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REDUCED MODEL IN H_∞ VIBRATION CONTROL USING LINEAR MATRIX INEQUALITIES

Fernando Sarracini Júnior
Department of Computational Mechanics, Faculty of Mechanical Engineering, State University of Campinas - UNICAMP, Campinas, São Paulo, Brazil
e-mail: tato@soon.com

Alberto Luiz Serpa
Department of Computational Mechanics, Faculty of Mechanical Engineering, State University of Campinas - UNICAMP, Campinas, São Paulo, Brazil
e-mail: serpa@fem.unicamp.br

Abstract. *Many practical structural problems are modeled with a high number of degrees of freedom in order to properly describe the structure. A formulation that allows to design robust controllers is the H_∞ technique. But the H_∞ controller has the same order of the mathematical model used and this becomes unpractical and infeasible for most practical problems. One way to overcome this difficulty is to employ a model reduction technique. A reduced controller design can be done based on the reduced model. Although, this reduced controller should ensure a good performance to the non reduced model and to the real structure. This work investigates a reduced order controller obtained by a reduced model through the Guyan reduction. The H_∞ robust control and Linear Matrix Inequalities (LMIs) formulations are employed to the problem of controlling a flexible structure subjected to an external disturbance. Some simulations are done using a cantilever beam modeled through the finite element method. The results show that the Guyan reduced order model can be used to design a controller to the non reduced model with success.*

Keywords: *Vibration Control, Linear Matrix Inequalities, H_∞ Control, Guyan Reduction.*

1 Introduction

Real flexible structures present a great number of significant degrees of freedom and respective resonances. Controlling these structures is a difficult problem because the mathematical models are, in general, reduced. For these reduced models only a limited range of frequency is considered. In these cases, the controller is designed based on a reduced order model that should ensure a suitable performance and stability in frequency ranges related to real modes of vibration that were not considered in the mathematical model.

Dynamic uncertainties are originated due to differences between the real structure and a reduced model in terms of high frequency dynamics. Another kind of uncertainty is called parametric and is related to variations of estimated or identified parameters of the structure. Approximations of parameters such as mass, stiffness and damping determinates the degree of parametric uncertainty of the model. Experimental difficulties can also be responsible for these uncertainties due to measurements limitations. Although the robust controller was designed using the nominal model of the structure, it must be able to control the real structure under parametric and dynamic uncertainties (Maciejowski, 1989).

The mathematical model of a structure can be obtained efficiently through the finite element method (Desai and Abel, 1972).

The H_∞ control consists of a frequency domain method where the peak of singular value of the transfer matrix between the disturbance input and the performance output should be minimized (Geromel, Colaneri and Locatelli, 1997; Maciejowski, 1989). In this work, the H_∞ robust control solved by Linear Matrix Inequalities (LMIs) is used to design a controller that must increase the robust stability margin of the system (Gahinet, 1996; Gahinet and Apkarian, 1994; Iwasaki and Skelton, 1994). In other words, the closed loop system must minimize the external disturbances and the uncertainties effect for a desired performance output. The first author to formulate the H_∞ problem was Zames (Zames, 1981).

LMIs started to be studied in 1890 with Lyapunov, but only in 1940 Lur'e and Postnikov performed applications in control engineering filed. Nowadays, LMIs can be used to represent several kinds of control problems. Some good references about LMIs are (Boyd et al, 1993; Boyd et al, 1994; Nesterov and Nemirovski, 1994).

Designing and implementing a controller in practice with the same order that the original model is in most cases something infeasible. Most interesting and practical problems present a great number of degrees of freedom. In these cases it is desirable to design a reduced order controller in order to control the original model efficiently. One approach to handle with this difficulty is obtaining a reduced order controller based on a reduced model.

The Guyan reduction (Guyan, 1965) is a classical technique that provides a reduced model. This reduced model can be used to design a reduced order H_∞ controller, and the reduced H_∞ controller can be used to control the original non reduced model.

This paper investigates the behavior of a H_∞ reduced controller designed in terms of a reduced model by the Guyan reduction for a structure of beam elements (Guyan, 1965; Inman, 1989). To obtain the controller the problem is formulated using the techniques based on LMIs. The results are critically analyzed and show the potential of used approach.

2 Mathematical Model of the Structure

The Finite Element Method (FEM) is widely used in structural mechanics as a tool of mathematical modeling (Desai and Abel, 1972).

Two main steps of the FEM in structural modeling are: i) The structure is divided in finite elements that are connected by nodes with specific degrees of freedom. Each finite element has interpolation functions to describe the displacements, strain and stresses for example. A proper integration over the element allows to calculate the mass and stiffness matrices of the finite element; ii)The mass and stiffness matrices of each finite element are assembled into the global mass and stiffness matrices of the structure.

