NUMERICAL SOLUTION OF NON-ISOTHERMAL LAMINAR JET FLOW USING THE MULTIGRID METHOD

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Abstract. *Heated laminar jet flow is numerically investigated using the multigrid method. Numerical analysis is based on the finite volume discretization scheme applied to structure orthogonal regular meshes. Performance of the correction storage (CS) multigrid is compared for different Reynolds number at inlet and distinct number of grids. Up to three grids were used for both V- and W-cycles. Simultaneous and segregated temperature-velocity solution schemes were investigated. Advantages in using more than one grid are discussed. For simultaneous solution, results indicate an increase in the computational effort for higher inlet Reynolds numbers. Optimal number of intermediate relaxation sweeps within both V- and W-cycles are discussed.*

Keywords: *Sudden Expansion, Multigrid, CFD, Numerical Methods*

1. INTRODUCTION

Convergence rates in most single-grid solutions are greatest in the beginning of the iterative process but slow down as the procedure goes on. Effects like those get more pronounced, as the grid becomes finer. Large grid sizes, however, are often needed when resolving small recirculating regions or detecting high heat transfer spots. The reason for this behavior is that iterative methods can efficiently smooth out only those Fourier error components of wavelengths smaller than or comparable to the grid size. In contrast, multigrid methods aim to cover a broader range of wavelengths through relaxation on more than one grid.

The number of iterations and convergence criterion in each step along consecutive grid levels visited by the algorithm determines the cycling strategy, usually a *V*- or *W*-cycle. Within each cycle, the intermediate solution is relaxed before (*pre*-) and after (*post-smoothing*) the transportation of values to coarser (*restriction*) or to finer (*prolongation*) grids (Brandt, 1977, Stüben & Trottenberg, 1982, Hackbusch, 1985).

Accordingly, multigrid methods can be roughly classified into two major categories. In the CS formulation, algebraic equations are solved for the corrections of the variables whereas, in the full approximation storage (FAS) scheme, the variables themselves are handled in all grid levels. It has been pointed out in the literature that the application of the CS formulation is recommended for the solution of linear problems being the FAS formulation more suitable to non-linear cases (Brandt, 1977, Stüben & Trottenberg, 1982, Hackbusch, 1985). An exception to this rule seems to be the work of Jiang, et al, 1991, who reported predictions for the Navier-Stokes equations successfully applying the multigrid CS formulation. In the literature, however, not too many attempts in solving non-linear problems with multigrid linear operators are found.

Acknowledging the advantages of using multiple grids, Rabi & de Lemos, 1998a, presented numerical computations applying this technique to recirculating flows in several geometries of engineering interest. There, the correction storage (CS) formulation was applied to non-linear problems. Later, Rabi & de Lemos, 1998b, analyzed the effect of Peclet number and the use of different solution cycles when solving the temperature field within flows with a given velocity distribution. In all those cases, the advantages in using more than one grid in iterative solution were confirmed. Subsequently, de Lemos & Mesquita, 1999, introduced the solution of the energy equation in their multigrid algorithm for the flow field. Temperature distribution was calculated solving the whole equation set together with the flow field as well as segregating the momentum and energy equations. A study on optimal relaxation parameters was there reported. Next, Mesquita & de Lemos, 1999, considered non-isothermal laminar flow past a back-step in a parallel plate channel. In that paper, an in dept analysis on optimal multigrid cycle parameters was conducted. Recently, de Lemos & Mesquita, 2000 and Mesquita & de Lemos, 2000, considered the cases of flow in a heated thank and after an abrupt expansion, respectively. All of these papers are based on the development done by Mesquita, 2000.

The present contribution extends the early work on CS multigrid methods to the solution of the energy equation in laminar jet flows. More specifically, heated steady-state flows in confined jets are now calculated with up to three grids. A schematic of such configuration is show in Fig. 1.

