AN IMPROVED LUMPED DIFFERENTIAL APPROACH FOR THE SOLUTION OF THERMAL ABLATION PROBLEMS

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Abstract. *This work is aimed at proposing an improved lumped differential approach for ablative thermal protection, which involves the use of materials with low thermal diffusivity. The results obtained by the proposed technique for an one-dimensional thermal ablation problem in a finite slab are compared against those obtained by previously reported lumped differential solutions. Benchmark results obtained through the integral transform technique are utilized to verify the proposed solution in a realistic ablation problem consisting of a low thermal diffusivity material subjected to a known aerodynamic heating.*

Key-words: Thermal ablation, Heat conduction, Phase-change, Lumped Analysis

1. INTRODUCTION

The so-called coupled integral equations approach (CIEA) is a formulation simplification technique for diffusion problems. A mixed lumped differential formulation is obtained through the use of improved lumping procedures, such as Hermite-type approximations for integrals, on the independent variables selected to be removed. Such approach, recently reviewed by Cotta and Mikhailov (1997), has been already employed for the solution of different classes of heat transfer problems (Aparecido and Cotta, 1988 and 1989; Scofano Neto and Cotta, 1993; Cotta and Ramos, 1993), including the important class of non-linear phase-change problems here under consideration (Cotta *et al.*, 1990 and 1992; Romani *et al.*, 1995).

The most common thermal protection system used for ballistic hypersonic re-entries in the atmosphere involves the addition of an ablative heat shield to the regular structure of re-entry space vehicles. The accurate solution of the ablation problem is required for the proper estimation of the required thickness of the thermal protection layer. Purely numerical methods and hybrid numerical-analytical schemes have been previously employed to solve heat conduction with ablation in finite slabs (Blackwell, 1988; Diniz *et al.*, 1990). However, such accurate approaches are too costly for the optimization of thermal protection systems, when several simulations are required. In such a case, simple approximate formulations of the energy equation in the slab, such as those offered by the CIEA, become imperative. The CIEA was employed by Cotta *et al.* (1990) to solve a two-region phase-change problem, improving the results obtained by the classical lumped system analysis (CLSA) for increasing Stefan numbers. Nevertheless, such type of approach becomes less accurate when considering materials with low thermal diffusivity properties, as those employed in ablative heat shields.

This work is aimed at proposing an improved lumped differential approach for ablative thermal protection, which involves the use of materials with low thermal diffusivity. The improved technique is based upon a heat penetration depth concept (Hogge and Gerrekens, 1982 and 1985) and a front tracking method. The results obtained by the proposed technique for an one-dimensional thermal ablation problem in a finite slab are compared against those obtained by previously reported lumped differential solutions (Cotta *et al.*, 1992). Benchmark results obtained through the integral transform technique (Diniz *et al.*, 1990) are utilized to verify the proposed solution in a realistic ablation problem consisting of a low thermal diffusivity material subjected to an actual thermal load due to aerodynamic heating.

2. PROBLEM FORMULATION

We consider one-dimensional transient heat conduction with ablation in a planar layer, with constant physical properties. Initially, the slab has an uniform temperature , T_0, and a pre-ablation period exists up to time t_0, when the surface at $x=0$ reaches the phase-change temperature, T_{ab}. The initial condition for the ablation problem itself is, therefore, readily obtained from solution of the linear heat conduction problem at time t_0. Ablation is caused by a prescribed heat flux at $x=0$, while the other surface is kept insulated during the whole process. In dimensionless form the problem formulation is given by (Cotta *et al.*, 1992):

Pre-ablation period.

$$\frac{\partial\theta(\eta,\tau)}{\partial\tau} = \frac{\partial^2\theta(\eta,\tau)}{\partial\eta^2}, \qquad 0<\eta<1, \qquad 0<\tau<\tau_0 \tag{1.a}$$

$$\theta(\eta,0) = 0, \qquad 0<\eta<1 \tag{1.b}$$

$$\frac{\partial\theta(1,\tau)}{\partial\eta} = 0, \qquad 0<\tau<\tau_0 \tag{1.c}$$

$$-\frac{\partial\theta(0,\tau)}{\partial\eta} = Q(\tau), \qquad 0<\tau<\tau_0 \tag{1.d}$$

