Proceedings of the XII International Symposium on Dynamic Problems of Mechanics (DINAME 2007),
P.S.Varoto and M.A.Trindade (editors), ABCM, Ilhabela, SP, Brazil, February 26 - March 2, 2007

Wheel Dynamics

Georg Rill[1]

[1] University of Applied Sciences Regensburg, Galgenberstr. 30, 93053 Regensburg, Germany

Abstract: The dynamics of the wheel rotation is mainly influenced by the driving or the braking torque and the longitudinal tire force. Today different tire models are available. Structural tire models are very complex. Here, the dynamics of the wheel rotation is dominated by the complex tire model. Within handling models, the steady state tire forces and torques are generated as functions of the longitudinal and lateral slip. Depending on the slip definition the dynamics of a wheel depends now on the vehicle velocity or the angular velocity of the wheel. Here, the wheel dynamics will become more and more stiff if the vehicle slows down or the wheel is close to a locking situation. This causes problems in drive away and braking to stand still maneuvers.
In this paper a simple quarter car model is used to study the wheel dynamics. Different model approaches and the performance of explicit and implicit ode solvers are investigated.

Keywords: Vehicle Dynamics, Wheel, Tire, ode-Solver

INTRODUCTION

Today, numerical simulation of system dynamics is a standard in the design of cars and trucks. Usually the vehicles are modeled by Multibody Systems (Rauh, 2003 as well as Weinfurter et al., 2004). A typical model for a passenger car consists of several subsystems, Fig. 1. Complex vehicle models are used to investigate the ride comfort and the handling

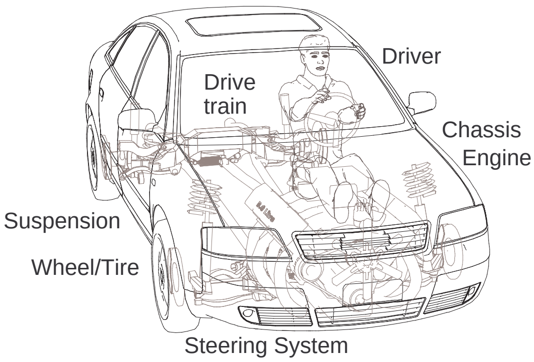

Figure 1 – Vehicle modeled by subsystems

performance as well as to generate load data for life time prediction analysis. Additionally to these off-line simulation tasks real-time applications, typically used for the design of vehicle control systems and the test of electronic control units, become more and more popular. In order to achieve real time performance either the model complexity has to be reduced (Pankiewicz, 2003) or sophisticated modeling techniques and appropriate ode-solvers (Rill, 2006c) have to be used. Among many other problems (Rill and Chucholowski, 2005 as well as Rill 2006a) the wheel dynamics becomes critical in particular in drive away and braking to stand still maneuvers. The dynamics of the wheel rotation is mainly influenced by the driving or the braking torque and the longitudinal tire force. Usually the driving torque, if present, is transmitted via the driving shaft to the wheel. The torsional stiffness of the driving shaft provides a simple torque-based interface between the wheel rotation and a separate drive train model. This interface is moderately stiff and can therefore be mastered by standard ode-solvers. The dynamics of a braked wheel is quite delicate, because a wheel may lock in an instant. Here, either a soft braking torque model or an implicit ode solver is needed.

SIMPLE VEHICLE MODEL

Equations of Motion

In order to focus on the dynamics of the wheel motion a quite simple vehicle model will be used, Fig. 2. The linear momentum for the chassis and the angular momentum for the wheel result in

$$m\dot{v} \quad = \quad F_x\,, \tag{1}$$

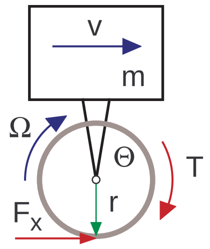

Figure 2 – Simple chassis/wheel model

$$\Theta\dot{\Omega} \;=\; T - rF_x\,, \tag{2}$$

where m is the corresponding chassis mass, Θ and r denote the inertia and the radius of the wheel. Within this simple model the wheel suspension and the tire deflection was not taken into account. Finally, T describes the driving or braking torque.

