DISCRETE-ORDINATES SOLUTIONS TO SOME CLASSICAL FLOW PROBLEMS IN THE RAREFIED GAS DYNAMICS

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Abstract. *A recently developed version of the discrete-ordinates method is used to solve in a unified manner, for plane and cylindrical geometry, some classical flow problems based on the Bhatnagar, Gross and Krook model in the theory of rarefied-gas dynamics. In particular, the thermal-creep problem for the case of a semi-infinite medium and the Poiseuille-flow problem, for a wide range of the Knudsen number, are solved. Analytical solutions for the discrete-ordinates problem are obtained based on a "half-range" quadrature scheme which results in simplified eigenvalue problems. Numerical results are presented to show that the solutions are specially accurate and easy to implement.*

Key Words: *Rarefied Gas Dynamics, BGK Model, Discrete-Ordinates*

1. INTRODUCTION

In a series of recent works (Barichello & Siewert, 1999a; Rodrigues, 1999; Barichello et. al., 2000a), a newly developed version (Barichello & Siewert, 1999b) of the discrete-ordinates method (Chandrasekhar, 1960) was used to solve in an unified and precise manner some classical flow problems in the theory of rarefied gas dynamics, in a plane channel. The solution is based on the use of a "half-range" quadrature scheme that allows us to reduce the order of the systems and simplify the eigenvalue problems to be solved, resulting in an analytical solution for the discrete-ordinates problem. As it has already been shown by the application of this approach in some other problems (Barichello & Siewert, 2000a),

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the solution seems to be very effective, in the sense of obtaining accurate numerical results based on algorithms easy to implement. In this work, to emphasize the applicability of our discrete-ordinates approach, we would like to revisit and solve in a unified manner some of those classical problems in regard to flow in plane-parallel media and, following Siewert (2000a), include the case of a cylindrical tube, using a variation of the quadrature scheme.

2. THE STATEMENT OF THE PROBLEM

In regard to the behavior of a gas as it moves along a tube or between parallel plates, we can find a classical and detailed description in the books by Williams (1971) and Cercignani (1990). In addition, a very good review, derivation of the basic equations and analysis of physical parameters was recently done in the works by Sharipov & Seleznev (1998) and Williams (2000). In this work, we follow mostly these references, however we find convenient to point out some basic aspects for developing our solution. Thus, we start saying that the state of a monoatomic gas is described by the one - particle velocity distribution function $f(\boldsymbol{r}, \boldsymbol{v})$ that satisfies the nonlinear Boltzmann equation

$$\boldsymbol{v} \cdot \nabla_{\boldsymbol{r}} f(\boldsymbol{r}, \boldsymbol{v}) = \widehat{J}(f', f) \tag{1}$$

where $\widehat{J}$ is the collision operator as described by Williams (1971), $\boldsymbol{r}$ is the position vector and $\boldsymbol{v}$ is the particle velocity vector. In general, we are interested on the macrocharactheristics of the gas flow that can be defined via the distribution function, namely, the number density, the gas velocity, the pressure tensor and the heat flow, depending on the specific problem. To evaluate those quantities, it is usual, however, to write f as

$$f(\boldsymbol{r}, \boldsymbol{v}) = f_0(\boldsymbol{r}, \boldsymbol{v})[1 + h(\boldsymbol{r}, \boldsymbol{v})], \tag{2}$$

where h is a disturbance caused to the local Maxwellian $f_0(\boldsymbol{r}, \boldsymbol{v})$ (Williams, 2000) by the presence of the walls, to substitute Eq. (2) into Eq. (1), to use the BGK model (Bhatnagar et. al., 1954) as a form of expressing, in a linearized way, the collision term and finally to define an equation for $h(\boldsymbol{r}, \boldsymbol{v})$. Furthermore, some moments of the function h are defined (Williams, 2000) such that at the end we can get the basic problems to be solved in order to evaluate the number density, the heat flow and so on. In what follows we present the mathematical formulation and propose a discrete-ordinates solution for the homogeneous version of the equations that these moments satisfy, obtained by some appropriate change of variables (Barichello & Siewert, 1999a). These will be the basic problems we will have to deal to solve the flow problems.