The bi-dimensional Hermitian beam element with six degrees of freedom (see Fig. 1) is used in this work to test the control formulation (Desai and Abel, 1972; Kwon and Bang, 1997).

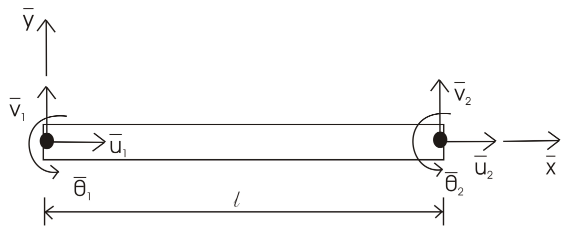

Figure 1: Bi-dimensional finite element.

The mass and stiffness matrices for this finite element are respectively (Desai and Abel, 1972; Kwon and Bang, 1997):

$$\bar{\mathbf{K}} = \frac{E}{l^3}\begin{bmatrix} Al^2 & 0 & 0 & -Al^2 & 0 & 0 \\ 0 & 12I & 6Il & 0 & -12I & 6Il \\ 0 & 6Il & 4Il^2 & 0 & -6Il & 2Il^2 \\ -Al^2 & 0 & 0 & Al^2 & 0 & 0 \\ 0 & -12I & -6Il & 0 & 12I & -6Il \\ 0 & 6Il & 2Il^2 & 0 & -6Il & 4Il^2 \end{bmatrix}, \quad \bar{\mathbf{M}} = \rho Al \begin{bmatrix} a & 0 & 0 & b & 0 & 0 \\ 0 & c & dl & 0 & h & -el \\ 0 & dl & fl^2 & 0 & el & -gl^2 \\ b & 0 & 0 & a & 0 & 0 \\ 0 & h & el & 0 & c & -dl \\ 0 & -el^2 & -gl^2 & 0 & -dl & fl^2 \end{bmatrix}, \quad (1)$$

where:

- $a = 1/3$, $b = 1/6$, $c = 13/35$, $d = 11/210$, $e = 13/420$, $f = 1/105$, $g = 1/140$, $h = 9/70$;
- ρ, A, I, E, l are material density, cross-sectional area, moment of inertia of cross-sectional area, Young's modulus and element length.

The local matrices defined in (1) need to be transformed from the local coordinate system ($\bar{u}_1$, $\bar{v}_1$, $\bar{\theta}_1$, $\bar{u}_2$, $\bar{v}_2$ and $\bar{\theta}_2$) to the global coordinate system (u_1, v_1, θ_1, u_2, v_2 and θ_2) if the element is rotated as in Fig. 2. This rotation is performed according to equation (2) (Kwon and Bang, 1997):

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}, \quad (2)$$

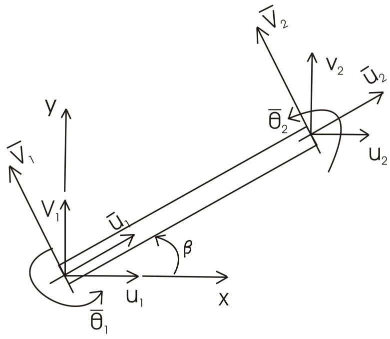

Figure 2: Element in global coordinates system.

where c=cos β and s=sin β.

After obtaining the mass and stiffness matrices for all finite elements of the mesh, these matrices should be assembled to obtain the structure mass and stiffness matrices. The boundary conditions can then be applied.

Structural modeling presents the difficulty in the determination of the damping. When possible, the damping can be determined experimentally through specific measurements in the identification phase. Although, when the structure is not available for identification, the damping should be estimated. In some cases it is usual to consider proportional (to mass and stiffness) damping because the mathematical treatment is simpler (Ewins, 1984).

Linear vibration problems present the dynamic equations as (Inman, 1989):

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}, \tag{3}$$

where $\mathbf{M}$, $\mathbf{C}$, $\mathbf{K}$ e $\mathbf{f}$ are mass matrix, damping matrix, stiffness matrix and external forces vector respectively.

This equation can be written in the state-space model form as (Kwon and Bang, 1997):

$$\dot{\mathbf{x}} \;=\; \mathbf{A} \;\; \mathbf{x} \;+\; \mathbf{B}\;\mathbf{f}; \qquad \mathbf{y} \;=\; \mathbf{C} \;\; \mathbf{x}\;, \tag{4}$$

where:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix}. \tag{5}$$

3 H_∞ Control Using Linear Matrix Inequalities

The H_∞ norm of a stable transfer matrix $\mathbf{G}(jw)$ is defined as the greatest singular value of this matrix regarding all the frequencies w (Maciejowski, 1989; Skogestad and Postlethwaite, 1996):

$$||\mathbf{G}(jw)||_\infty = sup_w \bar{\sigma}[\mathbf{G}(jw)], \tag{6}$$

where $\bar{\sigma}$ represents the greatest singular value of $\mathbf{G}(jw)$.