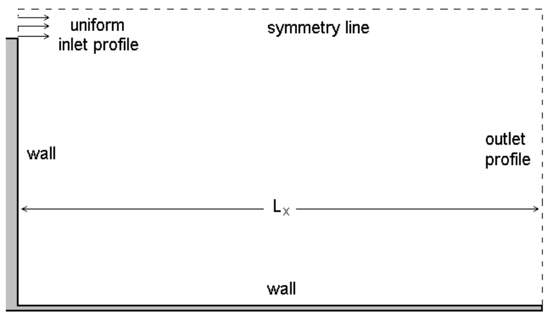


Figure 1 - Geometries and boundary conditions for confined jet.

2. ANALYSIS

2.1 Governing Equations and Numerics

The continuity, Navier-Stokes and energy equations describe fluid flow and heat transfer. They express mass, momentum and energy conservation principles respectively and, for a steady state condition in a two-dimension Cartesian coordinate frame, they are written in compact notation as:

$$\frac{\partial}{\partial x}(\rho U\phi)+\frac{\partial}{\partial y}(\rho V\phi)=\frac{\partial}{\partial x}\left(\Gamma_\phi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma_\phi\frac{\partial\phi}{\partial y}\right)+S_\phi \qquad \textbf{(1)}$$

where $\phi=\{1,U,V,T\}$, $\Gamma_\phi=\left\{0,\mu,\mu,\mu/\mathrm{Pr}\right\}$, $S_\phi=\left\{0,-\frac{\partial P}{\partial x},-\frac{\partial P}{\partial y},\ 0\right\}$, ρ is the fluid density, U and V are the x and y velocity components, respectively, T is the temperature, μ is the dynamic viscosity and Pr is the Prandtl number. In addition, in this work all fluid properties are held constant.

The solution domain is divided into a number of rectangular control volumes (CV), resulting in a structured orthogonal non-uniform mesh. Grid points are located according to a *cell-centered* scheme and velocities are stored in a *collocated* arrangement (Patankar, 1980). Integrating equation (1) over the CV of Figure 2, one gets,

$$\int_{\delta v}\left[\frac{\partial}{\partial x}(\rho U\phi)+\frac{\partial}{\partial y}(\rho V\phi)\right]\mathrm{d}v = \int_{\delta v}\left[\frac{\partial}{\partial x}\left(\Gamma_\phi\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\Gamma_\phi\frac{\partial\phi}{\partial y}\right)\right]\mathrm{d}v+\int_{\delta v}S_\phi\mathrm{d}v \qquad (2)$$

Integration of the three terms in (2), namely: convection, diffusion and source, leads to a set of algebraic equations. These practices are described elsewhere (*e.g.* Patankar, 1980) and for this reason they are not repeated here. In summary, convective terms are discretized using the upwind differencing scheme, diffusive fluxes make use of the central differencing scheme and pressures, needed at cell faces, are approximated by a linear interpolation of neighboring point values.

Substitution of all approximate expressions for interface values and gradients into the integrated transport equation (2), gives the final discretization equation for grid node P,

$$a_P\phi_P = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b \qquad (3)$$

with the east face coefficient, for example, being defined as

$$a_{\mathrm{E}} = \max[-C_e,0] + D_{\mathrm{e}} \qquad (4)$$

In Eq. (4) $D_{\mathrm{e}}=\mu_e\delta y/\Delta x_{\mathrm{e}}$ and $C_{\mathrm{e}}=(\rho U)_e\,\delta y$ are the diffusive and convective fluxes at the CV east face, respectively, and, as usual, the operator max[a,b] returns the greater of a and b.

2.2 Multigrid Technique

Assembling Eq. (3) for each control volume of Fig. 2 in the domain of Fig. **1** defines a linear algebraic equation system of the form,

$$\mathbf{A}_{\mathrm{k}}\mathbf{T}_{\mathrm{k}} = \mathbf{b}_{\mathrm{k}} \qquad (5)$$

where $\mathbf{A}_{\mathrm{k}}$ is the *matrix of coefficients*, $\mathbf{T}_{\mathrm{k}}$ is the *vector of unknowns* and $\mathbf{b}_{\mathrm{k}}$ is the vector accommodating source and extra terms. Subscript "k" refers to the grid level, with k=1 corresponding to the coarsest grid and k=M to the finest mesh.