Ablation period.

$$\frac{\partial\theta(\eta,\tau)}{\partial\tau} = \frac{\partial^2\theta(\eta,\tau)}{\partial\eta^2}, \qquad S(\tau)<\eta<1, \qquad \tau>\tau_0 \tag{2.a}$$

$$\theta(\eta,\tau_0) = \theta_0(\eta), \qquad S(\tau)<\eta<1 \tag{2.b}$$

$$\frac{\partial\theta(1,\tau)}{\partial\eta}=0\,, \qquad \tau>\tau_0 \tag{2.c}$$

$$\theta(S,\tau)=1, \qquad \tau>\tau_0 \tag{2.d}$$

where the heat balance at the moving boundary at η =S(τ), is written as:

$$\nu\frac{dS(\tau)}{d\tau}=Q(\tau)+\frac{\partial\theta(S,\tau)}{\partial\eta}\,, \qquad \tau>\tau_0 \tag{2.e}$$

with the initial condition

$$S(\tau_0)=0 \tag{2.f}$$

and $\theta_0(\eta)$ is the temperature profile within the slab at $\tau=\tau_0$ obtained from the solution of the pre-ablation problem. The various dimensionless groups are defined as follows:

$$\theta(\eta,\tau)=\frac{T(x,t)-T_0}{T_{ab}-T_0}\,;\ \eta=\frac{x}{L}\,;\ \tau=\frac{\alpha\,t}{L^2}\,;\ Q(\tau)=\frac{L\,q(t)}{k(T_{ab}-T_0)}\,;\ S(\tau)=\frac{s(\mathrm{t})}{L}\,;\ \nu=\frac{H}{c_p(T_{ab}-T_0)}$$

3. LUMPED DIFFERENTIAL SOLUTIONS

We seek an approximation for the partial differential system (2), through elimination of the spatial dependence, offering an ordinary differential system for the spatially averaged temperature. For this purpose, the integrals that define the average temperature and heat flux within the slab are written as:

$$\theta_{av}(\tau)=\frac{1}{1-S(\tau)}\int_{S(\tau)}^{1}\theta(\eta,\tau)d\eta \tag{3.a}$$

$$\theta(1,\tau)-\theta(S,\tau)=\int_{S(\tau)}^{1}\frac{\partial\theta(\eta,\tau)}{\partial\eta}d\eta \tag{3.b}$$

In this section one recalls some already reported CIEA analysis for ablation problems (Cotta *et al.*, 1992). Here, an improved CIEA formulation is specially developed to solve ablation problems for low thermal diffusivity materials. Herein, all the solutions are based on Hermite-type approximations for the integrals in Eqs. (3.a,b). Different degrees of approximation can be achieved, with increasing analytical involvement but also increasing overall accuracy depending on the order of approximation for each Hermite integration of Eqs. (3.a,b). The resulting systems of coupled ordinary differential equations and all the CIEA results were obtained through mixed symbolic-numerical computation by using the MATHEMATICA software system (Wolfram, 1996; Cotta and Mikhailov, 1997).

3.1 $H_{0,0}/H_{0,0}$ solution (Cotta *et al.*, 1992)

The integrals in Eqs. (3.a,b) are approximated, both, through the trapezoidal rule ($H_{0,0}$ formula) to yield:

$$\theta_{av}(\tau) \cong \frac{1}{2}\left[\theta(S,\tau)+\theta(1,\tau)\right] \tag{4.a}$$

$$\theta(1,\tau)-\theta(S,\tau) \cong \frac{(1-S(\tau))}{2}\left[\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S}+\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=1}\right] \tag{4.b}$$

Equations (4.a,b) and (2.c,d) are solved through symbolic manipulation yielding a solution for $\theta(S,\tau)$, $\theta(1,\tau)$, $\partial\theta(S,\tau)/\partial\eta$ and $\partial\theta(1,\tau)/\partial\eta$ with respect to $\theta_{av}(\tau)$ and $S(\tau)$.