In optimal H_∞ controller design, the controller is selected to reduce the H_∞ norm of the transfer matrix among the disturbance inputs and the desired outputs. The minimization of the H_∞ norm is desired because this reduces the effect of disturbances in the performance outputs. In other words, the main objective of the H_∞ control is to guarantee the system performance in presence of external disturbances (Doyle et al, 1989; Glover and Doyle, 1988; Shahian, 1993).

The solution for this problem can be found by two main ways: solving the associated Riccati equations, or solving an optimization problem that has Linear Matrix Inequalities (LMIs) as constraints equations (Gahinet et al, 1995; Li and Fu, 1997).

The solution using LMIs is more recent and it can be viewed as more general because many kinds of control problems can be solved using this approach (Boyd et al, 1994; Gahinet and Apkarian, 1994; Gahinet, 1996).

The LMIs solution is usually obtained through mathematical programming methods, mainly interior point methods. This is an interesting feature because if the iterative process finishes before the optimal solution was not found, the last point found can be regarded as a sub-optimal solution since it is a feasible point.

In the last years, LMIs has been widely used in control applications because the formulated problems with this approach can be treated as convex programming problems. Convexity is a property in mathematical programming area that allows a very efficient use of algorithms and computer methods (Boyd et al, 1994; Nesterov and Nemirovski, 1994).

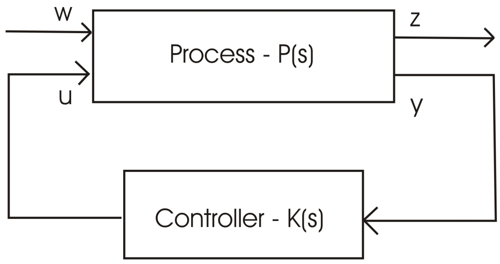

Figure 3: Process and H_∞ controller schemes.

The H_∞ problem can be formulated according to Fig. 3 and related to equations (7), (8) e (9) (Doyle et al, 1989; Geromel, Colaneri and Locatelli, 1997; Glover and Doyle, 1988):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, \tag{7}$$

$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u}, \tag{8}$$

$$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}, \tag{9}$$

where $\mathbf{x}$ is the state vector, $\mathbf{u}$ is the control signal vector, $\mathbf{w}$ is the external disturbance vector, $\mathbf{y}$ is the signal sent to controller and $\mathbf{z}$ is the desired output.

The transfer matrix of Fig. 3 system can be represented as:

$$\begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \tag{10}$$

where $\mathbf{P}_{11}$ is the transfer function between $\mathbf{w}$ input and $\mathbf{z}$ output, $\mathbf{P}_{12}$ is the transfer function between $\mathbf{u}$ input and $\mathbf{z}$ output, $\mathbf{P}_{21}$ is the transfer function between $\mathbf{w}$ input and $\mathbf{y}$ output and $\mathbf{P}_{22}$ is the transfer function between $\mathbf{u}$ input and $\mathbf{y}$ output.

It is also possible to write:

$$\mathbf{z} = \mathbf{P}_{11}\mathbf{w} + \mathbf{P}_{12}\mathbf{u}; \tag{11}$$

$$\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}; \tag{12}$$

$$\mathbf{u} = \mathbf{K}\mathbf{y}, \tag{13}$$

where equation (13) is the control law.

Relation between $\mathbf{w}$ and $\mathbf{z}$ in closed loop system can be obtained replacing (12) into (13) and (13) into (11):

$$\mathbf{z} = [\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}]\mathbf{w}. \tag{14}$$

The expression (14) is called lower linear fractional transformation and it is represented as (Skogestad and Postlethwaite, 1996):

$$\mathbf{F}_l(\mathbf{P},\mathbf{K}) = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}, \tag{15}$$

where $\mathbf{F}_l(\mathbf{P},\mathbf{K})$ is the transfer matrix between the disturbance input ($\mathbf{w}$) and the performance output ($\mathbf{z}$).

