# A SECOND-ORDER IN TIME SCHEME FOR EVOLUTION EQUATIONS MODELLING BENARD CONVECTION

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***Abstract.*** *Using the operator-splitting method a computationally efficient second-order in time implicit difference scheme is developed for the Swift-Hohenberg equation (S-H). For each time step the scheme involves internal iterations which improve the stability and increase the accuracy with which the Lyapunov functional for S-H is approximated. Different cases of pattern formation are treated and shown that the new scheme reaches the stationary pattern several times faster than the previously used first-order in time schemes. The results for the stationary pattern are compared with our previous results and shown to be in good qualitative and quantitative agreement.*

***Keywords:*** *Nonlinear Systems, Bénard Convection, Implicit Methods, Finite Difference Methods*

## 1. INTRODUCTION

Pattern formation in a thin layer of fluid heated from below is modelled as a rule by nonlinear parabolic equations containing fourth-order space-derivatives (generalized diffusion equations). To this class belong the Swift-Hohenberg (S-H), Kuramoto-Sivashinsky (K-S), and Knobloch (KE) equations, to mention a few.

In the present paper we concern ourselves with S-H [5] which describes pattern formation in fluid layers confined between horizontal good-conducting boundaries. Unlike the K-S and KE, the S-H equation possesses a Lyapunov functional which ensures the potential behaviour of the solution, i. e., a time-independent pattern as $t \longrightarrow \infty$. Solving S-H numerically is a challenge both because of the presence of higher-order spatial

derivatives and of nonlinearity (see, e.g., [1] where a first-order in time implicit scheme is developed).

It is important to have a second-order in time scheme not only because it provides better accuracy for a given time increment, but because it offers also the possibility to satisfy a discrete version of the Lyapunov functional which is simply impossible to achieve with a two-level first-order in time scheme. Here a finite-difference semi-implicit coordinate-splitting scheme of second order in time and space is developed to solve the S-H equation subject to generalized Dirichlet boundary conditions. The proposed scheme is nonlinear and iterations are used within each step with respect to physical time.

## 2. POSING THE PROBLEM

### 2.1 Generalized Diffusion Equations of Fourth Order

Consider the fourth-order PDE

$$\begin{aligned} \frac{\partial u}{\partial t} &= -\Delta\left[a_4(x,y,t)\Delta u\right] + \nabla\left[a_2(x,y,t)\nabla u\right] - a_0(x,y,t)u - b(x,y,t) + F(u), \\ \Delta &\equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\,, \quad F(u) \stackrel{\text{def}}{=} -\frac{dU(u)}{du}, \end{aligned} \tag{2.1}$$

which is a typical generalization of the diffusion equation when a higher-order diffusion is represented by the fourth-order derivatives. In its turn, the second-order operator can be either dissipative or energy pumping mechanism according to the sign of $a_2$.

Consider a rectangular region $D : \{x \in [0, L_x,],\ y \in [0, L_y]\}$ with boundary $\partial D$. From physical point of view different types of b.c. can be considered. In order to figure out the correct sets of b.c. we consider the difference $v = u_1 - u_2$, where $u_1, u_2$ are two solutions satisfying the same b.c. Whatever the b.c. for $u_1, u_2$ are, the b.c. for the function $v$ are homogeneous, namely

$$v = \frac{\partial v}{\partial n} = 0\,, \quad v = \Delta v = 0\,, \quad \frac{\partial v}{\partial n} = \frac{\partial \Delta v}{\partial n} = 0\,, \quad \Delta v = \frac{\partial \Delta v}{\partial n} = 0\,; \qquad (x,y) \in \partial D, \tag{2.2}$$

where $n$ stands for the outer normal direction to the boundary $\partial D$. For the function $v$ one has the same equation (2.1) but without the inhomogeneous term $b(x,y,t)$. Upon multiplying it by $v$ and integrating over the domain $D$ we get