Longitudinal Tire Force

Today, different tire models are available (Lugner and Plöchl, 2006). Structural tire models are very complex. They are computer time consuming and they need a lot a data. Usually, they are used for stochastic vehicle vibrations occurring during rough road rides and causing strength-relevant component loads. Here, the dynamics of the wheel rotation is dominated by the complex tire model. Handling models are characterized by a useful compromise between user-friendliness, model-complexity and efficiency in computation time on the one hand, and precision in representation on the other hand (Hirschberg et al., 2002). Here, the steady state tire forces and torques are generated as functions of the longitudinal and lateral slip. The longitudinal force F_x can be described as a function of the longitudinal slip s_x, Fig. 3. The longitudinal

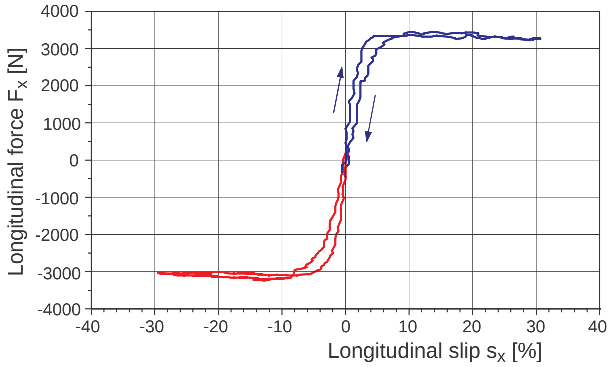

Figure 3 – Longitudinal force characteristics of a typical passenger car tire

slip is defined by

$$s_x \;=\; \frac{r\Omega - v}{r\,|\Omega|}\,, \tag{3}$$

where r serves in this simple approach also for the dynamic rolling radius.

Linearized Equations of Motion

In normal driving situations the longitudinal slip is very small and the wheel is close to the rolling condition

$$\Omega \approx \frac{v}{r}\,. \tag{4}$$

Hence, disturbances of a normal driving situation with $v = v_0$ and $\Omega = v_0/r$ may be described by

$$v \;=\; v_0 + \triangle v \tag{5}$$

and

$$\Omega \;=\; \frac{v_0}{r} + \triangle\Omega\,. \tag{6}$$

For small disturbances $\triangle v \ll v_0$ and $r\triangle\Omega \ll v_0$ the longitudinal slip can be simplified to

$$s_x \;=\; \frac{v_0 + r\triangle\Omega - v_0 - \triangle v}{|v_0 + r\triangle\Omega|} \;\approx\; \frac{r\triangle\Omega - \triangle v}{|v_0|}\,. \tag{7}$$

But, for small slip values the steady state force characteristics can be approximated by a linear function

$$F_x \approx dF_0 \, s_x \,, \tag{8}$$

where dF_0 describes the initial inclination of the longitudinal tire characteristics $F_x(s_x)$.

Assuming a constant driving velocity $v_0 = const$ the equations of motion can be written as

$$\triangle\dot{v} = \frac{1}{m}\,\frac{dF_0}{v_0}\,(r\triangle\Omega - \triangle v) \tag{9}$$

and

$$\triangle\dot{\Omega} = \frac{T}{\Theta} - \frac{r}{\Theta}\,\frac{dF_0}{v_0}\,(r\triangle\Omega - \triangle v)\,. \tag{10}$$

This set of linear differential equations can now be arranged in matrix form

$$\underbrace{\begin{bmatrix} \triangle\dot{v} \\ r\triangle\dot{\Omega} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\dfrac{dF_0}{v_0}\dfrac{1}{m} & \dfrac{dF_0}{v_0}\dfrac{1}{m} \\ \dfrac{dF_0}{v_0}\dfrac{r^2}{\Theta} & -\dfrac{dF_0}{v_0}\dfrac{r^2}{\Theta} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \triangle v \\ r\triangle\Omega \end{bmatrix}}_{x} + \begin{bmatrix} 0 \\ \dfrac{T}{\Theta} \end{bmatrix}, \tag{11}$$

where the disturbances in the vehicle velocity $\triangle v$ and the circumferential velocity of the wheel $r\triangle\Omega$ were used as state variables.