Following the formalism in the use of the CIEA for the approximate formulation of phase-change problems (Cotta *et al.*, 1990), the energy equation for the ablation period (Eq. 2.a) is integrated within the region $S(\tau) \le \eta \le 1$, to yield after application of Leibniz rule and utilization of the above solution:

$$\frac{d\theta_{av}(\tau)}{d\tau} = \frac{1}{1-S(\tau)}\left[Q(\tau)-\frac{(\nu+1-\theta_{av}(\tau))}{\nu}\left(\frac{4(\theta_{av}(\tau)-1)}{1-S(\tau)}+Q(\tau)\right)\right] \tag{5}$$

The equation for the moving boundary is obtained from Eq. (2.e):

$$\frac{dS(\tau)}{d\tau} = \frac{1}{\nu}\left[\frac{4(\theta_{av}(\tau)-1)}{1-S(\tau)}+Q(\tau)\right] \tag{6}$$

Equations (5) and (6) are two coupled ordinary differential equations that are solved simultaneously through the built in function NDSolve from *MATHEMATICA* (Wolfram, 1996), starting from the initial conditions:

$$\theta_{av}(\tau_0) = \int_0^{\tau_0} Q(\tau')d\tau' \quad \text{and} \quad S(\tau_0)=0 \tag{7}$$

3.2 $H_{1,1}/H_{0,0}$ solution (Cotta *et al.*, 1992)

The corrected trapezoidal rule ($H_{1,1}$ formula) is employed in approximating the average temperature integral, Eq. (3.a), and maintaining the $H_{0,0}$ formula for the average heat flux expression, Eq. (4.b):

$$\theta_{av}(\tau) \cong \frac{1}{2}\left[\theta(S,\tau)+\theta(1,\tau)\right]+\frac{(1-S(\tau))}{12}\left[\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S}-\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=1}\right] \tag{8}$$

The same procedure adopted in Section 3.1 is followed to yield the ODE system:

$$\frac{d\theta_{av}(\tau)}{d\tau} = \frac{1}{\nu(1-S(\tau))^2}\left[-Q(\tau)(1-S(\tau))(1-\theta_{av}(\tau)) + 6(\nu+1-\theta_{av}(\tau))(1-2\theta_{av}(\tau))\right] \quad (9.a)$$

$$\frac{dS(\tau)}{d\tau} = \frac{1}{\nu}\left[\frac{6(2\theta_{av}(\tau)-1)}{1-S(\tau)} + Q(\tau)\right] \quad (9.b)$$

with the initial conditions given by Eqs. (7).

3.3 $H_{1,1}/H_{1,1}$ solution

The corrected trapezoidal rule ($H_{1,1}$ approximation) is also employed on the average heat flux integral expression. Eq. (4.b) is then substituted by:

$$\theta(1,\tau) - \theta(S,\tau) \cong \frac{(1-S(\tau))}{2}\left[\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S} + \left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=1}\right] + \frac{(1-S(\tau))^2}{12}\left[\left.\frac{\partial^2\theta(\eta,\tau)}{\partial\eta^2}\right|_{\eta=S} - \left.\frac{\partial^2\theta(\eta,\tau)}{\partial\eta^2}\right|_{\eta=1}\right] \quad (10)$$

The second derivatives with respect to η in Eq. (10) are evaluated from the partial differential equation itself, Eq. (2.a), to yield:

$$\theta(1,\tau) - \theta(S,\tau) \cong \frac{(1-S(\tau))}{2}\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S} - \frac{(1-S(\tau))^2}{12}\frac{\partial\theta(1,\tau)}{\partial\tau} \quad (11)$$

Equations (8) and (2.c,d) are solved through symbolic manipulation yielding a solution for $\theta(S,\tau)$, $\partial\theta(S,\tau)/\partial\eta$ and $\partial\theta(1,\tau)/\partial\eta$ with respect to $\theta_{av}(\tau)$, $\theta(1,\tau)$ and $S(\tau)$.