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}\int_D v^2 dxdy = &- \oint_{\partial D} v\frac{\partial a_4(x,y,t)\Delta v}{\partial n}\mathrm{d}l - \oint_{\partial D} a_4(x,y,t)\Delta v\frac{\partial v}{\partial n}\mathrm{d}l \\ &- \oint_{\partial D} a_2(x,y,t)v\frac{\partial v}{\partial n}\mathrm{d}l - \int_D a_0(x,y,t)(\Delta v)^2\mathrm{d}x\mathrm{d}y \\ &- \int_D a_2(x,y,t)(\nabla v)^2\mathrm{d}x\mathrm{d}y - \int_D a_0(x,y,t)v^2\mathrm{d}x\mathrm{d}y \end{aligned} \tag{2.3}$$

The correct set of b.c. is the one which secures that the evolution of the "energy" $\int v^2$ depends only on its production or dissipation in the bulk, but not on the surface. The admissible b.c. are the first three b.c. in 2.2) while the last one can be used only if the original equation does not contain second-order operators , i.e., when $a_2 \equiv 0$.

We call $(2.2)_1$ and $(2.2)_2$ "generalized Dirichlet conditions" of first and second kind, respectively. The condition $(2.2)_3$ involves only derivatives at the boundary, hence the coinage – "generalized Neumann condition".

## 2.2 Swift-Hohenberg Equation

The improper sign of the second-order operator in (2.1) can bring about a linear bifurcation which is an important feature of the Generalized Nonlinear Diffusion Equation (GNDE) (2.1). Consider the following version of (2.1) (see, [5]):

$$\frac{\partial u}{\partial t} = -D(\Delta + \kappa^2)^2 + F(u) \equiv -D\Delta\Delta u - 2D\kappa^2\Delta u - D\kappa^4 + F(u)\,. \tag{2.4}$$

For the case of cubic nonlinearity one has

$$F(u) = -\frac{dU(u)}{du} = \varepsilon(x)u + gu^3\,, \quad U(u) = -\frac{\varepsilon(x)}{2}u^2 - \frac{g}{4}u^4\,. \tag{2.5}$$

Eq.(2.4) is the Swift-Hohenberg equation (S-H, for brevity) derived for the Rayleigh-Bénard convection to account for the formation of convective rolls in high Prandtl number fluid layers. The variable $u(x, y, t)$ describes the horizontal planform of the temperature deviation from the conductive profile. The main feature of (2.4) is that the damping of inhomogeneous perturbations occurs via the fourth-order spatial derivatives, while the perturbations are enhanced by the second-order spatial derivatives the latter having an anti-diffusive behavior. Here, the difference is that the S-H admits a non-increasing Lyapunov functional

$$\Psi = \int_D \left\{ -\frac{\varepsilon}{2}u^2 + \frac{g}{4}u^4 + \frac{D}{2}\left[(\Delta u)^2 - 2\kappa^2(\nabla u)^2 + \kappa^4 u^2\right] \right\} \mathrm{d}x\,\mathrm{d}y, \tag{2.6}$$

$$\frac{\partial \Psi}{\partial t} = -\int_D \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}y < 0\,. \tag{2.7}$$

The existence of Lyapunov function rules out complex temporal or spatio-temporal behavior (turbulent, oscillatory, or chaotic) in the long time and allows formation of steady convective patterns only which patterns branch out from a (generally motionless) reference state due to the interplay between the complicated linear operator and the nonlinearity. These steady patterns can be quite complicated in shape, e.g., spatially chaotic. From the perspective of GNDE it is clear that a bifurcation can take place only for sufficiently large domains whose size is commensurate with the length scales of the patterns.