Eigenvalues

The solution of the homogenous state equation $\dot{x} = Ax$ is given by $x(t) = x_0 e^{-\lambda t}$, where the eigen-values λ and eigen-vectors x_0 are defined by

$$(A - \lambda I)\,x_0 = 0\,. \tag{12}$$

Non-trivial solutions $x_0 \neq 0$ are possible if $det\,|A - \lambda I| = 0$ or

$$\left(-\frac{dF_0}{v_0}\,\frac{1}{m} - \lambda\right)\left(-\frac{dF_0}{v_0}\,\frac{r^2}{\Theta} - \lambda\right) - \left(\frac{dF_0}{v_0}\,\frac{1}{m}\right)\left(\frac{dF_0}{v_0}\,\frac{r^2}{\Theta}\right) = 0 \tag{13}$$

will hold. The resulting quadratic equation has the solution

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = -\frac{dF_0}{v_0}\left(\frac{r^2}{\Theta} + \frac{1}{m}\right). \tag{14}$$

The second eigenvalue and hence the dynamic of the system depends on the driving velocity. For small driving velocities the system becomes very stiff. Hence, numerical integration algorithms based on explicit formulas must reduce the step size with a dropping driving velocity in order to maintain the stability.

NUMERICAL INTEGRATION

MatLab-Function

The equations of motion (1) and (2) and a simple nonlinear longitudinal force characteristics are coded in the following MatLab function

```
 function xp=f(t,x)
%
% simple vehicle model including wheel dynamics
%
% data
  global mass Theta r dF0 Fxmax Torque
%
% states
  v=x(1);                   % vehicle velocity
  o=x(2);                   % wheel ang. vel.
%
% longitudinal slip (nonlinear)
  sx = (r*o-v)/abs(r*o);    %
```

```
%
% longitudinal force (simple nonlinear function)
  fx = dF0*sx;
  if fx > +Fxmax, fx=+Fxmax; end
  if fx < -Fxmax, fx=-Fxmax; end
%
% equations of motion
vp = fx/mass;              % linear momentum
op = (Torque-r*fx)/Theta;  % angular momentum
%
% state derivatives
xp = [ vp; op ];
```

as a system of first order differential equations.

Vehicle Data

Typical data for a passenger car are provided in Tab. 1.

Chassis mass (quarter car)	m	$=$	$400\,kg$
Inertia of wheel	Θ	$=$	$1.2\,kgm^2$
Wheel radius	r	$=$	$0.3\,m$
Initial incl. long. force char.	dF_0	$=$	$100\,000\,N/-$
Maximum long. force	F_x^{max}	$=$	$3200\,N$
Driving torque	T	$=$	$100\,Nm$

Table 1 – Typical data for passenger car

Explicit Euler Integration

The explicit Euler integration

$$x(t+h) \;=\; x(t) + h\,f(t,x) \tag{15}$$

is numerically stable as long as the integration step size h and the eigenvalues λ of the dynamic system will satisfy

$$|\,1 + h\lambda\,| < 1\,. \tag{16}$$

Applied to Eq. (14) one gets

$$h\left|\frac{dF_0}{v_0}\left(\frac{r^2}{\Theta}+\frac{1}{m}\right)\right| \le 2 \tag{17}$$

or

$$|v_0| \ge \frac{h}{2}\,dF_0\left(\frac{r^2}{\Theta}+\frac{1}{m}\right). \tag{18}$$

The step size of a stable explicit Euler solution depends on the vehicle data and the driving speed. Hence, in real-time applications where a constant step size must be used driving manoeuver with $v_0 \to 0$ cannot be handled by an explicit Euler algorithm.

Here, for an integration step size of $h = 0.5\,ms$ the explicit Euler algorithm is stable as long as

$$|v_0| > 1.9375\,m/s \tag{19}$$

will hold.