The ODE´s corresponding to $\theta_{av}(\tau)$, $\theta(1,\tau)$ and $S(\tau)$ are obtained from Eqs. (11), (2.e) and from the integration of Eq. (2.a). The resulting ODE system is given by:

$$\frac{d\theta_{av}(\tau)}{d\tau} = \frac{1}{\nu(1-S(\tau))^2}\left[-Q(\tau)(1-S(\tau))(1-\theta_{av}(\tau)) + 6(\nu+1-\theta_{av}(\tau))(1-2\theta_{av}(\tau)+\theta(1,\tau))\right] \quad (12.a)$$

$$\frac{dS(\tau)}{d\tau} = \frac{1}{\nu}\left[\frac{6(2\theta_{av}(\tau)-1-\theta(1,\tau))}{1-S(\tau)} + Q(\tau)\right] \quad (12.b)$$

$$\frac{d\theta(1,\tau)}{d\tau} = \frac{12}{(1-S(\tau))^2}\left[-1 + \frac{3(1-S(\tau))(1-2\theta_{av}(\tau)+\theta(1,\tau))}{(1-S(\tau))} + \theta(1,\tau)\right] \quad (12.c)$$

with the initial conditions given by Eq. (7) and $\theta(1,\tau_0) = \theta_0(1)$.

3.4 Improved $H_{1,1}/H_{0,0}$ solution

In this section one proposes an improved CIEA for the solution of such ablation problems, where the internal temperature gradients near the wall can range within the magnitude of several hundred degrees per millimeter.

Although in the solution of parabolic partial differential equations such as Eq. (2.a) the influence of any perturbation at one point is felt instantaneously by any other point in the domain, one can assume that there is a distance from the boundary beyond which the temperature within the domain is practically unaffected in one-dimensional situations. This distance is the so-called penetration depth (PD) (Hogge and Gerrekens, 1982 and 1985) which is located at $\eta=\delta(\tau)$.

Herein, one considers zero temperature and heat flux at $\eta=\delta(\tau)$. This condition allows to state an equation for a second moving boundary, which occurs between the pre-ablation period and the time before the PD reaches the second boundary ($\tau=\tau_\delta$). We propose here a new solution for this period of time, hereafter called PD $H_{1,1}/H_{0,0}$ solution.

The PD corresponding to the beginning of the ablation period can be determined from Hogge and Gerrekens (1982):

$$\delta_0 = \left[\frac{6\int_0^{\tau_0} Q(\tau')d\tau'}{Q(\tau_0)} \right]^{1/2} \tag{13}$$

The integrals that define the average temperature and heat flux within the slab are re-written considering the new space domain for $\tau_0 < \tau < \tau_\delta$:

$$\theta_{av}(\tau) = \frac{1}{\delta(\tau) - S(\tau)} \int_{S(\tau)}^{\delta(\tau)} \theta(\eta,\tau)d\eta \tag{14.a}$$

$$\theta(\delta,\tau) - \theta(S,\tau) = \int_{S(\tau)}^{\delta(\tau)} \frac{\partial\theta(\eta,\tau)}{\partial\eta} d\eta \tag{14.b}$$

The improved $H_{1,1}/H_{0,0}$ approximation is derived following the same steps in Section 3.2. The corrected trapezoidal rule ($H_{1,1}$ formula) is employed in approximating the average temperature integral and using the $H_{0,0}$ formula for the average heat flux expression:

$$\theta_{av}(\tau) \cong \frac{1}{2}\left[\theta(S,\tau) + \theta(\delta,\tau)\right] + \frac{(\delta(\tau) - S(\tau))}{12} \left[\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S} - \left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=\delta} \right] \tag{15.a}$$

$$\theta(\delta,\tau) - \theta(S,\tau) \cong \frac{(\delta(\tau) - S(\tau))}{2} \left[\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S} + \left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=\delta} \right] \tag{15.b}$$

where $\theta(S,\tau)=1$, from Eq. (2.d), $\theta(\delta,\tau)=0$ and $\partial\theta(\delta,\tau)/\partial\eta = 0$.