Consider the rectangle $x \in [0, L_x]$, $y \in [0, L_y]$. In the rectangular domain, the Dirichlet b.c. of the first and second kind read:

$$u = \frac{\partial u}{\partial x} = 0 \text{ for } x = 0,\ L_x; \qquad u = \frac{\partial u}{\partial y} = 0 \text{ for } y = 0,\ L_y, \tag{2.8}$$

$$u = \frac{\partial^2 u}{\partial x^2} = 0 \text{ for } x = 0,\ L_x; \qquad u = \frac{\partial^2 u}{\partial y^2} = 0 \text{ for } y = 0,\ L_y, \tag{2.9}$$

Respectively, the Neumann condition is

$$\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad x = 0,\ L_x, \qquad \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for} \quad y = 0,\ L_y, \tag{2.10}$$

There is no restriction to use mixed types of b.c. which are combinations of Dirichlet and Neumann conditions. If the scheme and algorithm perform properly for the "pure" cases (including the Neumann one), then they will do the same for the mixed cases, since any admissible (in the sense of (2.3)) mixture of conditions yields to a well posed boundary value problem. For the sake of simplicity we restrict ourselves in the present work to Dirichlet conditions of the first kind.

## 3 . DIFFERENCE SCHEME

### 3.1 Implicit Time-Stepping

First we show the desired form of the second-order in time implicit scheme for (2.4). In order to elucidate the splitting idea we present the derivations for the continuous spatial derivatives. The difference approximation of the spatial terms are discussed in what follows.

$$\begin{aligned}\frac{u^{n+1}-u^n}{\tau} &= D\left[-\frac{\partial^4}{\partial x^4}-\frac{\partial^4}{\partial y^4}-2k^2\frac{\partial^2}{\partial x^2}-2k^2\frac{\partial^2}{\partial y^2}-2\frac{\partial^4}{\partial x^2\partial y^2}-k^4\right]\frac{u^{n+1}+u^n}{2}\\ &\quad+\frac{U(u^{n+1})-U(u^n)}{u^{n+1}-u^n}\,,\end{aligned} \tag{3.1}$$

In the particular case under consideration we use the following representation of the nonlinear term

$$\frac{U(u^{n+1})-U(u^n)}{u^{n+1}-u^n}=\frac{\varepsilon(x)}{2}[u^{n+1}+u^n]+\frac{g}{4}\left[(u^{n+1})^3+(u^{n+1})^2u^n+u^{n+1}(u^n)^2+(u^n)^3\right]$$

### 3.2 Internal Iterations

The scheme (3.1) is nonlinear and can be solved by means of iterating the solution within a given time step. The iterations allow us to alleviate a possible problem that may arise in an implicit scheme connected with the inversion of the linear operators when they are not negative definite. Complications can be expected only when $a_2 < 0$, since depending on the size of region $D$ (and hence on the slopes of the solution itself), the linear spatial operator (involving both fourth- and second- order derivatives) may cease to be negative definite. A simple consequence of this fact is the occurrence of a linear bifurcation of the stationary problem. We tackle this complication by means of an explicit approximation of the second-order terms. We use the following scheme to find the consecutive iteration:

$$\begin{aligned}\frac{u^{n,k+1}-u^n}{\tau} &= D\left[-\frac{\partial^4}{\partial x^4}-\frac{\partial^4}{\partial y^4}-k^4\right]\frac{u^{n,k+1}+u^n}{2}\\ &+ D\left[-2k^2\frac{\partial^2}{\partial x^2}-2k^2\frac{\partial^2}{\partial y^2}-2\frac{\partial^4}{\partial x^2\partial y^2}\right]\frac{u^{n,k}+u^n}{2}\\ &+ \frac{\varepsilon}{2}[u^{n,k+1}+u^n]+\frac{g}{4}\left[u^{n,k+1}(u^{n,k})^2+u^{n,k+1}u^{n,k}u^n+u^{n,k+1}(u^n)^2+(u^n)^3\right].\end{aligned} \tag{3.2}$$

In the last scheme $u^{n,k+1}$ stands for the current iteration of the unknown grid function while the functions $u^{n,k}$ and $u^n$ are considered as known properties during a given internal

iteration: the former – from the previous iteration, the latter – from the previous time step. Unlike the original implicit scheme, the scheme with internal iterations is linear for $u^{n,k+1}$. The internal iterations are conducted until for certain $k = K$ the following criterion is satisfied

$$\frac{\max \|u^{n,K+1} - u^{n,K}\|}{\max \|u^{n,K+1}\|} < \delta, \tag{3.3}$$

and the last iteration gives the sought function on the new time stage, $u^{n+1} \stackrel{\text{def}}{=} u^{n,K+1}$.

