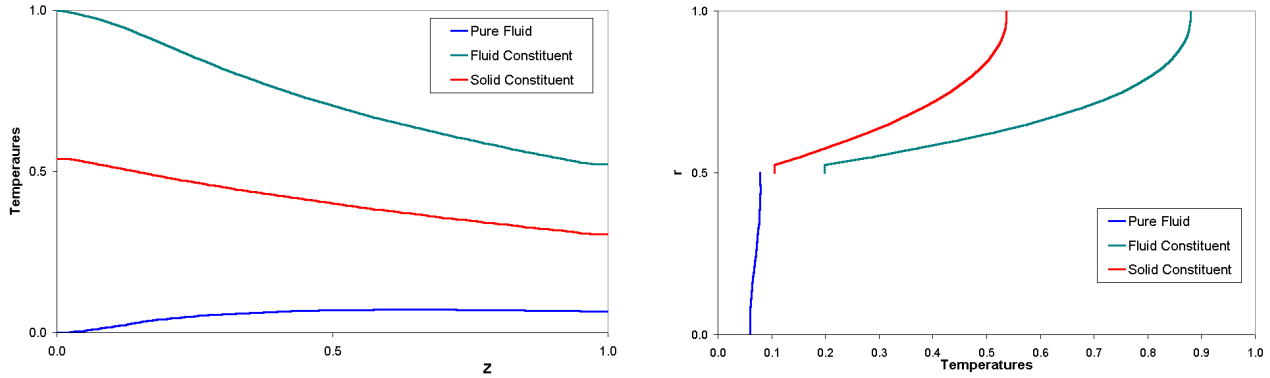
4. Numerical results
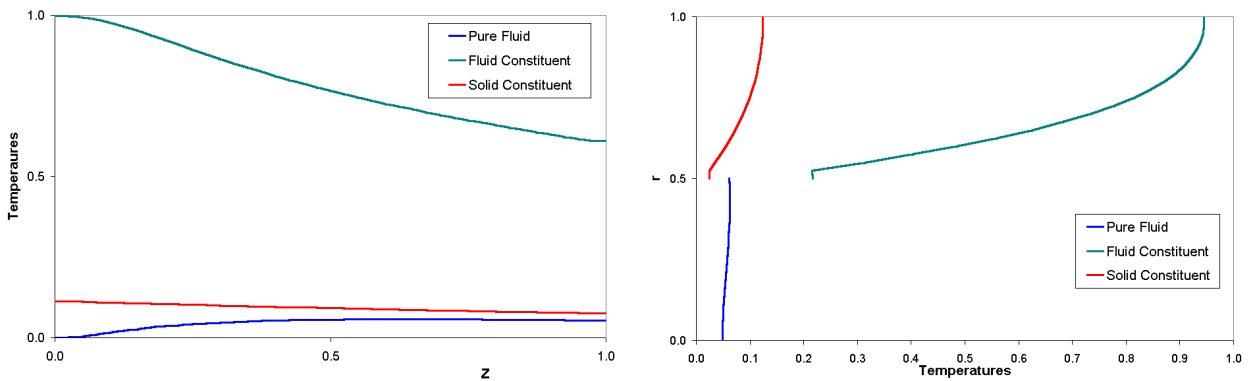
Figure 3. Fluid temperature at a section $r = r_a$ and fluid and solid constituents temperatures at $r = r_b$ along z and temperature profiles at a central section for $\boldsymbol{a} = \boldsymbol{a}_F = 10^5$ and $\boldsymbol{b}_F = \boldsymbol{b}_S = 10^0$.

Since no analytical solution for the heat transfer problem – given by equations (13)-(15) is known, a finite difference approach (Euvrard, 1987) is employed to find numerical approximations to its solution. For the diffusive terms, a central finite difference scheme discretization was used, while an upwind scheme was employed in the convective term discretization.