As mentioned, multigrid is here implemented in a *correction storage* formulation (CS) in which one seeks coarse grid approximations for the *correction* defined as $\delta_{\mathrm{k}}=\mathbf{T}_{\mathrm{k}}-\mathbf{T}_{\mathrm{k}}^*$ where $\mathbf{T}_{\mathrm{k}}^*$ is an *intermediate value* resulting from a small number of iterations applied to (5). For a

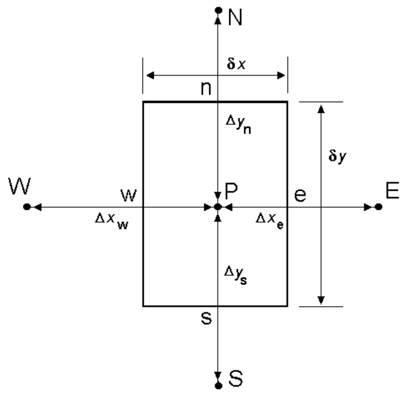


Figure 2 - Control volume for discretization.

linear problem, one shows that δ_k is the solution of (Brandt, 1977, Stüben & Trottenberg, 1982, Hackbusch, 1985),

$$\mathbf{A}_k \delta_k = \mathbf{r}_k \tag{6}$$

where the *residue* is defined as

$$\mathbf{r}_k = \mathbf{b}_k - \mathbf{A}_k \mathbf{T}_k^* \tag{7}$$

Eq. (10) can be approximated by means of a coarse-grid equation,

$$\mathbf{A}_{k-1} \delta_{k-1} = \mathbf{r}_{k-1} \tag{8}$$

with the *restriction operator* I_k^{k-1} used to obtain

$$\mathbf{r}_{k-1} = I_k^{k-1} \mathbf{r}_k \tag{9}$$

The residue restriction is accomplished by summing up the residues corresponding to the four fine grid control volumes that compose the coarse grid cell. Thus, equation (9) can be rewritten as,

$$r_{k-1}^{I\,J} = r_k^{i\,j} + r_k^{i\,j+1} + r_k^{i+1\,j} + r_k^{i+1\,j+1} \tag{10}$$

Diffusive and convection coefficients in matrix $\mathbf{A}_k$ need also to be evaluated when changing grid level. Diffusive terms are recalculated since they depend upon neighbor grid node distances whereas coarse grid mass fluxes (*convective terms*) are simply added up at control volume faces. This operation is commonly found in the literature (Peric, et al, 1989, Hortmann et al, 1990).

Once the coarse grid approximation for the correction δ_{k-1} has been calculated, the *prolongation* operator I_{k-1}^{k} takes it back to the fine grid as

$$\delta_k = I_{k-1}^{k} \delta_{k-1} \tag{11}$$

In order to update the intermediate value

$$\mathbf{T}_k = \mathbf{T}_k^* + \delta_k \qquad (12)$$

Fig. **3** illustrates a 4-grid iteration scheme for both the *V*- and *W*-cycles where the different operations are: *s=smoothing, r=restriction, cg=coarsest grid iteration* and *p=prolongation*. Also, the number of domain sweeps before and after grid change is denoted by ν^{pre} and ν^{post}, respectively. In addition, at the coarsest k level (k=1), the grid is swept ν^{cg} times by the error smoothing operator.

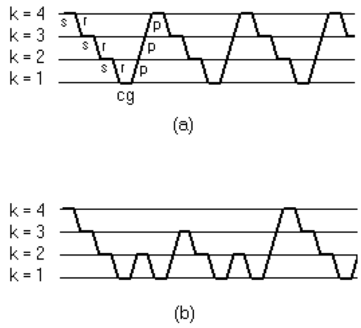


Figure 3 - Sequence of operations in a 4-grid iteration: (a) V-cycle; (b) W-cycle.

3. RESULTS AND DISCUSSION

3.1 Computational Details

The computer code developed was run on an IBM PC machine with a Pentium 166 MHz processor. Grid independence studies were conducted such that the solutions presented herein are essentially grid independent. For both *V*- and *W*-cycles, pre- and post-smoothing iterations were accomplished via the Gauss-Seidel algorithm whereas, at the coarsest-grid, the TDMA method has been applied (Patankar, 1980). Also, all flows in the geometry of Fig. **1** were run with a finest grid of 160x34 nodes.