The gist of the concept of internal iterations is that the same time step is repeated until convergence. Since the iterative process begins from an initial condition which is the value of the sought function on the previous time step, the number of internal iterations needed for convergence depends heavily on the magnitude of the time increment $\tau$. For smaller $\tau$ the initial condition is closer to the sought function and the number of iterations is expected to be small. The trade-off is that a very small $\tau$ requires a larger number of time steps which increases the overall number of arithmetic operations per nodal point. Conversely, an unappropriate large $\tau$ will bring about a larger number of internal iterations per time step increasing significantly the computational time needed to achieve a single time step dispelling thus the advantage of the larger "strides" (the faster time-stepping). The dependence of the the internal iterations on $\tau$ is nonlinear and there is a room for optimization. Our experience (numerical experiments) with schemes involving internal iterations shows that the calculations are cost effective if the number of internal iterations $4 \leq K \leq 16$. This estimate calls for a reduction of the time step when faster processes are treated for which the evolution from a given time stage to the next time stage involves a significant deformation of the field. This means that when faster temporal processes are involved, the usage of larger time steps $\tau$ leading to $K \gg 20$ is not justified regardless to the fact that formally speaking the implicit scheme is still stable.

## 3.3 The Splitting

The implicit scheme for time-stepping equation for a given internal iteration (3.3) can be recast in the form

$$\begin{aligned}
\frac{u^{n,k+1} - u^n}{\frac{1}{2}\tau} &= D\left[-\frac{\partial^4}{\partial x^4} - \frac{\partial^4}{\partial y^4} - k^4\right] u^{n,k+1} + D\left[-\frac{\partial^4}{\partial x^4} - \frac{\partial^4}{\partial y^4} - k^4\right] u^n \\
&+ D\left[-2\frac{\partial^4}{\partial x^2 \partial y^2} - 2k^2\frac{\partial^2}{\partial x^2} - 2k^2\frac{\partial^2}{\partial y^2} + \varepsilon(x)\right](u^{n,k} + u^n) + \frac{g}{2}(u^n)^3 \\
&+ \frac{g}{2}\left[\left(u^{n,k}\right)^2 + u^{n,k}u^n + (u^n)^2\right] u^{n,k+1}\,.
\end{aligned} \tag{3.4}$$

Spatial operators in Eq. (3.4) are discretized using central-difference approximations. The operator acting on the vector of variables has a three-dimensional "stereo-five diagonal" structure. The inversion of this kind of matrix is a rather costly procedure even though it is sparse. The 3D case is drastically more expensive. One should be noted that the internal iterations require the solution of (3.4) to be repeated several times during each time step. Then it is only natural to introduce operator splitting in order to minimize the operations per unit iteration and hence per one time step.

The first idea which comes to mind is to generalize the so-called Alternating Directions Implicit scheme (ADI) [3]. However, ADI does not work for non-commutative operators (see, e.g., [6]). Hence we prefer the *second Douglas scheme* [2] (also called "scheme of stabilizing correction" ) which gives the full-time-step approximation for non-commutative operators and is more robust for nonlinear problems than ADI (see, [6] for a review of the splitting schemes and strategies). Another advantage of the stabilizing correction is that for linear problems in 3D it is absolutely stable, while ADI is not. The only disadvantage is that the original scheme is first-order accurate in time. In what follows we both generalize the Douglas scheme for fourth-order operators and modify it to be second order accurate in time (a Crank-Nicolson like scheme). Then

