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THE GENERALIZED BOUNDARY LAYER THEORY AND THE CONVERGENT CHANNEL FLOW SIMILARITY ANALYSIS

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Abstract. *In the present work, a recently developed generalized version of the boundary Layer formulation will be used to analyze the convergent channel flow. This formulation combine the inviscid flow formulation and the classical boundary layer equations into a single and more general theory that disregards the needs for any type of viscid-inviscid interaction. The proposed formulation will be analyzed through similarity approach, and an ordinary differential equation will be derived which represents an extension of the classical sink flow solution equation.*

Keywords*. asymptotic analysis, generalized boundary layer, similarity analysis*

1. Introduction

The boundary layer theory, developed in the beginning of the past century, represents one of the most important results obtained in fluid mechanics history. In spite of its great success, there are some situations where the boundary layer formulation does not give good results, as the description of the flow pattern at separation region, for example. The reason for that lack of success is mainly related to the division of the flow into two regions, the near wall viscous region and the inviscid flow region far from the solid boundary. This division assumes initially that the inviscid flow can be solved independently of the viscous region. This approach gives good results only in some specific circumstances, where the displacement thickness is small. It is also recognized that the boundary layer theory yields only the first term, in an asymptotic solution of the Navier Stokes equation for large Reynolds number.

Those limitations stimulated the efforts to construct a higher order correction for the boundary layer formulation, which takes into account the effects of the displacement thickness. Many authors have analyzed the boundary layer displacement thickness-higher order correction. Van Dyke (1962 a, b) developed a systematic unified theory of higher order effects using the so-called method of inner and outer expansions. This technique was in great detail developed Kaplun (1954), Lagestron & Cole (1955) and basically consist in the construction of two complementary asymptotic expansions that can be matched in their overlap region of common validity. Many articles can be found in literature in which the viscous-inviscid interactive process is discussed for some practical situations (Whitfield et al, 1981; Strawn et al, 1984; Veldman, 1981; Kwon and Pletcher, 1986; DeJanette and Radcliffe, 1996; Tuncer et al, 1995; Smith, 1977; Smith et al, 1984) and where the singular perturbation character of the boundary layer formulation is analyzed in detail. However, the discussion of which set of differential equations corresponds to the asymptotic limit of the Navier Stokes equations, as the Reynolds number tends to infinity is not yet settled since neither the boundary layer equations nor the inviscid formulation, described by the Euler equations, can completely characterize the flow.

In a recent work Cruz (2002) proposed an alternative asymptotic formulation for the large Reynolds number flow. It this formulation, no previous division of the flow into a viscous and in a inviscid region is required, disregarding the need of any type of viscous-inviscid interaction.

In this work the Generalized Boundary Layer formulation proposed by Cruz (2002) is used to analyze the convergent channel flow. An ordinary differential equation is developed which represents a generalization of classical solution. Numerical solutions of this equation are presented and analyzed.

2. The Generalized Boundary Layer Equation

The asymptotic method described in Cruz (2002), will now be used to analyze the behavior of the Navier-Stokes equations as Reynolds number tends to infinity. The main objective is to determinate which set of differential equations can asymptotically describe the solution of the Navier-Stokes equations for high values of the Reynolds number. For a laminar incompressible, stationary and two-dimensional flow of a Newtonian fluid, the continuity and the momentum equations can be written as follows:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{2}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial P}{\partial y}+\frac{1}{Re}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{3}$$

In the above equations the dimensionless variables are defined using a characteristic length and a characteristic velocity of the flow; Re stands for the Reynolds number, which is assumed to be large, i.e. ($1/Re<<1$)

The intermediate variables are defined as:

$$\hat{y}=\frac{y}{\eta(\varepsilon)} \tag{4}$$

$$\hat{v}=\frac{v}{\eta(\varepsilon)} \tag{5}$$

where $\varepsilon=\frac{1}{Re}$.