The Integration was started at $t = 0$ with $v(t=0) = v_0 = -2\,ms$ and $\Omega(t=0) = v_0/r$. This describes a vehicle which is rolling backwards. The constant driving torque of $T = 100\,Nm$ will then accelerate the vehicle forward. The results are plotted in Fig. 4. As predicted the explicit Euler algorithm becomes numerically unstable if the absolute value of the velocity drops below a critical value. Here, the numerical instability produces non-physical oscillations in the circumferential wheel velocity at $r\Omega \approx -1.8\,m/s$. On the basis of the linearized equations the critical velocity was predicted in Eq. 19 with a slightly larger value.

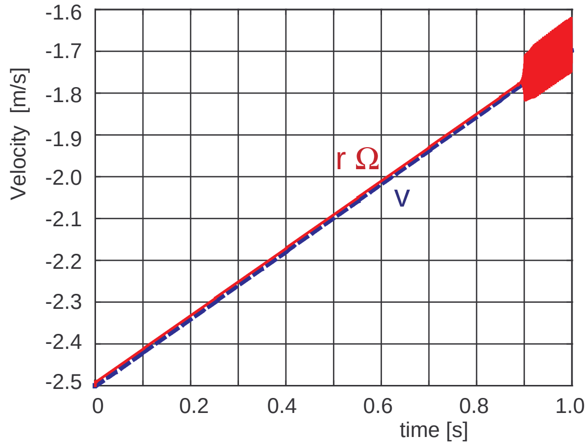

Figure 4 – Results of the explicit Euler algorithm

Comparison to Implicit Formula

A comparison of the MatLab solvers ode23 (explicit Runge-Kutta (2,3) pair of Bogacki and Shampine) with ode23s (a modified implicit Rosenbrock formula of order 2) is shown in Fig. 5. For both formulas the defaults error values of

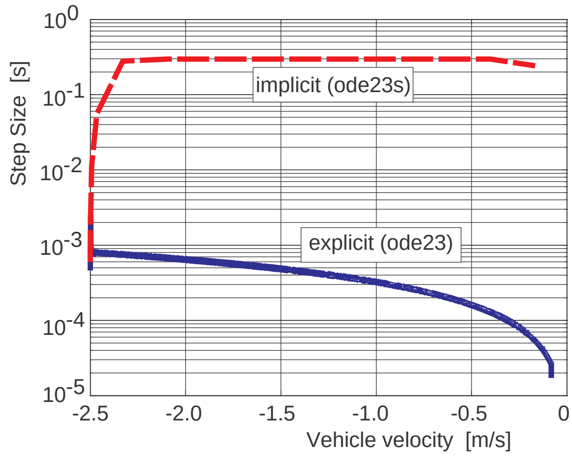

Figure 5 – Step sizes for explicit and implicit solver

RelTol $= 10^{-3}$ and AbsTol $= 10^{-6}$ were applied. In order to maintain stability the explicit Runge-Kutta algorithm has to reduce the step size nearly proportional to the absolut value of the vehicle velocity. Here, the implicit algorithm is approximately 50 times faster than the explicit one. Even at small absolute velocities it runs with considerable large step sizes.

Modified Slip

For a locked wheel or at stand still when $\Omega = 0$ will hold, the longitudinal slip in Eq. (3) is no longer defined. A small modification in the slip definition

$$s_x = \frac{r\Omega - v}{r|\Omega| + v_{num}}, \tag{20}$$

where a small but finite velocity $v_{num} > 0$ was added to denominator avoids this problem. Now, the explicit Euler algorithm is stable if

$$|v_0| + v_{num} > \frac{h}{2} dF_0 \left(\frac{r^2}{\Theta} + \frac{1}{m} \right). \tag{21}$$

will hold. Hence, by setting $v_{num} = 2\,m/s$ a stable transition from negative to positive velocities is possible here, Fig. 6. The simple explicit Euler algorithm can master drive away and stand still maneuvers. But, the modified slip definition slows down the wheel dynamics especially at low wheel angular velocities and may hence produce results with poor accuracy. Figure 6 shows that the longitudinal slip based on physics will tend to infinity at $r\Omega \rightarrow 0$ whereas the modified slip stay constant thus generating a constant longitudinal force which correspond to the constant driving torque. However,