As the temperature coefficients matrix associated to the system of equations (13) is sparse, a grid description, in which each temperature (fluid and fluid and solid constituents) possesses two indexes, according to its position on the grid, is used. Each iteration, represented in such a way to allow an effective storage scheme with memory reutilization, is then solved with the help of the Gauss-Seidel method.

In this section some results, considering a long permeable duct with a fixed geometry – namely a duct with unit external radius and 100 units length, are presented. Figure 1 shows the geometry of the problem characterized by velocity

profiles given by equation (12) and temperatures satisfying equations (13) to (15). In Figure 2 velocity profiles for the pure fluid and the fluid constituent – the former flowing in the region inside the duct $0 < r < R_1$ and the latter flowing in the porous region – given by $R_1 < r < R_2$ are presented, considering Darcy number – defined as the ratio between the porous matrix permeability and the square of a characteristic dimension $Da = K / R_2^2$ given by $Da = 10^{-4}$. The fluid and the fluid constituent are such that $v = v / \left[\dfrac{K}{m} \dfrac{\partial f}{\partial z} \right]$ and $v_F = v_F / \left[\dfrac{K}{mj} \dfrac{\partial f}{\partial z} \right]$.
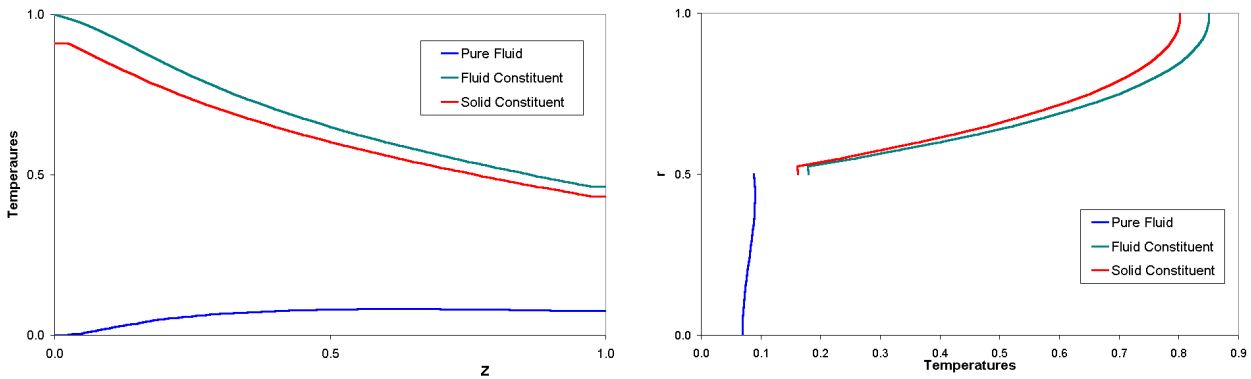

Figure 4. Fluid temperature at a section $r = r_a$ and fluid and solid constituents temperatures at $r = r_b$ along z and temperature profiles at a central section for $a = a_F = 10^5$ and $b_F = b_S = 10^1$.

Figure 5. Fluid temperature at a section $r = r_a$ and fluid and solid constituents temperatures at $r = r_b$ along z and temperature profiles at a central section for $a = a_F = 10^5$ and $b_F = b_S = 10^2$.

Figures 3 to 6 present – at the left-hand side – the temperature behavior for the fluid flowing inside the porous duct (in the so-called pure fluid region) by considering a central section characterized by $r_a = R_1 / 2$ as well as the thermal behavior of both the fluid and solid constituents, depicted in a central section of the porous region – namely at $r_b = R_1 + (R_2 - R_1)/2$, both versus longitudinal coordinate z. At the right-hand side, Figures 3 to 6 show temperature profiles at a cross section located in the middle of the porous duct ($z = L/2$) in which the pure fluid occupies the region $0 < r < R_1$ and the fluid and the solid constituents coexist superposed in the region $R_1 < r < R_2$. All these figures have been obtained by considering the fluid inlet temperature $T_0 = 0$ in the fluid region Ω_1 and the fluid constituent inlet temperature $T_{F0} = 1$ in the mixture region Ω_2 – representing the permeable duct. Besides, in all the shown examples a unitary ratio of heat conduction between both constituents – namely $\dfrac{k_f j}{k_s (1-j)} = 1$ has been considered.