In the H_∞ design it is desired to find the $\mathbf{K}(s)$ controller to minimize the H_∞ norm of $\mathbf{F}_l(\mathbf{P},\mathbf{K})$ (15). Thus, the optimal H_∞ controller requires the minimization of the peak value in singular diagram of transfer matrix between the disturbance input and the performance output, considering the frequency response (Shahian, 1993), i.e.:

$$min_w \ \|\mathbf{F}_l(\mathbf{P},\mathbf{K})\|_\infty. \tag{16}$$

A sub-optimal controller $\mathbf{K}(s)$ can be found when this norm is lower than a real value γ, i.e.:

$$\|\mathbf{F}_l(\mathbf{P},\mathbf{K})\|_\infty < \gamma. \tag{17}$$

An usual form to solve the H_∞ problem consists of defining a value to γ and checking the condition (17). If this condition is satisfied, it is possible to reduce γ and solve again the H_∞ problem. This process can be repeated until the restriction (17) is not satisfied.

In this work, it is designed an output feedback controller based on LMIs. The dynamic controller state-space model can be written as (Geromel, 2002):

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{y}, \tag{18}$$

$$\mathbf{u} = \mathbf{C}_c\mathbf{x}_c, \tag{19}$$

where $\mathbf{y}$ is the controller's input vector, $\mathbf{u}$ is a controller's output vector, $\mathbf{x}_c$ is the controller's state vector and $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{C}_c$ are the controller's matrices.

For the closed loop analysis it is necessary to obtain the extended state space model ((20) and (21)) from (7), (8), (9), (18) and (19):

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{w}, \tag{20}$$

$$\mathbf{z} = \tilde{\mathbf{C}}\tilde{\mathbf{x}}, \tag{21}$$

where:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix}; \quad \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix}; \quad \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_c\mathbf{D}_{21} \end{bmatrix}; \quad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{D}_{12}\mathbf{C}_c \end{bmatrix}.$$

It is possible to demonstrate that H_∞ control problem can be written in terms of a minimization problem (Boyd et al, 1993; Boyd et al, 1994; Li and Fu, 1997) as:

minimize γ
subjected to :

$$\tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \gamma^{-2}\tilde{\mathbf{P}}\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{C}}^T\tilde{\mathbf{C}} \prec 0, \tag{22}$$

$$\tilde{\mathbf{P}} \succ 0, \tag{23}$$

where the symbol $\prec$ means that the expression (22) is negative-definite and the symbol $\succ$ in (23) means that the matrix $\tilde{\mathbf{P}}$ is positive-definite. The expression (22) replaces the Riccati equation when it is desirable to solve the H_∞ problem using LMIs (Gahinet, 1996; Gahinet and Apkarian, 1994).

The matrix inequality (22) is nonlinear in $\tilde{\mathbf{P}}$. Applying Schur Complement (Boyd et al, 1994), it is possible to obtain:

$$\left[\begin{array}{cc|c} \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} & \tilde{\mathbf{C}}^T & \tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & -\mathbf{I} & \mathbf{0} \\ \hline \tilde{\mathbf{B}}^T\tilde{\mathbf{P}} & \mathbf{0} & -\gamma^2\mathbf{I} \end{array}\right] \prec 0. \tag{24}$$

Using a congruence transformation (Boyd et al, 1994), the linear matrix inequality (24) can be left-multiplied by $\begin{bmatrix} \tilde{\mathbf{T}}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$ and right-multiplied by $\begin{bmatrix} \tilde{\mathbf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$. This results in the equation:

$$\begin{bmatrix} \tilde{\mathbf{T}}^T(\tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}})\tilde{\mathbf{T}} & \tilde{\mathbf{T}}^T\tilde{\mathbf{C}}^T & \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{T}} & -\mathbf{I} & \mathbf{0} \\ \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} \prec 0 \Rightarrow \begin{bmatrix} \tilde{\mathbf{T}}^T(\tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}})\tilde{\mathbf{T}} & \tilde{\mathbf{T}}^T\tilde{\mathbf{C}}^T & \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{T}} & -\mathbf{I} & \mathbf{0} \\ \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} & \mathbf{0} & -\mu\mathbf{I} \end{bmatrix} \prec 0, \tag{25}$$

where:

$$\mu = \gamma^2; \quad \tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{X} & \mathbf{U}^T \\ \mathbf{U} & \hat{\mathbf{X}} \end{bmatrix}; \quad \tilde{\mathbf{P}}^{-1} = \begin{bmatrix} \mathbf{Y} & \mathbf{V}^T \\ \mathbf{V} & \hat{\mathbf{Y}} \end{bmatrix}; \quad \tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{V} & \mathbf{0} \end{bmatrix}.$$