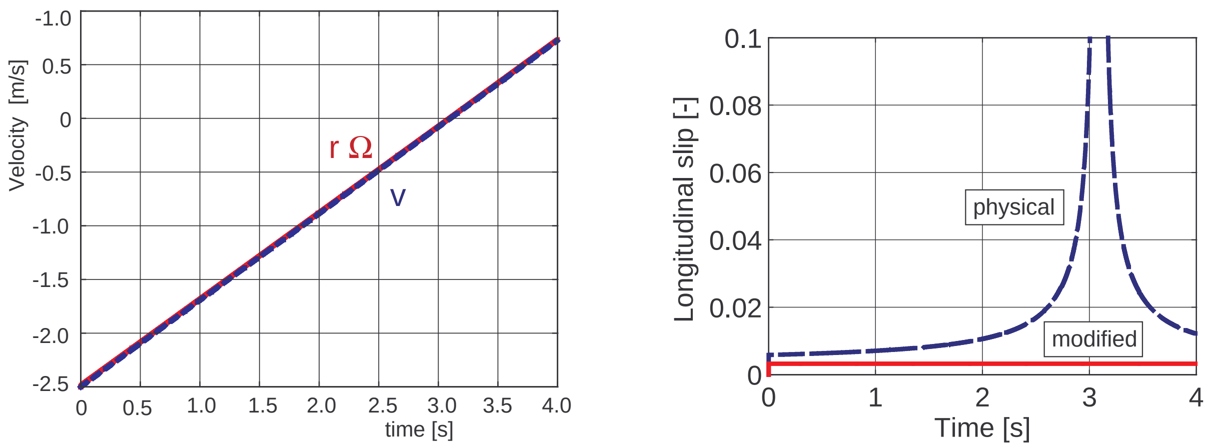

Figure 6 – Stable solution with explicit Euler algorithm for $\mathbf{v_{num} = 2\,m/s}$

in normal driving situations, where $r\,|\Omega| \gg v_{num}$ holds, the difference between the physically based slip and the modified slip will hardly be noticeable. In general, an implicit algorithm should be used to integrate the wheel rotation. Then, even the dynamics of a braked wheel can be mastered without problems (Rill, 2006a).

DYNAMIC TIRE FORCES

According to van der Jagt (2000) the tire forces show a dynamic behavior which can be approximated by a first order differential equation

$$T_x \dot{F}_x^D = F_x - F_x^D \,, \tag{22}$$

where the time constant T_x is derived from the relaxation length ℓ_x,

$$T_x = \frac{\ell_x}{r\,|\Omega|} \,. \tag{23}$$

Now, the dynamic tire force F_x^D instead of the steady state value F_x must be used in the equations of motion (1) and (2). According to Eqs. (4) to (8) the equations of motion can be linearized. One gets

$$\begin{aligned} m\,\triangle\dot{v} &= F_x^D \,, \\ \Theta\,\triangle\dot{\Omega} &= T - r\,F_x^D \,, \\ \tfrac{\ell_x}{|v_0|}\,\dot{F}_x^D &= dF_0\,\tfrac{r\triangle\Omega - \triangle v}{|v_0|} - F_x^D \,, \end{aligned} \tag{24}$$

where the term $r\,|\Omega|$ in the denominator of Eq. (23) was replaced by its linearized value $|v_0|$. Arranged in matrix form they read as

$$\underbrace{\begin{bmatrix} \triangle\dot{v} \\ r\triangle\dot{\Omega} \\ \dot{F}_x^D \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 0 & \dfrac{1}{m} \\ 0 & 0 & -\dfrac{r^2}{\Theta} \\ -\dfrac{dF_0}{\ell_x} & \dfrac{dF_0}{\ell_x} & -\dfrac{v_0}{\ell_x} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \triangle v \\ r\triangle\Omega \\ F_x^D \end{bmatrix}}_{x} + \begin{bmatrix} 0 \\ \dfrac{T}{\Theta} \\ 0 \end{bmatrix} , \tag{25}$$