$$\begin{aligned}
\frac{\tilde{u}-u^n}{\tau} &= L_{11}^{n,k}\tilde{u} + L_{22}^{n,k}u^n + \frac{1}{2}\left[-D\frac{\partial^4}{\partial x^4} - D\frac{\partial^4}{\partial y^4} - Dk^4 + \varepsilon(x) + \frac{g}{2}\left(u^n\right)^2\right]u^n \\
&+ \frac{1}{2}\left[-L_{12} - L_1 - L_2\right]\left[u^{n,k} + u^n\right] \\
\frac{u^{n,k+1}-\tilde{u}}{\tau} &= L_2^{n,k}(u^{n,k+1} - u^n)
\end{aligned}$$

where

$$L_{11}^{n,k} \stackrel{\text{def}}{=} -\frac{D}{2}\frac{\partial^4}{\partial x^4} - \frac{D}{4}k^4 + \frac{g}{8}\left[\left(u^{n,k}\right)^2 + u^{n,k}u^n + \left(u^n\right)^2\right] + \frac{\varepsilon(x)}{2} \tag{3.5}$$

$$L_{22}^{n,k} \stackrel{\text{def}}{=} -\frac{D}{2}\frac{\partial^4}{\partial y^4} - \frac{D}{4}k^4 + \frac{g}{8}\left[\left(u^{n,k}\right)^2 + u^{n,k}u^n + \left(u^n\right)^2\right] + \frac{\varepsilon(x)}{2} \tag{3.6}$$

$$L_{12} \stackrel{\text{def}}{=} 2D\frac{\partial^4}{\partial x^2 \partial y^2}, \qquad L_1 \stackrel{\text{def}}{=} 2Dk^2\frac{\partial^2}{\partial x^2}, \qquad L_2 \stackrel{\text{def}}{=} 2Dk^2\frac{\partial^2}{\partial y^2} \tag{3.7}$$

In order to show that the splitting scheme approximates the original implicit scheme we rewrite (3.5) as follows

$$\begin{aligned}
[E - \tau L_{11}^{n,k}]\tilde{u} &= [E + \tau L_{22}^{n,k}]u^n + \frac{\tau}{2}\left[-L_{12} - L_1 - L_2\right]\left[u^{n,k} + u^n\right] \\
&+ \frac{\tau}{2}\left[-\frac{\partial^4}{\partial x^4} - \frac{\partial^4}{\partial y^4} - k^4 + \varepsilon(x) + \frac{g}{2}\left(u^n\right)^2\right]u^n, \\
[E - \tau L_{22}^{n,k}]u^{n,k+1} &= \tilde{u} - \tau L_{22}^{n,k}u^n
\end{aligned} \tag{3.8}$$

Now we are prepared to eliminate the intermediate variable $\tilde{u}$. This is done after applying the operator $[E - \tau L_{11}^{n,k}]$ to the second of equations (3.9) and adding the result to the first one, namely

$$\begin{aligned}
[E - \tau L_{11}^{n,k}][E - \tau L_{22}^{n,k}]u^{n,k+1} &= [E + \tau L_{22}^{n,k}]u^n - \tau[E - \tau L_{11}^{n,k}]L_{22}^{n,k}u^n \\
&+ \frac{\tau}{2}\left[-L_{12} - L_1 - L_2\right]\left[u^{n,k} + u^n\right] \\
&+ \frac{\tau}{2}\left[-\frac{\partial^4}{\partial x^4} - \frac{\partial^4}{\partial y^4} - k^4 + \varepsilon(x) + \frac{g}{2}\left(u^n\right)^2\right]u^n,
\end{aligned}$$

or else,

$$[E+\tau^2 L_1^{n,k}L_2^{n,k}]\frac{u^{n,k+1}-u^n}{\tau} = (L_{11}^{n,k}+L_{22}^{n,k})u^{n,k+1} + \frac{1}{2}\left[-L_{12}-L_1-L_2\right]\left[u^{n,k}+u^n\right] + \frac{1}{2}\left[-D\frac{\partial^4}{\partial x^4}-D\frac{\partial^4}{\partial y^4}-Dk^4+\varepsilon(x)+\frac{g}{2}\left(u^n\right)^2\right]u^n. \tag{3.9}$$