It is possible to simplify (25) defining the following terms:

$$\tilde{\mathbf{P}}\tilde{\mathbf{P}}^{-1} = \mathbf{I} = \begin{bmatrix} \mathbf{XY} + \mathbf{U}^T\mathbf{V} & \mathbf{XV}^T + \mathbf{U}^T\hat{\mathbf{Y}} \\ \mathbf{UY} + \hat{\mathbf{X}}\mathbf{V} & \mathbf{UV}^T + \hat{\mathbf{X}}\hat{\mathbf{Y}} \end{bmatrix}; \quad \tilde{\mathbf{P}}^{-1}\tilde{\mathbf{P}} = \mathbf{I} = \begin{bmatrix} \mathbf{YX} + \mathbf{V}^T\mathbf{U} & \mathbf{YU}^T + \mathbf{V}^T\hat{\mathbf{X}} \\ \mathbf{VX} + \hat{\mathbf{Y}}\mathbf{U} & \mathbf{VU}^T + \hat{\mathbf{Y}}\hat{\mathbf{X}} \end{bmatrix}. \tag{26}$$

It is also possible to verify that $\mathbf{XY} + \mathbf{U}^T\mathbf{V} = \mathbf{YX} + \mathbf{V}^T\mathbf{U} = \mathbf{I}$ and $\mathbf{UY} + \hat{\mathbf{X}}\mathbf{V} = \mathbf{VX} + \hat{\mathbf{Y}}\mathbf{U} = \mathbf{0}$.
Besides, it is also possible to simplify (25) using $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{T}}$:

$$\tilde{\mathbf{C}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{C}_1\mathbf{Y} + \mathbf{D}_{12}\mathbf{F} & \mathbf{C}_1 \end{bmatrix}; \quad \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{B}_1^T & \mathbf{B}_1^T\mathbf{X} + \mathbf{D}_{21}^T\mathbf{L}^T \end{bmatrix}; \tag{27}$$

$$\tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{AY} + \mathbf{B}_2\mathbf{F} & \mathbf{A} \\ \mathbf{M} & \mathbf{XA} + \mathbf{LC}_2 \end{bmatrix}, \tag{28}$$

where:

$$\mathbf{F} = \mathbf{C}_c\mathbf{V}; \quad \mathbf{L} = \mathbf{U}^T\mathbf{B}_c; \quad \mathbf{M} = \mathbf{XAY} + \mathbf{XB}_2\mathbf{F} + \mathbf{LC}_2\mathbf{Y} + \mathbf{U}^T\mathbf{A}_c\mathbf{V}.$$

Considering that $\tilde{\mathbf{T}} \succ 0$ and remembering that $\tilde{\mathbf{P}}$ is positive-definite (as defined in (23)), the equations in (26) can be used to replace some terms in (29):

$$\tilde{\mathbf{P}} \succ 0 \Longleftrightarrow \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} \succ 0$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{V}^T \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{X} & \mathbf{U}^T \\ \mathbf{U} & \hat{\mathbf{X}} \end{bmatrix}\begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{bmatrix} \succ 0. \tag{29}$$

Therefore, the expression (29) should be satisfied to allow the solution of problem (22).

Substituting the terms of (27) and (28) in the expression (25), it is possible to formulate the the H_∞ problem with output feedback as a minimization problem subjected to LMIs constraints (Geromel, 2002; Li and Fu, 1997):

minimize μ
subjected to :

$$\begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \mathbf{N}_{13} & \mathbf{N}_{14} \\ \mathbf{N}_{12}^T & \mathbf{N}_{22} & \mathbf{N}_{23} & \mathbf{N}_{24} \\ \mathbf{N}_{13}^T & \mathbf{N}_{23}^T & \mathbf{N}_{33} & \mathbf{N}_{34} \\ \mathbf{N}_{14}^T & \mathbf{N}_{24}^T & \mathbf{N}_{34}^T & \mathbf{N}_{44} \end{bmatrix} \prec 0; \quad \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{bmatrix} \succ 0, \tag{30}$$

where: $\mathbf{N}_{11} = \mathbf{AY} + \mathbf{B}_2\mathbf{F} + \mathbf{YA}^T + \mathbf{F}^T\mathbf{B}_2^T$; $\mathbf{N}_{12} = \mathbf{A} + \mathbf{M}^T$; $\mathbf{N}_{13} = \mathbf{YC}_1^T + \mathbf{F}^T\mathbf{D}_{12}$; $\mathbf{N}_{14} = \mathbf{B}_1$; $\mathbf{N}_{22} = \mathbf{XA} + \mathbf{LC}_2 + \mathbf{A}^T\mathbf{X} + \mathbf{C}_2^T\mathbf{L}^T$; $\mathbf{N}_{23} = \mathbf{C}_1^T$; $\mathbf{N}_{24} = \mathbf{XB}_1 + \mathbf{LD}_{21}$; $\mathbf{N}_{33} = -\mathbf{I}$; $\mathbf{N}_{34} = \mathbf{0}$; $\mathbf{N}_{44} = -\mu\mathbf{I}$; $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{L}$, $\mathbf{F}$, $\mathbf{M}$, *and* μ are the unknowns of the problem.