Inserting Eqs. (4) and (5) into Eqs. (1), (2) and (3) the following expressions are derived.

$$\frac{\partial u}{\partial x}+\frac{\partial \hat{v}}{\partial \hat{y}}=0 \tag{6}$$

$$u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x}+\frac{1}{Re}\left[\frac{\partial^2 u}{\partial x^2}+\frac{1}{\eta(\varepsilon)^2}\frac{\partial^2 u}{\partial \hat{y}^2}\right] \tag{7}$$

$$\eta(\varepsilon)u\frac{\partial \hat{v}}{\partial x}+(\eta(\varepsilon))\hat{v}\frac{\partial \hat{v}}{\partial \hat{y}}=-\frac{1}{(\eta(\varepsilon))}\frac{\partial P}{\partial \hat{y}}+\frac{1}{Re}\left[(\eta(\varepsilon))\frac{\partial^2 \hat{v}}{\partial x^2}+\frac{1}{(\eta(\varepsilon))}\frac{\partial^2 \hat{v}}{\partial \hat{y}^2}\right] \tag{8}$$

Applying the η-limit onto Eqs. (7) and (8) respectively one gets:

For the momentum equation in x-direction:

$$\mathrm{ord}(\eta)=\mathrm{ord}(1): \quad u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x} \tag{9}$$

$$\mathrm{ord}(1)>\mathrm{ord}(\eta)>\mathrm{ord}(\sqrt{\varepsilon}): u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x} \tag{10}$$

$$\mathrm{ord}(\eta)=\mathrm{ord}(\sqrt{\varepsilon}): u\frac{\partial u}{\partial x}+\hat{v}\frac{\partial u}{\partial \hat{y}}=-\frac{\partial P}{\partial x}+\frac{\partial^2 u}{\partial \hat{y}^2} \tag{11}$$

$$\mathrm{ord}(\eta) < \mathrm{ord}\left(\sqrt{\varepsilon}\right): \frac{\partial^2 u}{\partial \hat{y}^2} = 0 \tag{12}$$

For the momentum equation in y-direction:

$$\mathrm{ord}(\ \eta) = \mathrm{ord}(1\ ): u\frac{\partial \hat{v}}{\partial x} + \ \hat{v}\frac{\partial \hat{v}}{\partial y} = -\frac{\partial P}{\partial \hat{y}} \tag{13}$$

$$\mathrm{ord}(\eta) < \mathrm{ord}(1\ ): \frac{\partial P}{\partial \hat{y}} = 0 \tag{14}$$

Each of the above set of expressions has a principal equation in the sense defined by Kaplun (1967). Equation (11) represents the principal relation obtained through application of the η-limit analysis, onto the mean momentum equation in x-direction, and Eq. (13) is the principal equation for the y direction. Since both original expressions for the two directions, x and y, have only one principal equation, the asymptotic behavior of the Navier-Stokes equations as $\varepsilon \to 0$ can be described by the solution of the following system:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial y^2} \tag{15}$$

$$u\frac{\partial v}{\partial x} + \ v\frac{\partial v}{\partial y} = -\frac{\partial P}{\partial y} \tag{16}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{17}$$

It is important to note that Eqs. (15) to (17) contain both the boundary layer and the inviscid flow formulation, as a particular case. The boundary layer equations can be obtained if one considers the case of ord(η)=ord($\sqrt{\varepsilon}$). For this particular value of η, the principal equation of the momentum expression for the x-direction is recovered, (Eq. (11)), for the y-direction, however, the equation that arises is not the principal one, it is Eq. (14). Therefore the boundary layer formulation is restricted to the near wall region since Eq. (14) can only be applied for ord(η)< ord(1); furthermore, because Eq. (14) is not a principal equation, its solution depends on the solution of the Eq. (13) which is not contained on the boundary layer formulation.

Hence, in this sense, the boundary layer formulation is not a self-contained theory. The same arguments can be used to show that the Euler equations are also not a self-contained theory for the no-slip boundary condition. Thus, none of the above-cited formulations can by itself be considered as the asymptotic limit (as $\varepsilon \longrightarrow 0$) of the momentum equations.