Upon acknowledging (3.5), (3.6), (3.7), it is readily shown that (3.9) is in fact (3.3) save the *positive definite* operator of norm larger than unity

$$B \equiv E+\tau^2 L_{11}^{n,k}L_{22}^{n,k} = E+O(\tau^2), \tag{3.10}$$

acting upon the time difference $(u^{n+1}-u^n)/\tau$. The latter means that it has no influence on the steady-state result. One sees that the splitting scheme approximates the desired scheme in full-time steps within the adopted order of approximation $O(\tau^2)$. Thus, employing a splitting does not degrade the temporal approximation of the scheme. In other words, the splitting scheme coincides with the original scheme within the order of approximation of the latter.

## 4. NUMERICAL RESULTS

In this section we report the results of our numerical simulations obtained with the proposed second-order difference scheme and compare the results to those obtained with the first-order scheme [1, 4]. We consider two different Rayleigh-Bénard systems. The first is a ramped system for which the forcing bifurcation parameter $\varepsilon = \frac{1}{2l}(x-0.5l)$, varies linearly along the $x$ direction, within the interval $-0.25 \le \varepsilon(x) \le 0.25$. The second is one a non-ramped system for which $\varepsilon = const = 0.2$.

For the sake of definiteness we take $\kappa = 3.1172$, $g = 12.9$, and $D = 0.015$ which values correspond to a typical Rayleigh-Bénard convection with pattern formation. The calculations are performed in a square box with a side wall of length $l = 20$, which corresponds roughly to 9 wave lengths. A staggered mesh containing $82 \times 82$ points was used, leading to a spatial resolution of approximately 9 points per wavelength. The second-order scheme was tested for four different values of the tolerances $\delta$, defined in eq. (3.3), namely $\delta = 10^{-3}$, $\delta = 10^{-4}$, $\delta = 10^{-6}$, and $\delta = 10^{-12.}$. During the simulation we track the rate of evolution of the pattern by monitoring the $L_1$ norm:

$$L_1 = \frac{1}{\tau}\frac{\sum_{i,j}|u_{i,j}^{n+1}-u_{i,j}^n|}{\sum_{i,j}|u^{n+1}|}, \tag{4.1}$$

The calculations begin from the same random initial condition and proceeded until $L_1 \le 5\times 10^{-7}$, when it can be assumed that the motion is virtually steady. The time increment is kept constant $\tau = 0.05$. In all of the cases the steady-state pattern obtained is the same. The upper panels of Fig. (1) show the evolution obtained with the second-order scheme for $\delta = 10^{-6}$. The lower panels depict the evolution according to the first-order scheme. It is clearly seen that the time evolution obtained with the second-order scheme

is smoother and much faster. This can be attributed to the presence of an artificial dispersion in the first-order scheme which makes the evolution more "undular".