The solution of Eqs. (15.a,b) with the above boundary conditions yields:

$$\theta_{av}(\tau)=\frac{1}{3} \tag{16.a}$$

$$\left.\frac{\partial\theta(\eta,\tau)}{\partial\eta}\right|_{\eta=S}=-\frac{2}{\delta(\tau)-S(\tau)} \tag{16.b}$$

The corresponding ODE system for $S(\tau)$ and $\delta(\tau)$ is obtained by introducing Eq. (16.b) into Eq. (2.e) and from the integration of Eq. (2.a):

$$\frac{d\delta(\tau)}{d\tau}=\frac{1}{\nu}\left[-2Q(\tau)+\frac{4+6\nu}{\delta(\tau)-S(\tau)}\right] \tag{17.a}$$

$$\frac{dS(\tau)}{d\tau}=\frac{1}{\nu}\left[Q(\tau)-\frac{2}{\delta(\tau)-S(\tau)}\right] \tag{17.b}$$

with the initial conditions given by:

$$\delta(\tau_0)=\delta_0 \quad \text{and} \quad S(\tau_0)=0 \tag{17.b}$$

The time for which the PD reaches the second boundary ($\tau=\tau_\delta$) is determined by solving the following transcendental equation:

$$\delta(\tau)-1=0 \tag{18}$$

For $\tau>\tau_\delta$ one recalls Eqs. (9.a,b) to predict the average temperature and the moving boundary position, with the initial conditions $\theta_{av}(\tau_\delta)=1/3$ and $S(\tau_\delta)=S_{\tau_\delta}$.

4. RESULTS AND DISCUSSION

A test-case was chosen in order to evaluate the performance of the proposed solution in the simulation of the thermal protection behavior for a typical re-entry flight situation under severe aerodynamic heating, Fig. 1. The relevant data for the application here considered are as follows (Blackwell, 1988; Cotta *et al.*, 1992):

L= 0.0065 m; k=0.22 W/(m K); ρ= 1922 kg/m^3; c_p=1256 J/(kg K); H=2326 kJ/kg;
T_{ab}= 833 K; T_0= 416 K

The CIEA results obtained through the *MATHEMATICA* software system, are shown in Fig. 2.a, for the moving boundary position, and in Fig. 2.b, for the rejected surface heat flux, $Q_r(\tau)=\nu\, dS(\tau)/d\tau$. In order to verify the CIEA solutions, a computer program was developed in Fortran 77 language to provide benchmark results employing the generalized integral transform technique (GITT) (Diniz *et al.*, 1990). The converged results, shown in Figs. 2.a and 2.b, were obtained considering a 400 terms series expansion. One can observe from Figs. 2.a and 2.b that the results provided by the PD $H_{1,1}/H_{0,0}$ solution are far more accurate than those obtained by the other lumped formulations, showing a very good agreement with the benchmark results. More than 10% relative difference is found between the results of both $H_{1,1}/H_{0,0}$ approximations at certain times in Figs. 2.a and 2.b.