Results below are focused on the behavior of the energy equation subjected to multigrid numerical methods. Analysis of velocity and pressure convergence characteristics have already been reported (Rabi & de Lemos, 1998a, Rabi & de Lemos, 1998b) and for that they are here not discussed.

3.2 Temperature field

Fig. 4 shows nondimensional temperature distribution patterns for flow in the confined jet flow of Fig. 1. All walls are kept at the same temperature, higher than the incoming flow temperature. The figure indicates the effect of increasing the inlet Reynolds number, $Re_{in}=\rho U_{in} L_{in}/\mu$, where the subscript "in" refers to inlet values. Cold fluid penetrates deeper into the flow core as the inlet Reynolds number increases. These and other results fully discussed in Mesquita, 2000, assured the accuracy and correctness of the computer code developed.

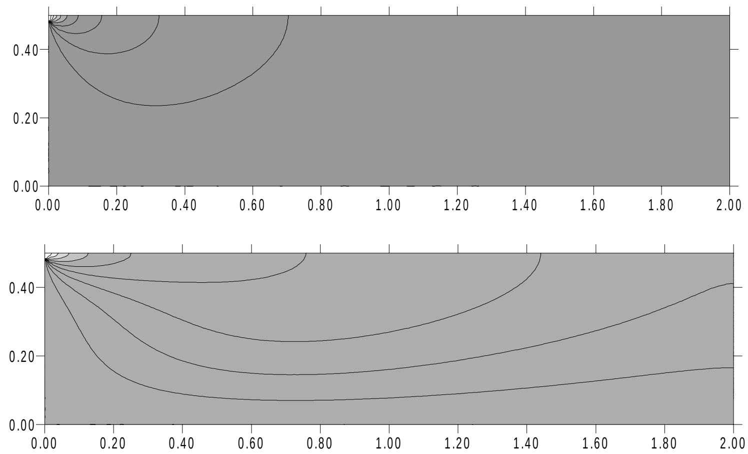


Figure 4- Effect of Re_{in} on temperature pattern for confined jet of Figure 1. From top to bottom: Re_{in} = 100, 300

3.3 Residues

Residues for the energy equation is defined as,

$$R_T = \sqrt{\sum_{\mathrm{ij}} \left(R_{\mathrm{ij}}^2 \right)} \qquad \text{with} \quad R_{\mathrm{ij}} = A_{\mathrm{P}} T_{\mathrm{P}} - \left(\sum_{\mathrm{nb}} A_{\mathrm{nb}} T_{\mathrm{nb}} \right) \tag{13}$$

where subscript *ij* identifies a given control volume on the finest grid and *nb* refers to its neighboring control volumes.

Fig. 5 is taken from de Lemos & Mesquita, 2000, and is included here for the same of completeness. It shows residue history for backward facing step case following the two cycles picture on Fig. **3**, namely the *V*- and *W*- cycles. The solution follows a simultaneous approach in the sense that the temperature is always relaxed after the flow field, within the multigrid