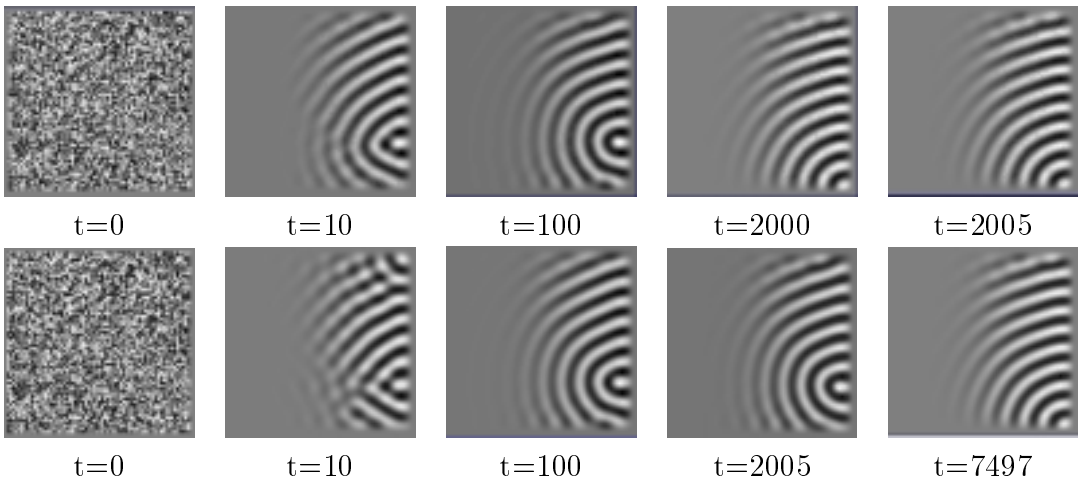



Figure 1: Time evolution of the pattern, starting from random initial conditions. Upper panels: second-order scheme with precision $\delta = 10^{-6}$ for the internal iterations. Lower panels: first-order scheme.

Fig. (2-a) shows $L_1$ as a function of time, obtained both with the first- and the second-order schemes. Figure (2-b) presents the number of internal iterations as function of time needed to from the second-order scheme to accomplish the accuracy within a given time step. One sees that as should be expected, the number of internal iterations increases in the time intervals of rapid evolution (the same intervals in which the $L_1$ norm increases).

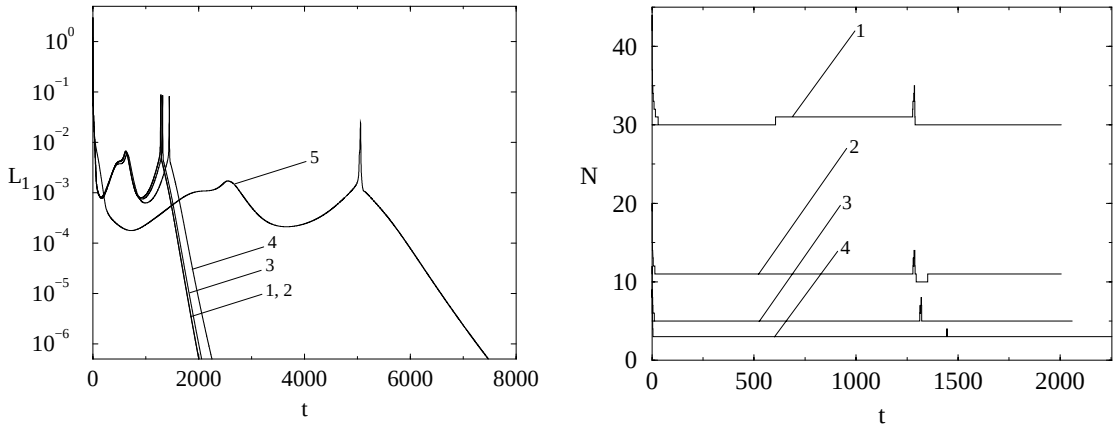



Figure 2: Left panel: $L_1 \times t$ curves $(a)$ for the second-order scheme for different time increments: lines 1 to 4: $\delta = 10^{-3}, 10^{-4}, 10^{-6}, 10^{-12}$ compared to the first-order scheme (line 5). Right panel: The number of iterations $N$ as function of time.

Initially the system displays a focus in the lower part of the right sidewall, but between $1440 \leq t \leq 1450$ it abruptly moves to the lower right corner. The evolution of the pattern accelerates at this moment, as precisely captured by the $L_1 \times t$ norm (line 1, Fig. 2-a). Figure 2-a shows also that the approach towards the steady state is qualitatively the same for each value of the precision $\delta$. The trajectory obtained with the first-order scheme is also qualitatively and quantitatively similar but the time required to attain the