In regard to the formulation of the problems, we follow Williams (2000), and we shall base our discussion on the case (for finite media) in which the walls are parallel plates lying at $x = \pm a$ with flow in the $z-$direction. There will be a gradient in the $x-$direction which will determine the profiles of the velocity and temperature between the plates. Variations in the $z-$direction will be due to pressure and temperature variations.

2.1. Half-Space Thermal-Creep Problem

Thermal creep is essentially a surface effect, that arises when there is a temperature gradient in the gas, and that can lead to a gas moving relative to a fixed plate (Williams, 2000). We can consider two problems of interest, which are when the gas occupies the half space $\tau > 0$ and when it is confined between two plates. For the half-space thermal-creep problem, we follow Loyalka et. al. (1975) and Williams (2000) and so seek a solution to (what we call) the reduced BGK equation

$$\xi \frac{\partial}{\partial \tau} Y(\tau, \xi) + Y(\tau, \xi) = \int_{-\infty}^{\infty} \Psi(u) Y(\tau, u) \, \mathrm{d}u, \tag{3}$$

for $\tau \in (0, \infty)$ and $\xi \in (-\infty, \infty)$, subject to

$$\lim_{\tau \to \infty} Y(\tau, \xi) = \frac{1}{2} A_T \tag{3a}$$

and the boundary condition

$$Y(0, \xi) - (1 - \alpha) Y(0, -\xi) = \frac{1}{2} \alpha (\xi^2 - \frac{1}{2}) \tag{3b}$$

for $\xi \in (0, \infty)$. Here (what we call) the characteristic function is

$$\Psi(u) = \pi^{-1/2} \exp\{-u^2\}, \tag{4}$$

A_T is the thermal-slip coefficient and $\alpha \in (0, 1]$ is the accommodation coefficient. In regard to the quantities of physical interest that we wish to establish we follow the definitions from Loyalka et. al. (1975) and thus will compute the thermal-slip coefficient A_T, related to the tangential velocity of the gas near the wall (Sharipov and Seleznev, 1998) and the macroscopic velocity profile

$$q_T(\tau) = 2 \int_{-\infty}^{\infty} \Psi(u) Y(\tau, u) \, \mathrm{d}u. \tag{5}$$

2.2. Poiseuille Flow in a Plane Channel

The plane Poiseuille flow, which arises from a pressure difference along the z-axis, is probably the most deeply investigated theoretically. The problem in a plane channel can be formulated in terms of the reduced BGK equation

$$\xi \frac{\partial}{\partial \tau} Y(\tau, \xi) + Y(\tau, \xi) = \int_{-\infty}^{\infty} \Psi(u) Y(\tau, u) \, \mathrm{d}u, \tag{6}$$

for $\tau \in (-a, a)$ and $\xi \in (-\infty, \infty)$, and the boundary conditions

$$Y(-a, \xi) - (1 - \alpha) Y(-a, -\xi) = \alpha \xi^2 + a(2 - \alpha)\xi \tag{6a}$$

and

$$Y(a, -\xi) - (1 - \alpha) Y(a, \xi) = \alpha \xi^2 + a(2 - \alpha)\xi \tag{6b}$$

for $\xi \in (0, \infty)$. Here $2a$ (the inverse Knudsen number) is the channel width (in nondimensional units). Making use of the definitions from Loyalka et. al. (1979), in this case, we will compute the macroscopic velocity profile and the flow rate.

We note that the solution we will describe in the next section has been also applied (Barichello et. al., 2000a) to solve other problems in finite and semi-infinite media (Couette flow, viscous slip problem, Kramers' problem). But since we need to keep this work in a short length and the approaches are very similar, we will restrict ourselves to the two problems described above.