From the point of view of asymptotic methods, it is common to consider the classical boundary layer formulation as the "inner" solution of the problem and the Euler equations as the "outer" solution. The global solution is then obtained by matching the two solutions, according to a "matching principle" (see, e.g., Van Dyke (1964)). Please note that the above mentioned procedure is usually adopted in problems where a single differential dominates the analysis, and in which the η-limit process furnishes two principal equations. In the present study, however, although we have two principal equations (Eq.(11) and Eq. (13)), they are not part of the same differential equation. Thus, since each of the momentum equations in the two principal directions has only one principal equation, valid for different values of the stretching variable η(ε), the concept of a global "inner" or "outer" solution valid for the Navier-Stokes equations (Eq. (6) to Eq.(8)) should be analyzed more carefully.

It should be noted that the classical boundary layer approach does not in fact use the matching principle. According to this principle, the inner limit of the outer solution (the inviscid flow solution) should be matched to the outer limit of the inner solution (the boundary layer solution). However in the Prandtl´s theory, the first step of the analysis is to solve the Euler equations, using the solid wall boundary condition. The inviscid flow result is then used as boundary conditions for the boundary layer equations.

Many authors have used different types of viscid-inviscid procedures in order to describe flow properties; these approaches use the boundary layer and the Euler equations to correct each other. Although those procedures have shown their applicability in many situations, as, for example, in the description of a separation region, their implementation is not always simple, and requires sometimes, the introduction of some artificial boundary conditions.

On the formulation described by the Eqs. (15) to (17), the viscid-inviscid interactions are not necessary, since no division of the flow into a viscous region near the wall, and inviscid region far from it is introduced.

3. The quasi-similar equation

The same approach used in Cruz (2002) to derive a for the deduction of the generalized Blasius equation will be used here to obtain a new version of the classical convergent channel flow. As a first step it is necessary to introduce stream function into the equations set (15)-(17) to obtain:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y^2\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}+\frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y\partial x^2}=\frac{1}{Re}\frac{\partial^4\psi}{\partial y^4} \tag{18}$$

Defining now the following variables:

$$\psi=\sqrt{\mu^2/Re}\,f(\eta) \tag{19}$$

$$\eta=\frac{y}{x}\sqrt{Re} \tag{20}$$

Inserting Eq. (19) and Eq. (20) into Eq. (18), the resulting transformed equation is obtained.

$$f''''=2f'f''+\frac{1}{Re}\left(4\eta\ f'^2+2\eta^2 f'f''\right) \tag{21}$$

Where,

$F^{(j)}=\dfrac{\partial^{(j)}F}{\partial\tilde{\eta}^{(j)}}$;

$Re=u_1/\nu$;

$u_1= Q/2\pi$;

and Q represents the flow rate.

The Eq. (21) must be solved according to the following boundary conditions:

$$F'(0,x)=0 \tag{22}$$
$$F(0,x)=0 \tag{23}$$
$$F'(\tilde{\eta}_\infty,x)=1 \tag{24}$$
$$F''(\tilde{\eta}_\infty,x)=0 \tag{25}$$

Where $\tilde{\eta}_\infty$ represents a value of η assumed to be far enough of the solid wall. The Eq. (22) to Eq. (24) represent the classical boundary condition of the similar boundary layer flows, the additional boundary condition characterize the asymptotic behavior of the velocity on η = $\tilde{\eta}_\infty$. It is important to note that Eq. (21) represents a generalized form of the classical channel flow equation (Goldstien, 1938) . The classical channel flow equation can be recovered if the limit when Re $\Rightarrow\infty$ in Eq. (21) is considered, and the resulting equation is integrated once.

4. Results and Discussion

The Eq. (21) was numerically solved using the BFPFD subroutine of the IMSL framework. In Fig. (1) the velocity profiles are show and, as mentioned before, as the Reynolds number increases, the numerical solutions approach the classical result. For lower values of the Reynolds number the numerical solutions of the velocity profile show an "overshoot". This phenomenon was already observed in other Navier-Stokes solutions of some boundary layer character situations for low values of the Reynolds number.