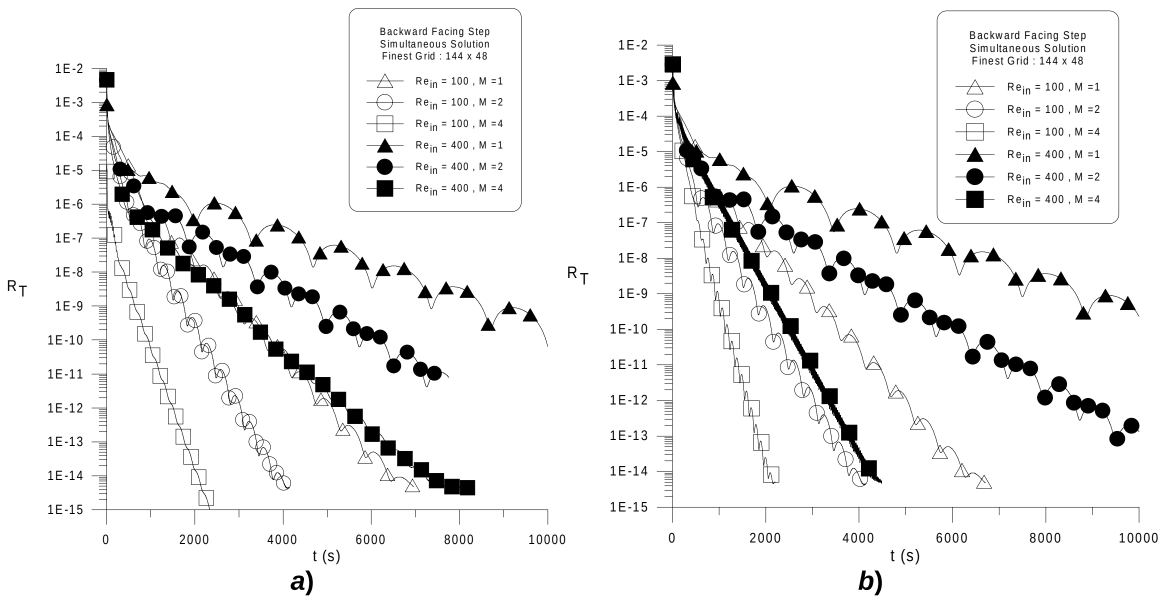


Figure 5 - Residue history for different number of grids and Re_{in} for backward facing step: *a*) *V*-cycle, *b*) *W*-cycle

cycle. One can readily notice that for lower Re_{in}, regardless of the number of grids used, faster solutions are obtained. In this case, relative importance of diffusion terms favors the stability of the system of equations. Increasing the number of grids for the same Reynolds number is also advantageous (see Fig. 5). This feature is what makes multigrid methods attractive, justifying their growing usage. Also interesting to note is that for the *V*-cycle and for Re_{in}=400 (Fig. 5a), the computational effort related to value transfer among too many grids became relevant. Using a *W*-cycle (Fig. 5b) for this Reynolds seems to bring more savings to the iterative simultaneous solution procedure. Although correspondent results for jets are not shown here, similar conclusion can be drawn for the geometry of Fig. 1.

Recognizing that the situation here investigated embraces physical uncoupling between momentum and heat transfer, *i.e.*, heated transfer depends on flow, not the other way around, a ***segregated*** algorithm has been devised. In this case, the flow field is obtained first by solving the momentum and continuity equations. Then the velocities were recorded. Subsequently, the multigrid method was again applied to the energy equation only, having the convection strength calculated with the stored flow field (i.e segregated solution). The other approach was to relax both momentum and energy equation within the same multigrid algorithm and was named the ***simultaneous*** solution.

A word of caution about the jargon here employed is timely. In the literature, **coupled** and **uncoupled** problems usually refer to the non-linearity among the variable involved. **Simultaneous** solutions are those in which all variables are solved together, so that the non-linearity due to the coupling among variables poses no difficulty. In **segregated** solutions, each variable is solved at a time while holding the others still. In this work, ***simultaneous*** and ***segregated*** solutions are those in which velocity and temperature are solved either within the same computer run or in separate calculations, respectively.

Fig. 6 presents residue history for the energy equation for the two situations considered, namely the ***simultaneous*** multigrid solutions for velocity and temperature and the sole solution of the energy equation, given the flow field (***segregated*** approach). As expected, the number of iterations needed in the segregated solutions case is lower. Consequently, the advantages in using multiple grids is felt stronger in simultaneous solutions where overall computing time are greater. Also interesting to note is that for the jet flow, the advantages in following a segregated approach are not as evident as in the case of other flows. In de Lemos