Since $\mathbf{XY} + \mathbf{U}^T\mathbf{V} = \mathbf{I}$, it is possible to consider $\mathbf{U}^T$ arbitrary and calculate $\mathbf{V}$. Thus $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C}_c$ (the controller matrices) can be obtained from:

$$\mathbf{F} = \mathbf{C}_c\mathbf{V}; \tag{31}$$

$$\mathbf{M} = \mathbf{XAY} + \mathbf{XB}_2\mathbf{F} + \mathbf{LC}_2\mathbf{Y} + \mathbf{U}^T\mathbf{A}_c\mathbf{V}; \tag{32}$$

$$\mathbf{L} = \mathbf{U}^T\mathbf{B}_c. \tag{33}$$

It is possible to note that the controller order is the same of the system model (matrices $\mathbf{A}$ and $\mathbf{A}_c$ have the same dimensions). This represents the main practical difficulty to apply the H_∞ in the control of structures. To overcome this, some alternatives can be employed: the model reduction or the controller reduction. In this work the model reduction by the well known Guyan reduction is investigated.

4 Guyan Model Reduction

Designing and implementing a controller with the same number of degrees of freedom that the model is sometimes unpractical. Most interesting problems are modeled with a great number of degrees of freedom at least when considering the respective control problem. An approach to work with this problem is the reduction of the dimension of the original model. This methodology is called model reduction.

A very well known technique in structural mechanics is the Guyan reduction. Through Guyan method, the degrees of freedom can be separated into master and slave degrees of freedom. In this way, the mass and stiffness matrices can be put in the form (Guyan, 1965):

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}}_m \\ \ddot{\mathbf{q}}_s \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix}\begin{bmatrix} \mathbf{q}_m \\ \mathbf{q}_s \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{34}$$

where $\mathbf{q}_m$ denotes the master degrees of freedom and $\mathbf{q}_s$ denotes the slave degrees.

It is possible to express the slave degrees of freedom as a function of the master degrees (considering no applied forces in slaves degrees):

$$\mathbf{q}_s = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\mathbf{q}_m, \tag{35}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_m \\ \mathbf{q}_s \end{bmatrix} = \mathbf{W}\mathbf{q}_m = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} \end{bmatrix}\mathbf{q}_m. \tag{36}$$

Therefore, it is possible to obtain the reduced mass and the reduced stiffness matrices (Inman, 1989):

$$\begin{aligned} \mathbf{W}^T\mathbf{M}\mathbf{W} &= \mathbf{M}_{mm} - \mathbf{K}_{sm}^T\mathbf{K}_{ss}^{-1}\mathbf{M}_{sm} - \mathbf{M}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} + \mathbf{K}_{sm}^T\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}; \\ \mathbf{W}^T\mathbf{K}\mathbf{W} &= \mathbf{K}_{mm} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}. \end{aligned}$$

In the case of proportional damping, a reduced damping matrix can also be obtained using the reduction transformation:

$$\mathbf{W}^T\mathbf{D}\mathbf{W} = \mathbf{W}^T(\eta\mathbf{K} + \alpha\mathbf{M})\mathbf{W} = \eta\mathbf{W}^T\mathbf{K}\mathbf{W} + \alpha\mathbf{W}^T\mathbf{M}\mathbf{W},$$

where η and α are the proportionality coefficients related to stiffness and mass respectively.