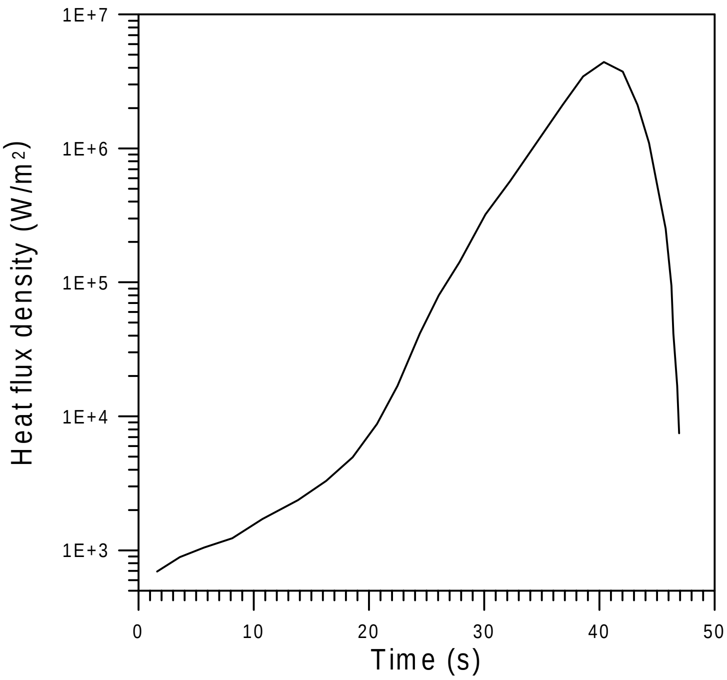

Figure 1 – Prescribed aerodynamic heating rate.

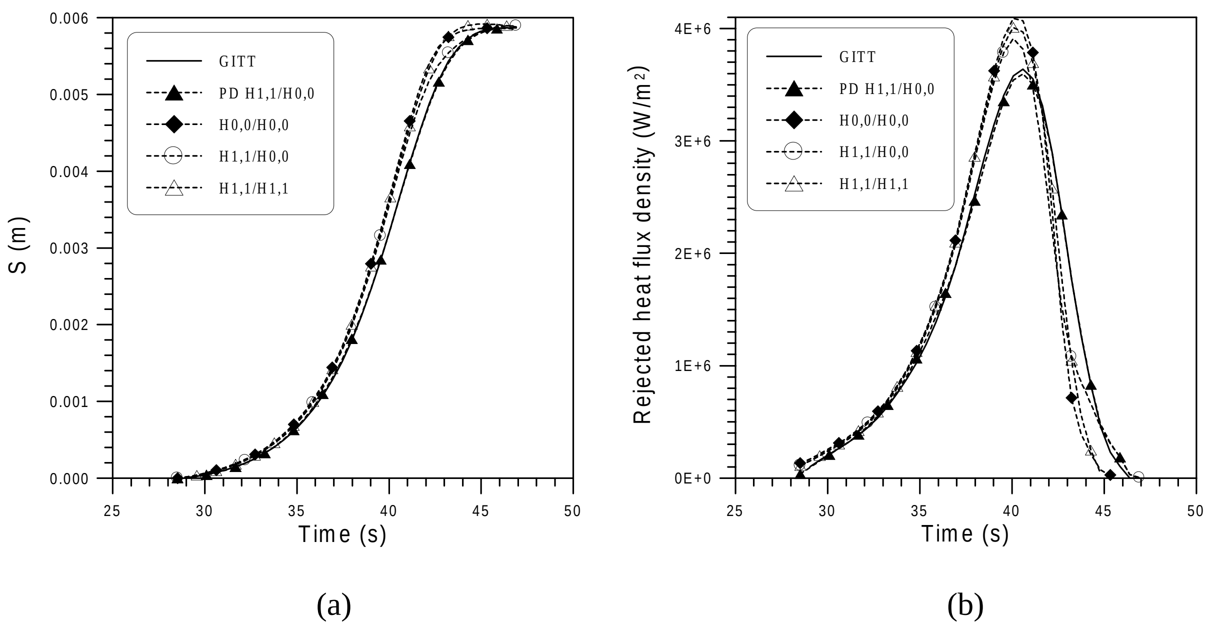

Figure 2 – Comparison of CIEA solutions for: (a) moving boundary position; (b) rejected surface heat flux.

Tables 1 and 2 present the results obtained through the use of the GITT with different truncation orders, and those provided by the PD $H_{1,1}/H_{0,0}$ solution, for the moving boundary position and for the rejected surface heat flux, respectively, at different times. One can notice that in the worst cases the PD $H_{1,1}/H_{0,0}$ results are close to those obtained by the GITT with

N=20. The PD $H_{1,1}/H_{0,0}$ is very accurate for intermediate times, loosing accuracy when the temperature gradient becomes very pronounced at the second boundary (t>44.5 s).

Table 1 – Comparison of moving boundary position (m) through the PD $H_{1,1}/H_{0,0}$ approximation against GITT solutions with different truncation orders, N.

