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EXTENSION OF THE STEGER AND WARMING AND RADESPIEL AND KROLL ALGORITHMS TO SECOND ORDER ACCURACY AND IMPLICIT FORMULATION APPLIED TO THE EULER EQUATIONS IN TWO-DIMENSIONS - THEORY

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Abstract. *In this work, the first part of this study, the Steger and Warming and the Radespiel and Kroll schemes are implemented following a MUSCL approach, aiming to guarantee second order accuracy and to achieve TVD properties, and employing an implicit formulation to solve the Euler equations in the two-dimensional space. These schemes are implemented according to a finite volume formulation and using a structured spatial discretization. Both schemes are flux vector splitting ones. The MUSCL approach employs five different types of nonlinear limiters, which assure TVD properties, namely: Van Leer limiter, Van Albada limiter, minmod limiter, Super Bee limiter and* β*-limiter. All variants of the MUSCL schemes are second order accurate in space. The implicit schemes employ an ADI approximate factorization to solve implicitly the Euler equations. Explicit and implicit results are compared, as also the computational costs, trying to emphasize the advantages and disadvantages of each formulation. The schemes are accelerated to the steady state solution using a spatially variable time step, which has demonstrated effective gains in terms of convergence rate according to Maciel. The algorithms are applied to the solution of the physical problem of the moderate supersonic flow along a compression corner. The results have demonstrated that the most accurate solutions are obtained with the Steger and Warming TVD scheme using Van Leer and minmod nonlinear limiters, when implemented in its explicit version.*

Keywords: *Steger and Warming algorithm, Radespiel and Kroll algorithm, MUSCL procedure, Implicit formulation, Euler equations.*

1. INTRODUCTION

Conventional non-upwind algorithms have been used extensively to solve a wide variety of problems (Kutler, 1975). Conventional algorithms are somewhat unreliable in the sense that for every different problem (and sometimes, every different case in the same class of problems) artificial dissipation terms must be specially tuned and judicially chosen for convergence. Also, complex problems with shocks and steep compression and expansion gradients may defy solution altogether.

Upwind schemes are in general more robust but are also more involved in their derivation and application. Some first order upwind schemes that have been applied to the Euler equations are:

Steger and Warming (1981) method, whose authors used the remarkable property that the nonlinear flux vectors of the inviscid gasdynamic equations in conservation law form are homogeneous functions of degree one of the vector of conserved variables to develop their algorithm. This property readily permitted the splitting of the flux vectors into subvectors by similarity transformations so that each subvector had associated with it a specified eigenvalue spectrum. As a consequence of flux vector splitting, new explicit and implicit dissipative finite-difference schemes were developed for first-order hyperbolic systems of equations.

Radespiel and Kroll (1995) emphasized that the Liou and Steffen Jr. (1993) scheme had its merits of low computational complexity and low numerical diffusion as compared to other methods. They also mentioned that the original method had several deficiencies. The method yielded local pressure oscillations in the shock wave proximities, adverse mesh and flow alignment problems. In the Radespiel and Kroll (1995) work, a hybrid flux vector splitting scheme, which alternated between the Liou and Steffen Jr. (1993) scheme and the Van Leer (1982) scheme, in the shock wave regions, is proposed, assuring that the resolution of strength shocks was clear and sharp.

Second order spatial accuracy can be achieved by introducing more upwind points or cells in the schemes. It has been noted that the projection stage, whereby the solution is projected in each cell face (i-1/2,j; i+1/2,j) on piecewise constant states, is the cause of the first order space accuracy of the Godunov schemes (Hirsch, 1990). Hence, it is sufficient to modify the first projection stage without modifying the Riemann solver, in order to generate higher spatial approximations. The state variables at the interfaces are thereby obtained from an extrapolation between neighboring cell averages. This method for the generation of second order upwind schemes based on variable extrapolation is often referred to in the literature as the MUSCL (“Monotone Upstream-centered Schemes for Conservation Laws”) approach. The use of nonlinear limiters in such procedure, with the intention of restricting the amplitude of the gradients appearing in the solution, avoiding thus the formation of new extrema, allows that first order upwind schemes be transformed in TVD high resolution schemes with the appropriate definition of such nonlinear limiters, assuring monotone preserving and total variation diminishing methods.