3. A DISCRETE-ORDINATES SOLUTION

Following previous works (Barichello & Siewert, 1999a, 1999b) we start approximating the integral term in Eq. (3) by a quadrature formula and write our discrete-ordinates equations as

$$\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, \xi_i) + Y(\tau, \xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[Y(\tau, \xi_k) + Y(\tau, -\xi_k)] \tag{7a}$$

and

$$-\xi_i \frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau, -\xi_i) + Y(\tau, -\xi_i) = \sum_{k=1}^{N} w_k \Psi(\xi_k)[Y(\tau, \xi_k) + Y(\tau, -\xi_k)] \tag{7b}$$

for $i = 1, 2, \ldots, N$. In writing Eqs. (7) we have taken into account the fact that the characteristic function defined by Eq. (4) is an even function. In addition, we are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \infty)$. We note that it is to this feature of using a "half-range" quadrature scheme that we partially attribute the especially good accuracy we have obtained from the solution reported here.

Seeking exponential solutions, we substitute

$$Y(\tau, \pm\xi_i) = \phi(\nu, \pm\xi_i) \mathrm{e}^{-\tau/\nu} \tag{8}$$

into Eqs. (7) to find

$$\frac{1}{\nu} \boldsymbol{M} \boldsymbol{\Phi}_+ = (\boldsymbol{I} - \boldsymbol{W}) \boldsymbol{\Phi}_+ - \boldsymbol{W} \boldsymbol{\Phi}_- \tag{9a}$$

and

$$-\frac{1}{\nu} \boldsymbol{M} \boldsymbol{\Phi}_- = (\boldsymbol{I} - \boldsymbol{W}) \boldsymbol{\Phi}_- - \boldsymbol{W} \boldsymbol{\Phi}_+ \tag{9b}$$

where ν is a separation constant, $\boldsymbol{I}$ is the $N \times N$ identity matrix,

$$\boldsymbol{\Phi}_\pm = \left[\phi(\nu, \pm\xi_1), \phi(\nu, \pm\xi_2), \ldots, \phi(\nu, \pm\xi_N)\right]^{\mathrm{T}}, \tag{10}$$

the superscript T denotes the transpose operation, the elements of the matrix $\boldsymbol{W}$ are

$$(\boldsymbol{W})_{i,j} = w_j \Psi(\xi_j) \tag{11}$$

and

$$\boldsymbol{M} = \mathrm{diag}\{\xi_1, \xi_2, \ldots, \xi_N\}. \tag{12}$$

If we now let

$$\boldsymbol{U} = \boldsymbol{\Phi}_+ + \boldsymbol{\Phi}_- \tag{13}$$

then we can eliminate between the sum and the difference of Eqs. (9) to find

$$(\boldsymbol{D} - 2\boldsymbol{M}^{-1} \boldsymbol{W} \boldsymbol{M}^{-1}) \boldsymbol{M} \boldsymbol{U} = \frac{1}{\nu^2} \boldsymbol{M} \boldsymbol{U} \tag{14}$$

where

$$\boldsymbol{D} = \operatorname{diag}\{\xi_1^{-2}, \xi_2^{-2}, \dots, \xi_N^{-2}\}. \tag{15}$$

Multiplying Eq. (14) by a diagonal matrix $\boldsymbol{T}$, we find

$$(\boldsymbol{D} - 2\boldsymbol{V})\boldsymbol{X} = \frac{1}{\nu^2}\boldsymbol{X} \tag{16}$$

where

$$\boldsymbol{V} = \boldsymbol{M}^{-1}\boldsymbol{T}\boldsymbol{W}\boldsymbol{T}^{-1}\boldsymbol{M}^{-1} \tag{17}$$

and

$$\boldsymbol{X} = \boldsymbol{T}\boldsymbol{M}\boldsymbol{U}. \tag{18}$$

As discussed by Barichello & Siewert (1999a) , we can define the elements $T_1, T_2, \dots, T_N$ of $\boldsymbol{T}$ so as to make $\boldsymbol{V}$ symmetric; and therefore, since $\boldsymbol{V}$ is a symmetric, rank one matrix, we can write our eigenvalue problem in the form

$$(\boldsymbol{D} - 2\boldsymbol{z}\boldsymbol{z}^T)\boldsymbol{X} = \lambda\boldsymbol{X} \tag{19}$$

where $\lambda = 1/\nu^2$ and

$$\boldsymbol{z} = \left[\frac{\sqrt{w_1\Psi(\xi_1)}}{\xi_1}, \frac{\sqrt{w_2\Psi(\xi_2)}}{\xi_2}, \dots, \frac{\sqrt{w_N\Psi(\xi_N)}}{\xi_N}\right]^{\mathrm{T}}. \tag{20}$$