5 Numerical Results

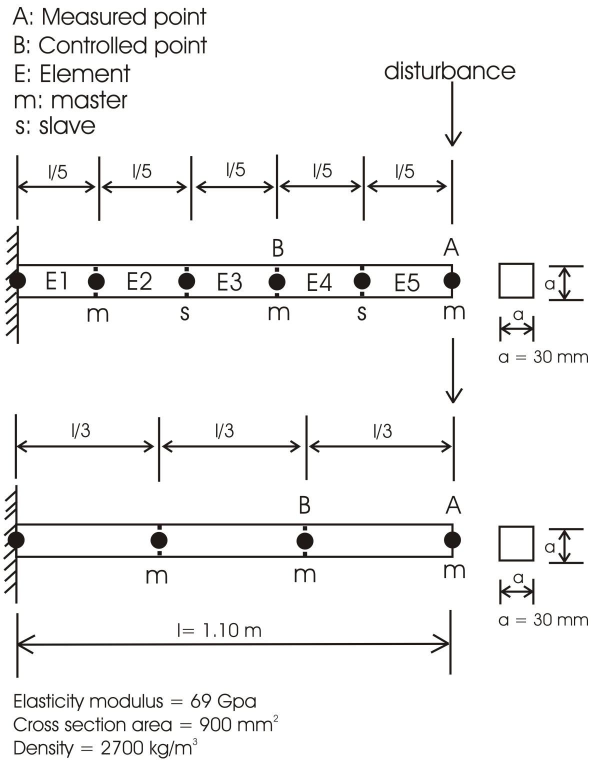

Figure 4: Original structure with 15 degrees of freedom that was reduced to 9 master degrees of freedom.

To test the performance of H_∞ controller designed using the LMI point of view a simple beam is employed.

The mass, stiffness and proportional damping matrices for a cantilever beam (Fig. 4) with proportional damping (0.001% proportional to mass matrix and 0.001% proportional to stiffness matrix) was obtained by finite element calculations.

The original beam model has 15 degrees of freedom and was reduced to a 9 degrees of freedom model (Fig. 4) using the Guyan reduction.

The state-space models for the original and the reduced models were obtained. In order to compare the dynamic response of the original and reduced models, the singular value diagrams were determined (see Fig. 5). It is possible to verify that the reduced model can be considered a good approximation to the original model in this case.

The flexible structure presents an external disturbance applied on point A (see Fig. 4) and the control acts at point B. The desired performance output is the vertical displacement of point A. This is a non collocated control problem, since the control force acts in a different point from that one to be controlled.

To analyze the model reduction effects, two H_∞ controllers were obtained by the solution of the respective minimization problems: one based on the original model and another based on the reduced model.

The controller based on the reduced model (9 degrees of freedom) was employed to control the original model (15 degrees of freedom). It is possible to verify from Fig. 6 that the peak of singular value of the original model was reduced by the reduced order controller. The original model without control presented -22.80 dB as peak singular value and the original model controlled by the reduced controller presented -34.60 dB as peak value.

The controller with the same order of the original model was also used to control the original model. It can be verified in Fig. 7 that the full order controller is more efficient than the reduced controller. This result was expected since the controller of full order does not have any information truncation. The peak of singular value obtained with the full order controller was -35.60 dB.

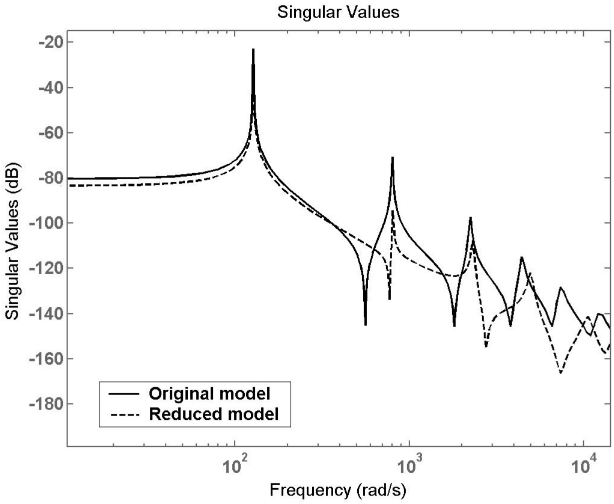

Figure 5: Comparison of singular value diagrams: original model and reduced model.

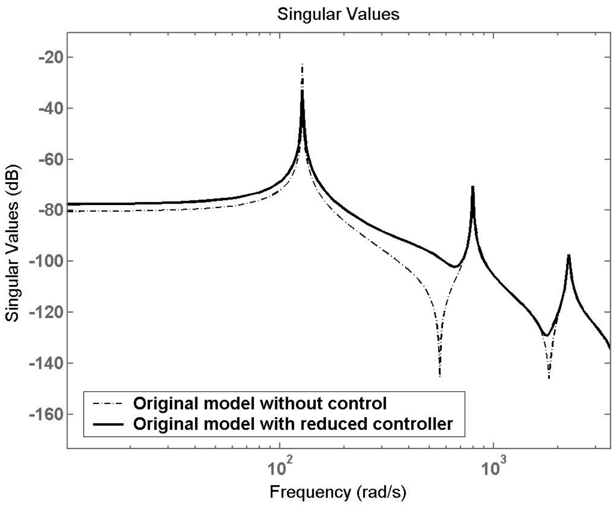

Figure 6: Comparison of singular value diagrams: original model without control and controlled by the reduced controller.