Time (s)	*N=5*	*N=10*	*N=20*	*N=50*	*N=100*	*N=200*	*N=400*	*PD*
30.09	5.395E-5	4.742E-5	4.558E-5	4.498E-5	4.483E-5	4.477E-5	4.474E-5	4.334E-5
33.24	3.489E-4	3.346E-4	3.289E-4	3.258E-4	3.248E-4	3.243E-4	3.240E-4	3.265E-4
35.87	9.545E-4	9.263E-4	9.110E-4	9.014E-4	8.981E-4	8.963E-4	8.955E-4	9.166E-4
39.02	2.665E-3	2.588E-3	2.531E-3	2.489E-3	2.473E-3	2.465E-3	2.461E-3	2.469E-3
42.18	5.323E-3	5.161E-3	5.036E-3	4.942E-3	4.908E-3	4.890E-3	4.881E-3	4.859E-3
44.80	6.296E-3	6.107E-3	5.972E-3	5.872E-3	5.835E-3	5.817E-3	5.807E-3	5.786E-3
45.85	6.390E-3	6.183E-3	6.037E-3	5.931E-3	5.894E-3	5.875E-3	5.865E-3	5.860E-3

Table 2 – Comparison of rejected heat flux density at the surface (W/m^2) through the PD $H_{1,1}/H_{0,0}$ approximation against GITT solutions with different truncation orders, N.

Time (s)	*N=5*	*N=10*	*N=20*	*N=50*	*N=100*	*N=200*	*N=400*	*PD*
30.09	2.287E5	2.177E5	2.145E5	2.125E5	2.119E5	2.115E5	2.113E5	2.090E5
33.24	6.716E5	6.570E5	6.478E5	6.417E5	6.396E5	6.385E5	6.380E5	6.520E5
35.87	1.482E6	1.445E6	1.419E6	1.399E6	1.392E6	1.389E6	1.387E6	1.436E6
39.02	3.498E6	3.385E6	3.285E6	3.204E6	3.174E6	3.158E6	3.150E6	3.089E6
42.18	3.064E6	2.998E6	2.954E6	2.922E6	2.911E6	2.904E6	2.901E6	2.893E6
44.80	6.271E5	5.065E5	4.691E5	4.576E5	4.548E5	4.538E5	4.540E5	4.802E5
45.85	2.248E5	1.955E5	1.438E5	1.169E5	1.100E5	1.069E5	1.055E5	1.863E5

5. CONCLUSIONS

In this work, we present two already reported CIEA formulations for ablation problems (Cotta *et al.*, 1992), the Hermite's $H_{0,0}/H_{0,0}$ and $H_{1,1}/H_{0,0}$ approximations, and a more involved solution, the $H_{1,1}/H_{1,1}$ approximation. A new $H_{1,1}/H_{0,0}$ solution, based on a penetration depth concept, appropriate to solve ablation problems for low thermal diffusivity materials, is proposed. A test-case was chosen in order to evaluate the performance of the approximate solutions in the simulation of the thermal protection behavior for a typical re-entry flight situation under severe aerodynamic heating, and the results were compared against benchmarks obtained through the generalized integral transform technique (GITT). The results provided by the PD $H_{1,1}/H_{0,0}$ solution are far more accurate than those obtained by the other lumped formulations, showing a very good agreement with the benchmark results. The PD $H_{1,1}/H_{0,0}$ solution was also compared against GITT solutions with different truncation orders, showing that in the worst cases the PD $H_{1,1}/H_{0,0}$ results are close to those obtained with a lower order eigenfunction expansion. Since high order GITT solutions involve the simultaneous solution of several coupled ODEs, a substantial reduction in computational effort is obtained when considering the PD $H_{1,1}/H_{0,0}$ approach, which requires the solution of an ordinary differential system with only two coupled equations. The results show that the proposed approximation can be utilized as a fast engineering tool for the thermal design of space vehicles, motivating further developments of this technique to study the thermal ablation of more complex structures.

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