We note that the eigenvalue problem defined by Eq. (19) is of a form that is encountered when the so-called "divide and conquer" method is used to find the eigenvalues of tridiagonal matrices (Golub, 1989). In addition, we see from Eq. (15) that, because of the way our basic eigenvalue problem is formulated, we must exclude zero from the set of quadrature points.

Considering that we have found the required eigenvalues from Eq. (19), we impose the normalization condition

$$\sum_{k=1}^{N} w_k\Psi(\xi_k)[\phi(\nu, \xi_k) + \phi(\nu, -\xi_k)] = 1 \tag{21}$$

so that we can write our discrete-ordinates solution as

$$Y(\tau, \pm\xi_i) = \sum_{j=1}^{N}\left[A_j\frac{\nu_j}{\nu_j \mp \xi_i}\mathrm{e}^{-(a+\tau)/\nu_j} + B_j\frac{\nu_j}{\nu_j \pm \xi_i}\mathrm{e}^{-(a-\tau)/\nu_j}\right] \tag{22}$$

where the arbitrary constants $\{A_j\}$ and $\{B_j\}$ are to be determined from the boundary conditions and the separation constants $\{\nu_j\}$ are the reciprocals of the positive square roots of the eigenvalues defined by Eq. (19). It is clear from Eq. (22) that we cannot allow any separation constant to be equal to one of the quadrature points. In addition, the scaling constant a in Eq. (22) is, at this point, also arbitrary (we will use $a = 0$ for half-space applications and $2a$ equal to the full channel width for the finite channel-width problems).

At this point it is convenient to modify slightly the discrete-ordinates solution we reported by Barichello & Siewert (1999b). We note that problems based on Eq. (3) are "conservative" and so we expect that one of the eigenvalues defined by Eq. (19) should tend to zero as N tends to infinity. We choose to take this fact into account by explicitly neglecting ν_N, the largest of the computed separation constants $\{\nu_j\}$ and, subsequently, by writing Eq. (22) as

$$Y(\tau, \pm\xi_i) = A + B(\tau \mp \xi_i) + \sum_{j=1}^{N-1} \Big[A_j \frac{\nu_j}{\nu_j \mp \xi_i} \mathrm{e}^{-(a+\tau)/\nu_j} + B_j \frac{\nu_j}{\nu_j \pm \xi_i} \mathrm{e}^{-(a-\tau)/\nu_j} \Big]. \tag{23}$$

The constants A, B, $\{A_j\}$ and $\{B_j\}$ that are present in Eq. (23) will, as discussed later on this work, be determined by fixing the behavior of $Y(\tau, \xi_i)$ at infinity (for half-space problems) and/or by constraining $Y(\tau, \xi_i)$ to meet discrete-ordinates versions of the relevant boundary conditions. In writing Eq. (23) we have not used the largest (infinite) separation constant ν_N and have replaced the two "missing" solutions by the two "exact" terms that appear as the first elements in Eq. (23). Considering subsequently that Eq. (23) is a mixture of exact terms and discrete-ordinates terms, we will, when it will be necessary, integrate the exact terms exactly and the discrete-ordinates terms by making use of our numerical quadrature scheme.

Having developed our discrete-ordinates formalism, we are now ready to solve the specific problems defined in Section 2.