Traditionally, implicit numerical methods have been praised for their improved stability and condemned for their large arithmetic operation counts (Beam and Warming, 1978). On the one hand, the slow convergence rate of explicit methods become they so unattractive to the solution of steady state problems due to the large number of iterations required to convergence, in spite of the reduced number of operation counts per time step in comparison with their implicit counterparts. Such problem is resulting from the limited stability region which such methods are subjected (the Courant condition). On the other hand, implicit schemes guarantee a larger stability region, which allows the use of CFL numbers above 1.0, and fast convergence to steady state conditions. Undoubtedly, the most significant efficiency achievement for multidimensional implicit methods was the introduction of the Alternating Direction Implicit (ADI) algorithms by Douglas (1955), Peaceman and Rachford (1955), and Douglas and Gunn (1964), and fractional step algorithms by Yanenko (1971). ADI approximate factorization methods consist in approximating the Left Hand Side (LHS) of the numerical scheme by the product of one-dimensional parcels, each one associated with a different spatial coordinate direction, which retract nearly the original implicit operator. These methods have been largely applied in the CFD community and, despite the fact of the error of the approximate factorization, it allows the use of large time steps, which results in significant gains in terms of convergence rate in relation to explicit methods.

In this work, the first part of this study, the Steger and Warming (1981) and the Radespiel and Kroll (1995) schemes are implemented following a MUSCL approach, aiming to guarantee second order accuracy and to achieve TVD properties, and employing an implicit formulation to solve the Euler equations in the two-dimensional space. These schemes are implemented according to a finite volume formulation and using a structured spatial discretization. Both schemes are flux vector splitting ones. The MUSCL approach employs five different types of nonlinear limiters, which assure TVD properties, namely: Van Leer limiter, Van Albada limiter, minmod limiter, Super Bee limiter and β-limiter. All variants of the MUSCL schemes are second order accurate in space. The implicit schemes employ an ADI approximate factorization to solve implicitly the Euler equations. Explicit and implicit results are compared, as also the computational costs, trying to emphasize the advantages and disadvantages of each formulation. The schemes are accelerated to the steady state solution using a spatially variable time step, which has demonstrated effective gains in terms of convergence rate according to Maciel (2005, 2008a). The algorithms are applied to the solution of the physical problem of the moderate supersonic flow along a compression corner. The results have demonstrated that the most accurate solutions are obtained with the Steger and Warming (1981) TVD scheme using Van Leer and minmod nonlinear limiters, when implemented in its explicit version.

2. EULER EQUATIONS

The fluid movement is described by the Euler equations, which express the conservation of mass, of the linear momentum and of the energy to an inviscid mean, heat non-conductor and compressible, in the absence of external forces. These equations can be represented, in the integral and conservative forms, to a finite volume formulation, by:

$$\partial/\partial t \int_V Q dV + \int_S \left[E n_x + F n_y\right] dS = 0\,, \qquad (1)$$

where Q is the vector of conserved variables written to a Cartesian system, V is the cell volume, n_x and n_y are components of the normal unit vector to the flux face, S is the flux area, and E and F are components of the convective flux vector. The Q, E and F vectors are represented by:

$$Q = \begin{Bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{Bmatrix}, \quad E = \begin{Bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (e+p)u \end{Bmatrix} \quad \text{and} \quad F = \begin{Bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (e+p)v \end{Bmatrix}, \qquad (2)$$

with ρ being the fluid density; u and v are Cartesian components of the velocity vector in the x and y directions, respectively; e is the total energy per unit volume of the fluid mean; and p is the static pressure of the fluid mean.

The Euler equations were nondimensionalized in relation to the freestream density, ρ_∞, and in relation to the freestream speed of sound, a_∞, to the studied problem in this work. The matrix system of the Euler equations is closed with the state equation of a perfect gas

$$p = (\gamma - 1)\left[e - 0.5\rho(u^2 + v^2)\right], \qquad (3)$$

considering the ideal gas hypothesis. γ is the ratio of specific heats. The total enthalpy is determined by $H = (e+p)/\rho$.

3. NUMERICAL ALGORITHMS

The Steger and Warming (1981) and the Radespiel and Kroll (1995) first order schemes are described in details in Maciel (2008b,c). In the present work, only the numerical flux vector of these schemes is presented:

3.1 Steger and Warming (1981) Algorithm

The Steger and Warming (1981) numerical flux vector is constructed by the following normal flux projection:

$$\tilde{F}_{\pm}^{(n)} = \frac{\rho}{2\gamma}\left\{\begin{matrix} \alpha \\ \alpha u + a\left(\lambda_2^{\pm} - \lambda_3^{\pm}\right)n_x \\ \alpha v + a\left(\lambda_2^{\pm} - \lambda_3^{\pm}\right)n_y \\ \alpha\frac{u^2+v^2}{2} + a v_n\left(\lambda_2^{\pm} - \lambda_3^{\pm}\right) + a^2\frac{\lambda_2^{\pm} + \lambda_3^{\pm}}{\gamma - 1} \end{matrix}\right\}, \tag{4}$$

where the eigenvalues of the normal Jacobian matrix are defined as

$$\lambda_1 = \vec{v} \bullet \vec{n} \equiv v_n,\ \lambda_2 = \vec{v} \bullet \vec{n} + a \quad \text{and} \quad \lambda_3 = \vec{v} \bullet \vec{n} - a, \tag{5}$$

with $\vec{v}$ being the flow velocity vector, a is the speed of sound ($a = \sqrt{\gamma p/\rho}$), $\pm$ sign indicates the positive or negative parts of the projection flux vector and of the eigenvalues. The eigenvalue separation is defined to the ξ direction, for example, as:

$$\left(\lambda_\xi^l\right)^{\pm} = 0.5\left(\lambda_\xi^l \pm \left|\lambda_\xi^l\right|\right), \tag{6}$$

with "l" varying from 1 to 4 (two-dimensional space). The parameter α is defined as

$$\alpha = 2(\gamma - 1)\lambda_1^{\pm} + \lambda_2^{\pm} + \lambda_3^{\pm}. \tag{7}$$

The numerical flux vectors of the Steger and Warming (1981) scheme, based on a finite volume formulation, is calculated as:

$$\tilde{F}_{i,j-1/2} = \left(\tilde{F}_{i,j-1}^{-} + \tilde{F}_{i,j}^{+}\right)S_{i,j-1/2},\ \tilde{F}_{i+1/2,j} = \left(\tilde{F}_{i+1,j}^{-} + \tilde{F}_{i,j}^{+}\right)S_{i+1/2,j},\ \tilde{F}_{i,j+1/2} = \left(\tilde{F}_{i,j+1}^{-} + \tilde{F}_{i,j}^{+}\right)S_{i,j+1/2}; \tag{8}$$

$$\tilde{F}_{i-1/2,j} = \left(\tilde{F}_{i-1,j}^{-} + \tilde{F}_{i,j}^{+}\right)S_{i-1/2,j}, \tag{9}$$

where S is the flux area described in Maciel (2008b,c). More details of the theory in the implementation of the Steger and Warming (1981) algorithm, the reader is encouraged to read Maciel (2008b).

3.2 Radespiel and Kroll (1995) Algorithm

The definition of the residual or the numerical flux vector at the (i+1/2,j) interface of the Radespiel and Kroll (1995) scheme proceeds as follows:

$$R_{i+1/2,j}^{e} = |S|_{i+1/2,j}\left\{\frac{1}{2}M_{i+1/2,j}\left(\begin{bmatrix}\rho a\\ \rho au\\ \rho av\\ \rho aH\end{bmatrix}_L + \begin{bmatrix}\rho a\\ \rho au\\ \rho av\\ \rho aH\end{bmatrix}_R\right) - \frac{1}{2}\phi_{i+1/2,j}\left(\begin{bmatrix}\rho a\\ \rho au\\ \rho av\\ \rho aH\end{bmatrix}_R - \begin{bmatrix}\rho a\\ \rho au\\ \rho av\\ \rho aH\end{bmatrix}_L\right)\right\} + \left\{\begin{matrix}0\\ S_x p\\ S_y p\\ 0\end{matrix}\right\}_{i+1/2,j}, \tag{10}$$

with R [cell (i+1,j)] and L [cell (i,j)] related to right and left states; S, S_x and S_y defining the flux area and its x and y components; these parameters are defined in Maciel (2008c); the superscript "e" defines Euler equations; ϕ is the dissipation function which defines the particular numerical scheme. $M_{i+1/2,j}$ defines the advection Mach number at the (i+1/2,j) face of the (i,j) cell, which is calculated according to Liou and Steffen Jr. (1993) as:

$$M_{i+1/2,j} = M_L^+ + M_R^-, \tag{11}$$

where the separated Mach numbers $M^{+/-}$ are defined by the Van Leer (1982) formulas:

$$M^+ = \begin{bmatrix} M, & if\ M \geq 1; \\ 0.25(M+1)^2, & if\ |M| < 1; \\ 0, & if\ M \leq -1; \end{bmatrix} \quad \text{and} \quad M^- = \begin{bmatrix} 0, & if\ M \geq 1; \\ -0.25(M-1)^2, & if\ |M| < 1; \\ M, & if\ M \leq -1. \end{bmatrix} \tag{12}$$