The eigenvalues of A are now given by

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_{2/3} = -\frac{v_0}{2\ell_x} \pm i\sqrt{\frac{dF_0}{\ell_x}\left(\frac{1}{m} + \frac{r^2}{\Theta}\right) - \left(\frac{v_0}{2\ell_x}\right)} \,. \tag{26}$$

For $v_0 = 0$ the eigenvalues $\lambda_{2/3}$ are purely imaginary. Hence, at stand still the wheel and the longitudinal tire force will perform undamped motions. Figure 7 shows the eigenvalues calculated with the vehicle data given in Tab. 1 and a relaxation length of $\ell_x = 0.7\,m$ which is a typical value for a normal passenger car tire. The stability region of the explicit Euler algorithm is a circle with radius $R = 1/h$ touching the origin from the left. Again, to achieve stable solutions with the

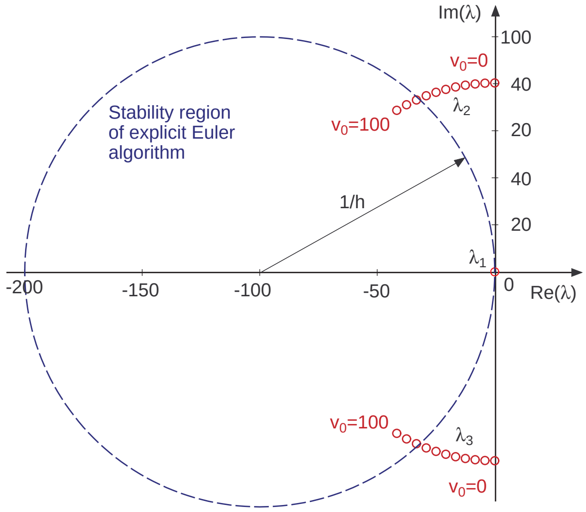

Figure 7 – Eigenvalues of vehicle with dynamic tire force and stability region of the explicit Euler algorithm