3.1. Half-Space Problems

Considering the half-space problems defined by Eqs. (3) we set the constants B and $\{B_j\}$ in Eq. (23) all equal to zero and write the desired solution as

$$Y(\tau, \pm\xi_i) = A + \sum_{j=1}^{N-1} A_j \frac{\nu_j}{\nu_j \mp \xi_i} \mathrm{e}^{-\tau/\nu_j}\,. \tag{24}$$

Now substituting Eq. (24) into the boundary condition, Eq. (3b), evaluated at the quadrature points, we find the system of linear algebraic equations

$$\alpha A + \sum_{j=1}^{N-1} M_{i,j} A_j = F(\xi_i) \tag{25}$$

for $i = 1, 2, \ldots, N$. Here

$$M_{i,j} = \nu_j \Big[\frac{\alpha\nu_j + \xi_i(2-\alpha)}{\nu_j^2 - \xi_i^2} \Big] \tag{26}$$

and

$$F(\xi) = \frac{1}{2}\alpha(\xi^2 - \frac{1}{2}). \tag{27}$$

Now all we have to do is to define a quadrature scheme, solve the eigenvalue problem defined by Eq. (19), thus obtaining the separation constants $\{\nu_j\}$, and solve the linear system defined by Eq. (25). In this way all that we seek here is established, *viz.*

$$A_T = 2A, \tag{28}$$

the thermal-slip coefficient, and the macroscopic velocity for the thermal-creep problem,

$$q_T(\tau) = 2[A + \sum_{j=1}^{N-1} A_j \mathrm{e}^{-\tau/\nu_j}]. \tag{29}$$

We note that in the case of solving the half-space viscous slip problem, the basic difference will be the right-hand side in Eq. (27) (Barichello et. al., 2000a).

3.2. Finite Channel-Width Problems

Looking now at the problem defined in Section 2 to describe flow in a plane channel, we consider the boundary conditions, subject to which we must solve Eq. (6), written as

$$Y(-a,\xi) - (1-\alpha)Y(-a,-\xi) = F_P(\xi) \tag{30a}$$

and

$$Y(a,-\xi) - (1-\alpha)Y(a,\xi) = F_P(\xi) \tag{30b}$$

for $\xi \in (0,\infty)$. To be explicit, we note that

$$F_P(\xi) = \alpha\xi^2 + a(2-\alpha)\xi \tag{31}$$

for Poiseuille flow. To solve this problem we substitute Eq. (23) into Eqs. (30) evaluated at the quadrature points to find the system of linear algebraic equations

$$\sum_{j=1}^{N-1} \Big\{ M_{i,j}A_j + N_{i,j}B_j \mathrm{e}^{-2a/\nu_j} \Big\} + \alpha A - B[\alpha a + \xi_i(2-\alpha)] = F_P(\xi_i) \tag{32a}$$

and

$$\sum_{j=1}^{N-1} \Big\{ M_{i,j}B_j + N_{i,j}A_j \mathrm{e}^{-2a/\nu_j} \Big\} + \alpha A + B[\alpha a + \xi_i(2-\alpha)] = F_P(\xi_i) \tag{32b}$$

for $i = 1, 2, \dots, N$. Here the matrix elements $M_{i,j}$ are given by Eq. (26) and

$$N_{i,j} = \nu_j \Big[\frac{\alpha\nu_j - \xi_i(2-\alpha)}{\nu_j^2 - \xi_i^2} \Big]. \tag{33}$$

Of course once we have solved Eqs. (32) to find the constants A, B and $\{A_j, B_j\}$ we are able to establish some quantities of interest, the macroscopic velocity and the flow rate, respectively as

$$q_P(\tau) = \frac{1}{2}\big(1 - a^2 + \tau^2\big) - A - B\,\tau - \sum_{j=1}^{N-1} \Big[A_j \mathrm{e}^{-(a+\tau)/\nu_j} + B_j \mathrm{e}^{-(a-\tau)/\nu_j} \Big]. \tag{34}$$

and

$$Q_P = \frac{1}{2a^2} \Big[2aA + \sum_{j=1}^{N-1} \nu_j \big(A_j + B_j\big)\big(1 - \mathrm{e}^{-2a/\nu_j}\big) \Big] - \frac{1}{2a}\big(1 - \frac{2}{3}a^2\big), \tag{35}$$

for the Poiseuille-flow problem. We note again that for solving the Couette flow and thermal-creep problem in a plane channel we basically will change the right-hand side in Eqs. (30) (Barichello et. al., 2000a).