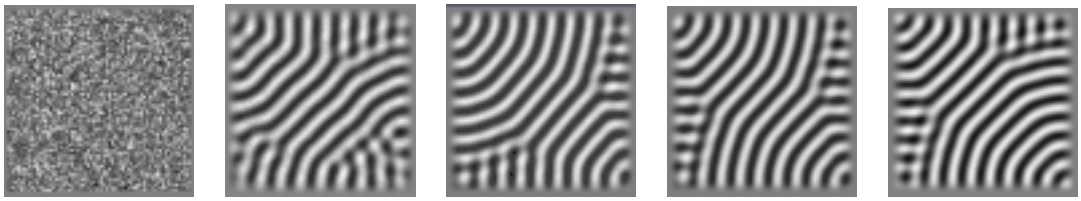
steady-state is about four times larger. In addition, Fig. 2-a shows a small decrease in the time required to attain the steady-state as the specified precision increases. However more internal iterations are required to match tighter precisions, which results in higher computational costs. In addition, no further reduction in the time required to attain the steady-state was observed, for $\delta < 10^{-6}$. Figure 2-b shows that a larger number of internal iterations is needed in the very beginning of the simulation, when the pattern evolves from the random initial condition. This number rapidly decreases as a pattern is formed and increases again only when a rearranging of the latter begins. In the case under consideration, the focus slides toward the right -bottom corner and the dynamics of the evolution accelerates. Table 1 summarizes the results concerning the number of internal iterations and the time $t_s$ required to attain the steady-state. Here $N_{max}$ is the maximal number, $N_{most}$ is the number needed in most of the time steps, and $N_{col}$ is the number of iterations needed during the collapse or rearranging of a subpattern.

Table 1: The number of internal iterations as function of time $t$.

| Precision | $N_{max}$ | $N_{most}$ | $N_{col}$ | $t_s$ |
|---|---|---|---|---|
| $1 \times 10^{-12}$ | 44 | 30 | 35 | 2005 |
| $1 \times 10^{-6}$ | 20 | 11 | 14 | 2005 |
| $1 \times 10^{-4}$ | 12 | 5 | 8 | 2059 |
| $1 \times 10^{-3}$ | 9 | 3 | 4 | 2253 |
| First-order | – | – | – | 7429 |

Here is to be mentioned that we monitored the difference approximation of the Lyapunov functional (2.7). It is indeed conserved within the truncation error and the adopted precision $\delta$.

As a second example we consider a uniformly loaded (non-ramped) system (see Fig. 3). The behaviour of the solution is essentially the same for the ramped system and the only differences are connected with the fact that larger number of solutions with different symmetries can appear. Clearly, due to the different approximation properties (truncation errors) of the two schemes, they present different routs the stationary solution (compare the last two panels in Fig.3).



t=0 t=10 t=100 t=2000 t=8938

Figure 3: Time evolution of a uniformly loaded system ($\varepsilon = 0.2$) as obtained with the second-order scheme for $\delta = 10^{-6}$. The last panel shows the first-order result.

The second-order scheme selects a steady solution which is symmetric with respect to two simultaneous flips around the vertical middle line and horizontal middle line of the box. At the time, the first-order scheme selects the solution which is symmetric with respect to the second diagonal of the box which connects the top-left and bottom-right corners of the box.

## 5. CONCLUSIONS

A second-order in time operator-splitting difference scheme is implemented for the numerical solution of parabolic equations containing fourth-order space-derivatives. The scheme is fully implicit owing to the use of internal iterations. It is not subject to the severe limitations on the time increment typical of explicit methods, while the memory storage requirements are kept at a level comparable to the one of explicit methods. The main advantage of the second-order scheme is that it does not contain artificial dispersion and the disturbances are quickly attenuated.

The performance of the scheme is featured on the Swift-Hohenberg equation which models the Rayleigh-Bénard convection in a horizontal layer. Two different systems are treated: a ramped one, and a uniformly loaded. The results are compared to the findings of a first-order in time scheme, when the same time increment is used in both cases. The same steady state is attained with the first- and the second-order schemes but former requires times about four times longer to reach the steady state than the latter. The discrete version of the Lyapunov functional of S-H is conserved within the adopted precision $\delta$.

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