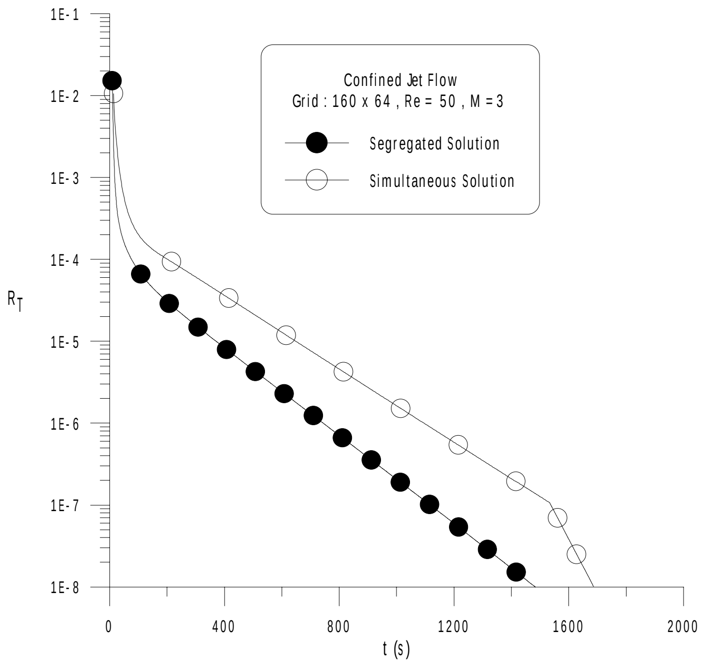


Figure 6 - Residue history for segregated and simultaneous solutions for confined jet.

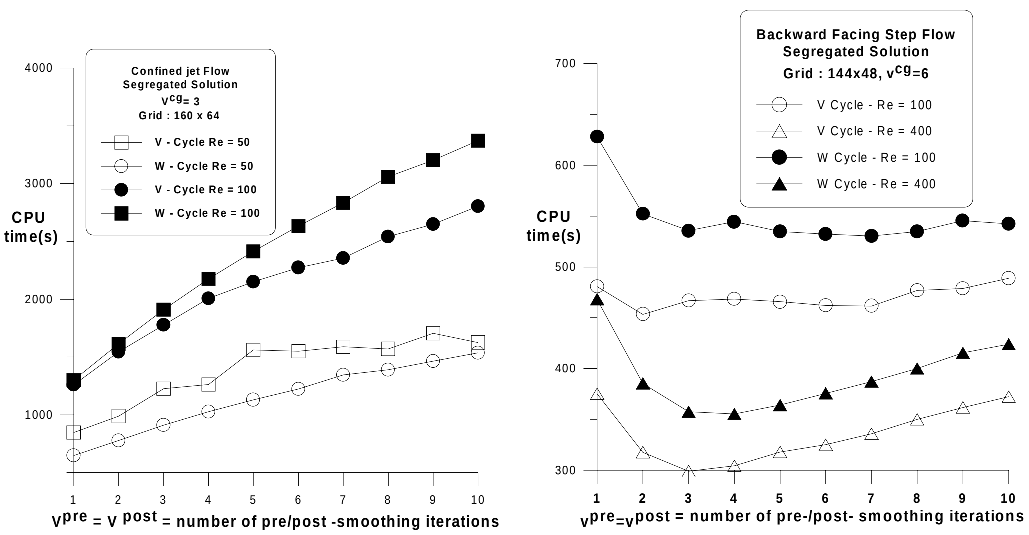
& Mesquita, 2000 and de Lemos & Mesquita, 1999, much greater savings were obtained when the segregated scheme was applied to flows in an abrupt expansion and in a heated tank.

3.4 Optimal relaxation parameters

In a series of papers (de Lemos & Mesquita, 1999, Mesquita & de Lemos, 1999, de Lemos & Mesquita, 2000Mesquita & de Lemos, 2000), a study was carried out to investigate optimal values for the parameters ν^{pre}, ν^{post} and ν^{cg} in several flows of engineering interest. Since the intermediate solutions, before and after grid changes, are not fully solved but are rather relaxed ν^{pre} and ν^{post} times, a question about their optimal values for increasing overall algorithm performance arises. As restriction and prolongation operations introduce imprecision to values being transferred, one should expect that the computational effort is sensitive to the number of smoothing sweeps. In other words, once the intermediate numerical solution has been relaxed a number of times, removing errors introduced by the transfer operators and further reducing the residue, it is of no use to keep iterating at a certain grid level. The results shown below extend such analysis for jet flows.