4. CYLINDRICAL GEOMETRY

In regard to developing a solution for the Poiseuille and the thermal creep flow in a cylindrical tube, we consider the BGK equation written as

$$\xi\Big(\mu\frac{\partial}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\Big)G(r,\xi,\mu)+G(r,\xi,\mu)=\int_{-1}^{1}\int_{0}^{\infty}\Psi(\xi',\mu')G(r,\xi',\mu')\mathrm{d}\xi'\mathrm{d}\mu' + Q(\xi), \quad (36)$$

for $\mu\in[-1,1]$, $\xi\in[0,\infty)$ and $r\in(0,R)$, and

$$G(R,\xi,-\mu)=F(\xi,\mu),\quad \mu\in(0,1]\quad\text{and}\quad \xi\in[0,\infty). \quad (37)$$

Here

$$\Psi(\xi,\mu)=\frac{2\xi\mathrm{e}^{-\xi^2}}{\pi(1-\mu^2)^{1/2}} \quad (38)$$

and again our basic equation is defined in terms of a perturbation of the particle distributions from a local Maxwellian (Williams, 1971). The solution of the integral form of this equation (Barichello et. al., 2000b) is related to the quantities of physical interest, for example the desired macroscopic velocity profile. However, using a convenient transformation (Mitsis, 1963), the solution of the integral form of Eq. (36) can be found in terms of the function

$$\Phi(r,\xi)=Y(r,\xi)-\frac{1}{4}\pi^{1/2}(r^2-R^2+4\xi^2) \quad (39)$$

where $Y(r,\xi)$ must satisfy

$$\xi^2\frac{\partial^2}{\partial r^2}Y(r,\xi)+\frac{\xi^2}{r}\frac{\partial}{\partial r}Y(r,\xi)-Y(r,\xi)+2\int_0^{\infty}\Psi(u)Y(r,u)\mathrm{d}u=0 \quad (40)$$

for $\xi\in(0,\infty)$ and the boundary condition

$$Y(R,\xi)+\xi\Gamma(\xi)\frac{\partial}{\partial r}Y(r,\xi)\Big|_{r=R}=\frac{1}{2}\pi^{1/2}\xi\big[2\xi+R\Gamma(\xi)\big] \quad (41)$$

for $\xi\in(0,\infty)$, for the Poiseuille-flow problem. Here $\Psi(u)$ is defined as in Eq. (4) and

$$\Gamma(\xi)=\frac{K_0(R/\xi)}{K_1(R/\xi)}, \quad (42)$$

where we used K_n to denote the modified Bessel functions of the second kind. We note also that in defining the thermal-creep problem, the right-hand side of Eq. (41) will be a different function (Siewert, 2000a). Thus, all we need is to solve what we call the Y-problem. To start, we repeat what we did in Section 3 and we approximate the integral term in Eq. (40) by a quadrature formula and write our discrete-ordinates equations as

$$\xi_i^2\frac{\mathrm{d}^2}{\mathrm{d}r^2}Y(r,\xi_i)+\frac{\xi_i^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}Y(r,\xi_i)-Y(r,\xi_i)+2\sum_{k=1}^{N}w_k\Psi(\xi_k)Y(r,\xi_k)=0 \quad (43)$$

for $i = 1, 2, \ldots, N$. Again, in writing Eq. (43) as we have, we are considering that the N quadrature points $\{\xi_k\}$ and the N weights $\{w_k\}$ are defined for use on the integration interval $[0, \infty)$. Seeking a Bessel function solution (bounded as $r \to 0$) of Eq. (43), we substitute

$$Y(r, \xi_i) = \phi(\nu, \xi_i) I_0(r/\nu) \tag{44}$$

into Eq. (43), where we used I_0 to denote the modified Bessel function of the first kind, and we follow an analogous procedure as the one presented in Section 3, so we can write