Table 1 presents a comparison among the H_∞ norm (peak value in singular value diagram) obtained for each case.

Table 1: Comparison among H_∞ norm obtained for studied each case.

Case	$\|H_\infty\|$
Simulation without control	-22.80 dB
Simulation with the reduced controller	-34.60 dB
Simulation with full order controller	-35.60 dB

To solve the optimization problem (30) related to the H_∞ control problem the function *mincx* of Matlab (LMI toolbox) was employed. The objective is to minimize a linear objective function (μ) under LMI constraint equations. The criterion of convergence used was based on the relative variation of μ, i.e.:

$$\frac{\mu(k+1) - \mu(k)}{\mu(k)} < 10^{-1}, \tag{37}$$

where k is the number of iterations.

Figure 8 shows the vertical displacement at point A of the beam without control and with the reduced controller when a random disturbance amplified by the factor 100 was applied. The random disturbance consists

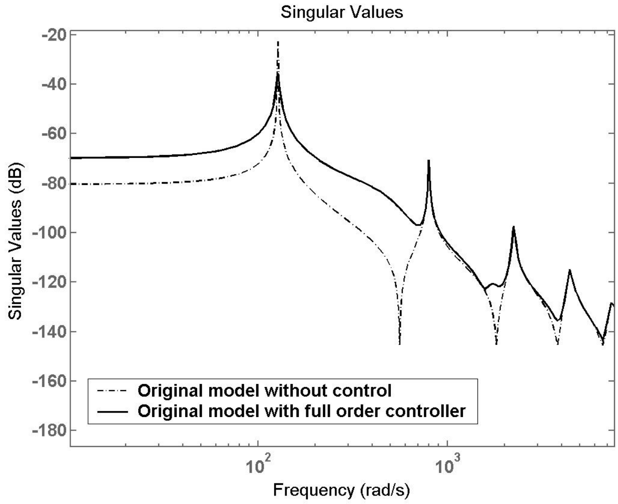

Figure 7: Comparison of singular value diagrams: original model without control and controlled by a full order controller.

of a normal distribution with mean zero, variance and standard deviation one. It is possible to verify that the disturbance is minimized when the system is controlled.

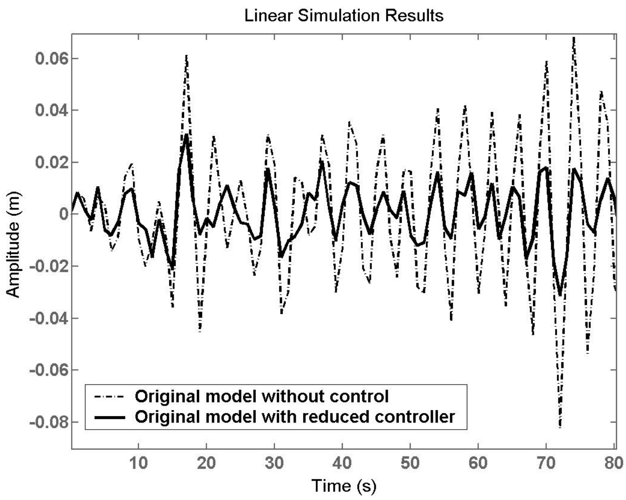

Figure 8: Comparison of time responses for the displacement at non fixed end: original model without control and controlled by the reduced order controller.

The computational elapsed time to obtain the full order controller was about 8 hours, and the elapsed time to get the reduced order controller was about 5 minutes. It was used a computer *Pentium* 4 of 1.8 GHz processor.

6 Conclusion

It was verified in this work some aspects of the flexible structure control using a reduced order controller that was designed based on a reduced model obtained by the Guyan reduction. The formulation used is the H_∞ control under the point of view of Linear Matrix Inequalities.

The reduced order controller (reduction from 15 degrees of freedom to 9 masters degrees of freedom for the beam example) was efficient in the H_∞ norm reduction (a reduction of 11.80 dB from disturbance input to performance output transfer function was found). Obviously, the full order controller presented a more efficient performance since it works with no information reduction.

One important aspect to be verified is the stability of the closed loop system. Since the model reduction truncates some information of the model, a risk of instability always exists. Other constraint equations can be imposed in the minimization problem in order to ensure pole placements in specific regions. This can also be put in terms of LMIs (Chilali and Gahinet, 1996).

In terms of computational processing effort, the full order controller has demanded a great computational processing time (8 hours) and for the reduced order controller a significantly time reduction was achieved (5 minutes). It is important to mention that these computational times can be considered too high and more efficient computational procedures should be investigated, since the structure considered as the example is very simple. For real structures, this processing time could be unpractical.

7 Acknowledgments

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