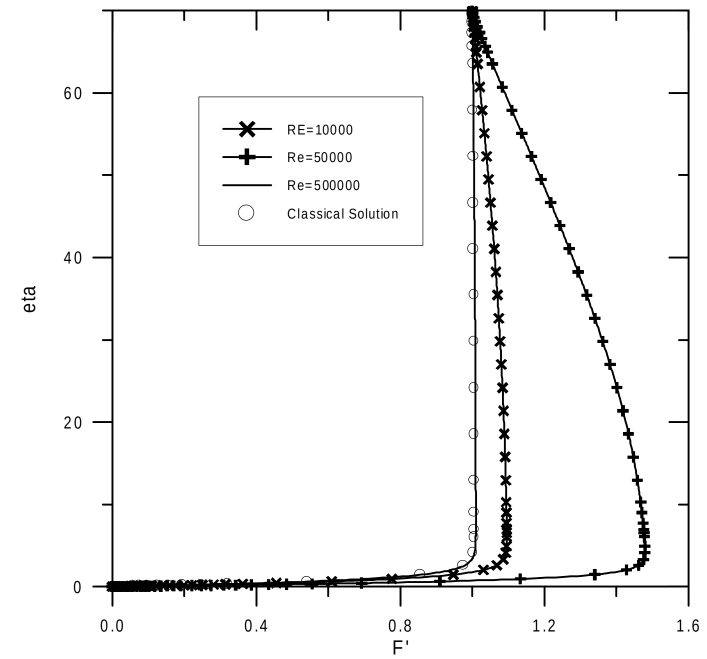

Figure 1 : Nondimensional velocity profile

In Fig. (2) the nondimensional value of the velocity gradient at the wall is show. For low values of the Reynolds number, the velocity gradient at the wall is very high, indicating, also, high values of the wall shear stress. This behavior can be better understood through the analysis of Fig. (1). For low values of the Reynolds number, the velocity profile exhibits very elevated values in the near wall region, causing the appearance of increased velocity gradients in that region. As the Reynolds number decreases the wall shear stress tends asymptotically to the classical situation.

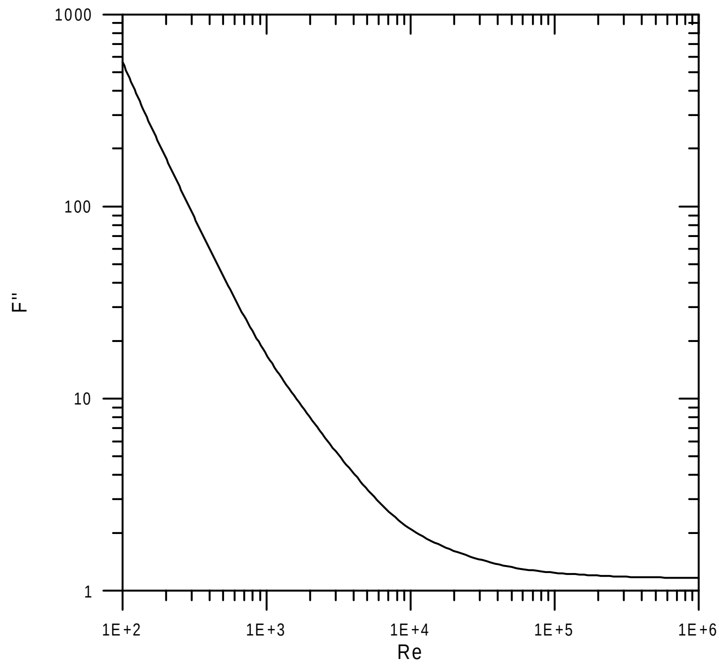

Figure 2: Nondimesional wall velocity gradient

5. Conclusion

In the present work a recently proposed generalized boundary layer formulation was used to analyze the convergent channel flow for a 180^0 situation. This new formulation constitutes a self contained theory and represents asymptotic behavior of Navier Stokes equation as the Reynolds number tends to infinity. One of the main characteristics of the generalized boundary layer theory is that no previous division of flow is required, making unnecessary any type of viscous-inviscid interaction.

A similarity approach was used to obtain a ordinary differential equation which describes the convergent channel and contains the classical solution as a particular case. Numerical solutions of the proposed equation were presented and it was show that for low values of the Reynolds number the velocity profile exhibits a velocity "overshoot" in the near wall region. This "overshoot" causes an increase on the velocity gradients in the near wall region which was correctly described by the present theory.

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