$$\phi(\nu_j, \xi_i) = \frac{\nu_j^2}{\nu_j^2 - \xi_i^2} \tag{45}$$

where clearly, as discussed earlier in Section 3, we cannot allow $\nu_j = \xi_i$. As in the previous section, here the separation constants ν_j are defined ($\nu_j = \lambda_j^{-1/2}$) by solving the eigenvalue problem given by Eq. (19). Continuing, we "sum up" our solutions and write

$$Y(r, \xi_i) = \sum_{j=1}^{N} A_j \phi(\nu_j, \xi_i) I_0(r/\nu_j) \tag{46}$$

where the arbitrary constants $\{A_j\}$ are to be determined from the boundary condition of our problem.

At this point we wish to modify slightly the discrete-ordinates solution given by Eq. (46), as we also did in the previous section, once this is a conservative problem. We then write

$$Y(r, \xi_i) = A + \sum_{j=1}^{N-1} A_j \phi(\nu_j, \xi_i) \widehat{I}_0(r/\nu_j) \mathrm{e}^{-(R-r)/\nu_j} \,. \tag{47}$$

The constants A and $\{A_j\}$ in Eq. (47) are to be determined by constraining $Y(r, \xi_i)$ to meet a discrete-ordinates version of the relevant boundary condition. To complete our discussion of Eq. (47) we note that, following Siewert (2000a), we have "rescaled" the solution by introducing in general, $\widehat{I}_n(x) = I_n(x)\mathrm{e}^{-x}$ and $\widehat{K}_n(x) = K_n(x)\mathrm{e}^{x}$, in order to keep "underflows/overflows" in our numerical work from degrading our calculation.

To complete the solution we substitute Eq. (47) into the boundary condition evaluated at the quadrature points to obtain a linear system as Eq. (25), for $\alpha = 1$, but here

$$M_{i,j} = \nu_j \Big[\frac{\nu_j \widehat{I}_0(R/\nu_j) + \xi_i \Gamma(\xi_i) \widehat{I}_1(R/\nu_j)}{\nu_j^2 - \xi_i^2} \Big] \tag{48}$$

and $F(\xi_i)$ is , for the Poiseuille-flow problem,

$$F(\xi_i) = \frac{1}{2} \pi^{1/2} \xi_i \big[2\xi_i + R\Gamma(\xi_i) \big]. \tag{49}$$

We can then evaluate, the macroscopic velocity and the heat flow as

$$q_P(r) = \pi^{-1/2} \Big[A + \sum_{j=1}^{N-1} A_j \widehat{I}_0(r/\nu_j) \mathrm{e}^{-(R-r)/\nu_j} \Big] - \frac{1}{4}(r^2 - R^2 + 2) \tag{50}$$

and

$$Q_P = \frac{2\pi^{-1/2}}{R^2} \Big[AR + 2 \sum_{j=1}^{N-1} A_j \nu_j \widehat{I}_1(R/\nu_j) \Big] + \frac{1}{4R}(R^2 - 4) \tag{51}$$

for Poiseuille flow.

5. NUMERICAL RESULTS

We first must define the quadrature scheme to be used in our discrete-ordinates solution. In this (and other) work we have used one of the (nonlinear) transformations

$$u(\xi) = \exp\{-\xi\} \tag{52a}$$

or

$$u(\xi) = \frac{1}{1+\xi} \tag{52b}$$

to map $\xi \in [0, \infty)$ into $u \in [0, 1]$, and we then used a Gauss-Legendre scheme mapped (linearly) onto the interval $[0, 1]$. Of course other quadrature schemes could be used. However, we have found the use of a mapping defined by either of Eqs. (52) followed by the use of the Gauss-Legendre integration formulas to be so simple and effective that we have not developed any special-purpose quadrature schemes.