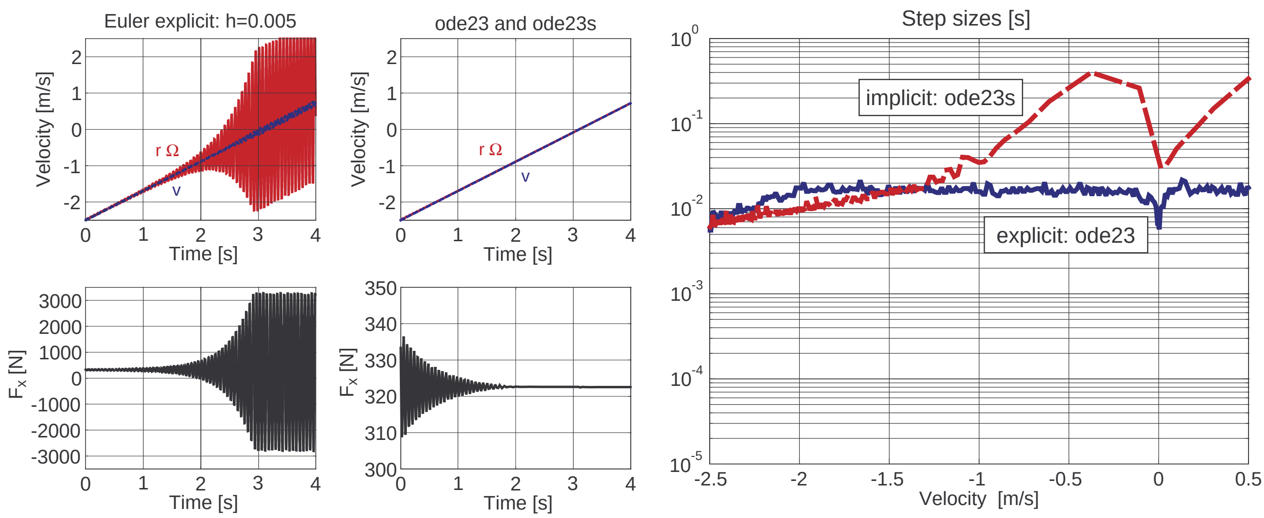

Figure 8 – Simulation results and step sizes of explicit and implicit algorithms

explicit Euler algorithm a very small step size must be used. For a step size of $h = 0.0005\,s$ the solution becomes unstable if the absolute circumferential velocity drops below $1\,m/s$, Fig. 8. However, the dynamics of the vehicle with dynamic tire forces can be handled quite well with any kind of implicit algorithm or higher order explicit formula, Fig. 8. This simple dynamic tire model covers the whole velocity range. At stand still it automatically changes to a stick slip model. An enhanced dynamic tire model can be found in Rill, 2006b. Hence, braking to stand still and drive away manoeuvres can be simulated in good accuracy because here, no changes in the wheel/tire dynamics at low velocities by modifications in the slip definition are necessary.

CONCLUSION

In this paper a simple quarter car model was used to analyze the wheel/tire dynamics. Using steady state tire models the wheel/tire dynamics strongly depends on the velocity of the vehicle. A slip modification which slows down the wheel/tire dynamics at low velocities is needed to overcome the singularity at stand still. Dynamic tire models operate in the whole velocity range. It was shown further that the explicit Euler algorithm is not suitable for solving the wheel/tire dynamics especially at low velocities. Here, implicit algorithms perform best. In real time applications where a constant step size is needed the implicit Euler algorithm has proofed its efficiency, Rill, 2006b. If the overall vehicle model cannot be integrated by an implicit solver a co-simulation of the overall vehicle with the subsystem wheel/tire is recommended.

REFERENCES

Hirschberg, W, Rill, G., Weinfurter, H., “User-Appropriate Tyre-Modeling for Vehicle Dynamics in Standard and Limit Situations”, Vehicle System Dynamics, Vol. 38, No. 2, pp. 103-125. Lisse: Swets & Zeitlinger, 2002

van der Jagt, P., “The Road to Virtual Vehicle Prototyping; new CAE-models for accelerated vehicle dynamics development”, PhD-Thesis, Tech. Univ. Eindhoven, 2000

Lugner, P.; Pacejka, H. and Plöchl, M., “Recent advances in tyre models and testing procedures”, Vehicle System Dynamics, Vol. 43, No. 67, pp. 413-436, 2005

Pankiewicz, E. and Rulka, W., “From Off-Line to Real Time Simulations by Model Reduction and Modular Vehicle Modeling”, in: Proceedings of the 19th Biennial Conference on Mechanical Vibration and Noise Chicago, Illinois, 2003

Rauh, J., “Virtual Development of Ride and Handling Characteristics for Advanced Passenger Cars”, Vehicle System Dynamics, Vol. 40, Nos. 1-3, pp. 135-155, 2003

Rill, G., “A Modified Implicit Euler Algorithm for Solving Vehicle Dynamics Equation”, Multibody System Dynamics, Vol. 15, No. 1, pp 1-24, 2006

Rill, G., “First Order Tire Dynamics”, Proceedings of the III European Conference on Computational Mechanics Solids, Structures and Coupled Problems in Engineering, Lisbon, Portugal, 2006

Rill, G., “Vehicle Modeling by Subsystems” To appear in: Journal of Brazilian Society of Mechanical Sciences and Engineering - RBCM, 2006

Rill, G. and Chucholowski, C., “Modeling Concepts for Modern Steering Systems”, ECCOMAS Multibody Dynamics, Madrid, Spain, 2005

Weinfurter, H., Hirschberg, W., Hipp, E., “Entwicklung einer Störgrößenkompensation für Nutzfahrzeuge mittels Steer-by-Wire durch Simulation”, in: Berechnung und Simulation im Fahrzeugbau, VDI-Bericht 1846, S. 923-941, VDI Verlag Düsseldorf, 2004

RESPONSIBILITY NOTICE

The author is the only responsible for the printed material included in this paper.