It is worth of notice that two distinct temperature profiles are shown in the permeable duct region, since thermal nonequilibrium has been assumed.

Figures 3 to 5 show the influence of the b_i - parameter on the temperature behavior. This parameter represents a ratio of the heat transfer between both solid and fluid constituents – taken into account in the energy generation term y_i that plays the role of an internal heat source to the i -th constituent heat conduction. A decrease in the b_i -parameter results in a tendency to a common value for both the solid and the fluid constituents' temperatures in the porous region. In other words, as b_i decreases the tendency toward thermal equilibrium between both constituents increases.
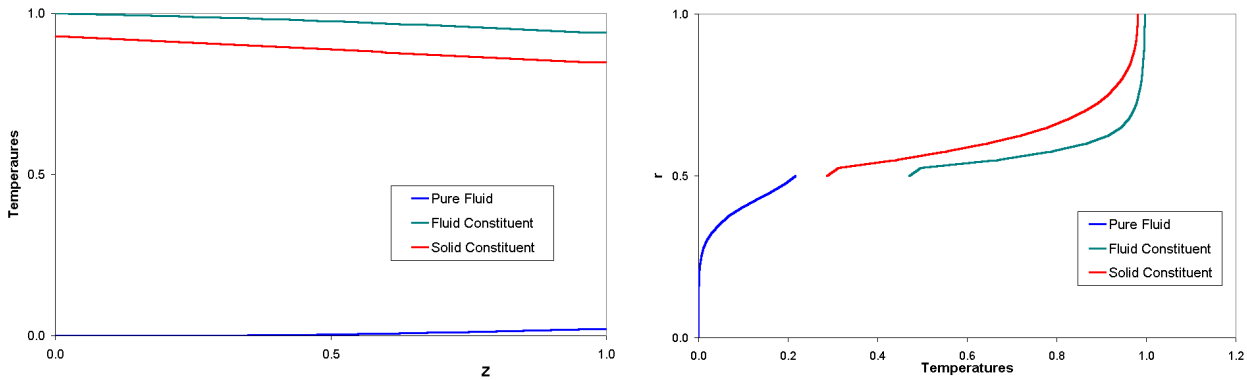

Figure 6. Fluid temperature at a section $r = r_a$ and fluid and solid constituents temperatures at $r = r_b$ along z and temperature profiles at a central section for $a = a_F = 10^6$ and $b_F = b_S = 10^2$.

The influence of the thermal diffusivity – represented by a parameter $1/a$ in the pure fluid region and redefined by a parameter $1/a_F$ in the mixture region where Λ has been included to account for the mixture structure and kinematics – may be noticed by comparing Figures 5 and 6.

As the a - parameter increases (and the thermal diffusivity decreases) both constituents' temperatures at a central section $r = r_b$ in the porous region tend to the fluid constituent inlet temperature ($T_{F0} = 1$) while in the pure fluid region the fluid temperature at $r = r_a$ tend to the fluid inlet temperature ($T_0 = 0$). An effect of the inlet boundary condition being carried out through the flow is suggested by this behavior. This effect is more pronounced as the a - parameter increases.

5. Final remarks

A mixture theory model has been employed to describe the momentum and energy transport in a permeable duct with thermal nonequilibrium being allowed. Two distinct flow regions were considered – in the porous region (the duct), fluid and solid (the porous matrix) were treated as continuous constituents of a binary mixture while the classical continuum mechanics equations have been recovered in the pure fluid region – namely the region inside the duct, where the mixture was reduced to a single constituent. Adequate conditions at the interface pure fluid-mixture were imposed.

6. Acknowledgements

7. Copyright Notice

8. References

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