Having defined our quadrature scheme and in developing a FORTRAN implementation of our solution, we found the required separation constants $\{\nu_j\}$ by using the special numerical package DZPACK (Siewert & Wright, 1999) that was developed to take advantage of the special structure of Eq. (19) to solve our eigenvalue problem. The required separation constants were then available as the reciprocals of the square roots of these eigenvalues. We then used the subroutines DGECO and DGESL from the LINPACK package (Dongarra et. al., 1979) to solve the linear system defined by Eqs. (25) and (32) – for the plane and cylindrical geometry cases – and so the solutions to the various problems were considered established.

We find important to note that since the function $\Psi(u)$ defined by Eq. (4) can be zero, from a computational point-of-view, we can have some, say a total of N_0, of the separation constants $\{\nu_j\}$ equal to some of the quadrature points $\{\xi_i\}$. This is not allowed either in Eq. (23) or Eq. (45), and so, since the quadrature points where $\Psi(\xi_k)$ is effectively zero make no contribution to the right-hand side of Eqs. (7) and the last term in Eq. (43), we have seen that we can simply omit these quadrature points from our calculation. In omitting these N_0 quadrature points we have effectively changed N to $N - N_0$ in some aspects of our final solution.

In Tables 1 and 2 we present the results for the problems in a plane channel and in Table 3 we show the results for Poiseuille flow in a cylindrical tube. In regard to these results we show in Tables 1–3, we have typically used $N = 50$ to generate them and they are given with what we believe to be seven figures of accuracy. While we found agreement that varied from three to six significant figures with results from the literature (Loyalka et. al., 1975; Loyalka et. al., 1979; Valougeorgis & Thomas Jr., 1986) we believe the results presented here should be considered more definitive than those in the mentioned earlier works – where in general different approaches were developed to solve each one of the problems. In addition, differently of the results presented in the papers by Barichello et. al. (2000a) and Siewert (2000a), we used here, instead of Eq. (52a) the transformation given by. Eq. (52b), in regard to the quadrature scheme, to generate the results we present in Tables 1 – 3, and found complete agreement with those previous results.

6. FINAL COMMENTS

In regard to additional work in the general area of rarefied gas dynamics, we note that our variation of the discrete-ordinates method has been used to solve the temperature-jump problem (Barichello & Siewert, 2000b) a heat-transfer problem in a plane channel where the coupled effects of temperature and density must be resolved simultaneously (Siewert, 1999), and new works devoted to binary gas mixtures also have been completed (Siewert, 2000b). And so in this basic work, we believe we have shown our unified discrete-ordinates solutions to be very effective (especially accurate and easy to implement) for what we consider to be a set of classical problems based on the BGK model. It seems, therefore, that we are justified in believing that the methods reported here can now be extended to solve even more challenging problems based on improved physical models derived from the Boltzmann equation.

Table 1: The thermal-slip coefficient A_T

α	A_T
0.1	5.283566(-1)
0.2	5.563021(-1)
0.3	5.838476(-1)
0.4	6.110039(-1)
0.5	6.377813(-1)
0.6	6.641898(-1)
0.7	6.902391(-1)
0.8	7.159384(-1)
0.9	7.412966(-1)
1.0	7.663225(-1)

Table 2: The Poiseuille flow rate Q_P

$2a$	$\alpha = 0.50$	$\alpha = 0.80$	$\alpha = 0.88$	$\alpha = 0.96$	$\alpha = 1.00$
0.05	5.223297	3.089712	2.738340	2.437355	2.302257
0.10	4.556406	2.707741	2.406046	2.148241	2.032714
0.30	3.778472	2.244771	2.001067	1.794509	1.702474
0.50	3.544371	2.102266	1.876620	1.686342	1.601874
0.70	3.437669	2.038767	1.822011	1.639850	1.559186
0.90	3.383887	2.009241	1.797636	1.620223	1.541800

Table 3: The velocity slip $q_P(R)$ and the flow rate Q_P

R	$q_P(R)$	Q_P
1.0	4.048069(-1)	1.458291
2.0	7.651726(-1)	1.657647
3.0	1.119114	1.879988
4.0	1.471454	2.111623
5.0	1.823461	2.348327
6.0	2.175514	2.588211

Acknowledgement

Is is noted that the work of LBB is supported in part by CNPq of Brazil.

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