For a fixed number of sweeps at the coarse grid (ν^{cg}), Figure 7 reproduces the necessary time to convergence when the number of pre- and post-smoothing iterations was allowed to vary, keeping the same value for $\nu^{pre}=\nu^{post}$. For comparisons, correspondent calculations for flow past a back step are also shown in the figure (de Lemos & Mesquita, 2000). One can clearly detect optimal values for those relaxation parameters. Additional sweeps past those values consume extra computing time. On the other hand, too few pre- and post-relaxation passes will demand also a higher computational effort.

In Fig. 8 the number of pre- and post-smoothing iterations was fixed $\nu^{pre}=\nu^{post}=1$ for jet confined, whereas the number of coarsest-grid sweeps ν^{cg} was free to vary. Here also additional calculations already presented in Mesquita & de Lemos, 1999, are plotted in Fig. 8b for comparison. Fig. 8a clearly shows optimal values for the confined jet case and a slightly better performance of the *V*-cycle, for both Reynolds analyzed and for ν^{cg} greater than 4. On the other hand, results in Fig. 8b are quite spread and no optimal value seems to be detected. This figure clearly shows that no general rule or definite conclusion can be drawn as far as obtaining an optimal value to be used in all flows. Ultimately, both Figs. 7 and 8 suggest a delicate balance between all parameters involved when minimum CPU consumption is sought.



a) *b*)

Figure 7- Influence of the number of pre/post-smoothing interations on the computational effort: *a*) confined jet, *b*) backward facing step

4.0 CONCLUSIONS

In all cases investigated, the used of more than one grid for relaxing the solution was advantageous for reducing the overall computing time. The *W*-cycle applied with four grids was more efficient than the *V*-cycle (Fig. 5). In the *W* strategy, more time per cycle is spent in coarse grids (Fig. 3) causing low frequency error to be more efficiently swept out. For the jet flow, the use of a segregate approach for solving uncoupled problems did not bring as much computational savings as it did for other types of flows (Fig. 6).

As far as optimal values, this paper has suggested that at the coarsest grid level, the solution should be relaxed no more than 1 to 2 times per cycle. Intermediate relaxation should also be limited to 10 times at the most (Fig. 8). However, optimal parameters can not be generalized or easily determined *a priori* and adaptive strategies have been proposed in the literature.

Accordingly, the ratio of residues after two successive sweeps can be monitored and used as a criterion for switching grids. Hortmann et al, 1990, points out that this practice is preferred for single equation systems but, when solving the full equation set as done here, such practice is not easy to implement. In this case, most works in the literature specify a fixed number of sweeps, as in the cases here reported (Sathyamurthy & Patankar, 1994, Hutchinson et al 1988).

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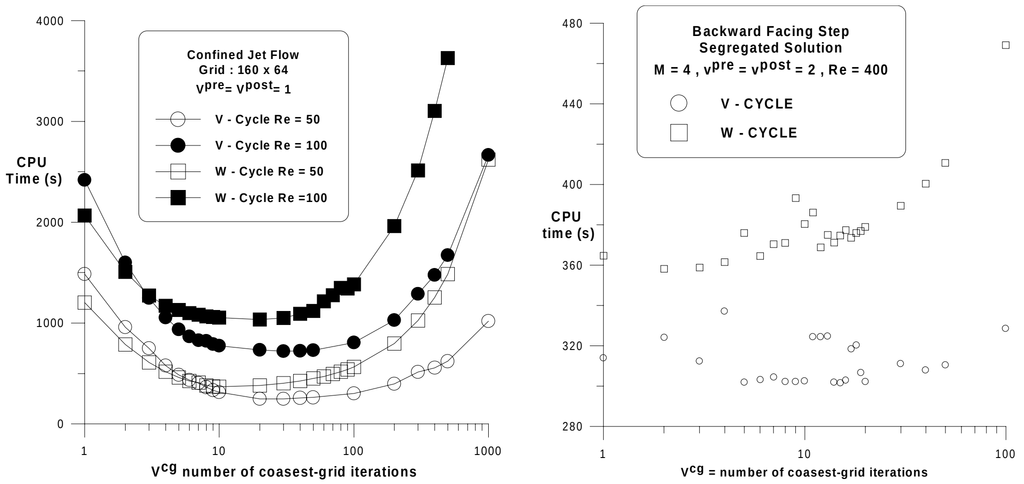
REFERENCES

Brandt, A., 1977, Multi-Level Adaptive Solutions To Boundary-Value Problems, *Math. Comp.*, vol. 31, No. 138, pp. 333-390.

de Lemos, M.J.S, Mesquita, M.S., 2000, Acceleration of Multigrid Solutions of Non-Isothermal Recirculating Flows, Advances in Computational Heat and Mass Transfer, Paper **NHTC2000-12232**, *34th National Heat Transfer Conference*, August 20-22, 2000, Pittsburgh, Pennsylvania, USA.



***a*)** ***b*)**

Figure 8- Influence of the number of coarsest-grid iterations , ν^{cg}, on the computational effort: *a*) Confined Jet, *b*) Backward Facing Step

de Lemos, M.J.S., Mesquita, M.S., 1999, Multigrid Numerical Solutions Of Non-Isothermal Laminar Recirculating Flows, *Applications of Computational Heat Transfer*, **ASME-HTD-vol. 364-3**, ISSN: 0272-5673, ISBN: 0-7918-1656-7, Ed..L.C. White, pg. 323-330.

Hackbusch, W., 1985, Multigrid Methods And Applications, Springer-Verlag, Berlin.

Hortmann, M., Peric, M., Scheuerer, G., 1990, Finite Volume Multigrid Prediction Of Laminar Convection: Bench-Mark Solutions, *Int. J. Num. Meth. Fluids*, vol. 11, pp. 189-207.

Hutchinson, B.R., Galpin, P.F., Raithby, G.D., 1988, Application Of Additive Correction Multigrid To The Coupled Fluid Flow Equations, *Num. Heat Transfer*, vol. 13, pp. 133-147.

Jiang, Y., Chen, C.P., Tucker, P.K., 1991, Multigrid solutions of unsteady Navier-Stokes equations using a pressure method, Num. Heat Transfer - part A, vol. 20, pp. 81-93.

Mesquita, M.S., 2000, Solução Numérica de Escoamentos Bidimensionais Não-Isotérmicos Usando o Método Multigrid, *Master Thesis*, IEME-ITA.

Mesquita, M.S., de Lemos, M.J.S., 1999, Convergence Characteristics of Multigrid Numerical Solutions of Non-Isothermal Recirculating Flows, COBEM 99 – XV Congresso Brasileiro de Engenharia Mecânica, Águas de Lindóia, SP, Brasil, 22 a 26 Nov.

Mesquita, M.S., de Lemos, M.J.S., 2000, Multigrid Numerical Solutions of Laminar Back Step Flow, *accepted for presentation at CONEM2000 - National Mechanical Engineering Congress*, Natal, Rio Grande do Norte, Brazil, August 7-11.

Patankar, S.V., 1980, Numerical Heat Transfer And Fluid Flow, Mc-Graw Hill.

Peric, M., Rüger, M., Scheuerer, G., 1989, A Finite Volume Multigrid Method For Calculating Turbulent Flows, In: *Seventh Symposium on Turbulent Shear Flows*, pp. 7.3.1-7.3.6, Standford University.

Rabi, J.A., de Lemos, M.J.S., 1998a, Multigrid Numerical Solution Of Incompressible Laminar Recirculating Flows, *ENCIT98 - Proc. of 7th Braz. Cong. Eng. Th. Sci.*, vol. 2, pp. 915-920, Rio de Janeiro, RJ, Nov. 3-6.

Rabi, J.A., de Lemos, M.J.S., 1998b, The Effects Of Peclet Number And Cycling Strategy On Multigrid Numerical Solutions Of Convective-Conductive Problems, *7th AIAA/ASME Jnt Thermcs & HT Conf*, **Paper AIAA-98-2584**, Albuquerque, New Mexico, USA, June 15-18.

Sathyamurthy, P.S., Patankar, S.V., 1994, Block-Correction-Based Multigrid Method For Fluid Flow Problems, *Numerical Heat Transfer - Part B*, Vol. 25, Pp. 375-394.

Stüben, K., Trottenberg, U., 1982, Multigrid Methods, In *Lect. Notes Math.*, Vol. 960, Pp. 1-76, Berlin