M_L and M_R represent the Mach number associated with the left and right states, respectively. The Mach number at the cell interface to the calculation of the separated Mach numbers is defined by:

$$M = (S_x u + S_y v)/(|S|a); \tag{13}$$

The pressure at the (i+1/2,j) face of the (i,j) cell is calculated by a similar way:

$$p_{i+1/2,j} = p_L^+ + p_R^-, \tag{14}$$

with $p^{+/-}$ denoting the pressure separation defined according to the Van Leer (1982) formulas:

$$p^+ = \begin{bmatrix} p, & if\ M \geq 1; \\ 0.25p(M+1)^2(2-M), & if\ |M| < 1 \\ 0, & if\ M \leq -1; \end{bmatrix} \quad \text{and} \quad p^- = \begin{bmatrix} 0, & if\ M \geq 1; \\ 0.25p(M-1)^2(2+M), & if\ |M| < 1; \\ p, & if\ M \leq -1. \end{bmatrix} \tag{15}$$

The definition of the ϕ dissipation term which determines the Radespiel and Kroll (1995) scheme combines the Van Leer (1982) and the Liou and Steffen Jr. (1993) (AUSM) schemes. Hence,

$$\phi_{i+1/2,j} = (1-\omega)\phi_{i+1/2,j}^{VL} + \omega\phi_{i+1/2,j}^{LS}, \tag{16}$$

with:

$$\phi_{i+1/2,j}^{VL} = \begin{cases} |M_{i+1/2,j}|, & if\ |M_{i+1/2,j}| \geq 1; \\ |M_{i+1/2,j}| + \frac{1}{2}(M_R - 1)^2, & if\ 0 \leq M_{i+1/2,j} < 1; \\ |M_{i+1/2,j}| + \frac{1}{2}(M_L + 1)^2, & if\ -1 < M_{i+1/2,j} \leq 0; \end{cases} \quad \text{and} \quad \phi_{i+1/2,j}^{LS} = \begin{cases} |M_{i+1/2,j}|, & if\ |M_{i+1/2,j}| > \tilde{\delta} \\ \frac{(M_{i+1/2,j})^2 + \tilde{\delta}^2}{2\tilde{\delta}}, & if\ |M_{i+1/2,j}| \leq \tilde{\delta} \end{cases}, \tag{17}$$

where $\tilde{\delta}$ is a small parameter, $0 < \tilde{\delta} \leq 0.5$, and ω is a constant, $0 \leq \omega \leq 1$. In this work, the values used to $\tilde{\delta}$ and ω were 0.2 and 0.5, respectively. More details of the theory in the implementation of the Radespiel and Kroll (1995) algorithm, the reader is encouraged to read Maciel (2008c).

4. MUSCL PROCEDURE

Second order spatial accuracy can be achieved by introducing more upwind points or cells in the schemes. It has been noted that the projection stage, whereby the solution is projected in each cell face (i-1/2,j; i+1/2,j) on piecewise constant states, is the cause of the first order space accuracy of the Godunov schemes (Hirsch, 1990). Hence, it is sufficient to modify the first projection stage without modifying the Riemann solver, in order to generate higher spatial approximations. The state variables at the interfaces are thereby obtained from an extrapolation between neighboring cell averages. This method for the generation of second order upwind schemes based on variable extrapolation is often referred to in the literature as the MUSCL approach. The use of nonlinear limiters in such procedure, with the intention of restricting the amplitude of the gradients appearing in the solution, avoiding thus the formation of new extrema, allows that first order upwind schemes be transformed in TVD high resolution schemes with the appropriate definition of such nonlinear limiters, assuring monotone preserving and total variation diminishing methods. Details of the present implementation of the MUSCL procedure, as well the incorporation of TVD properties to the schemes, are found in Hirsch (1990). The expressions to calculate the fluxes following a MUSCL procedure and the nonlinear flux limiter definitions employed in this work, which incorporates TVD properties, are defined as follows.

The conserved variables at the interface (i+1/2,j) can be considered as resulting from a combination of backward and forward extrapolations. To a linear one-sided extrapolation at the interface between the averaged values at the two upstream cells (i,j) and (i-1,j), one has:

$$Q_{i+1/2,j}^{L} = Q_{i,j} + \frac{\varepsilon}{2}\left(Q_{i,j} - Q_{i-1,j}\right), \text{ cell (i,j);} \quad (18)$$

$$Q_{i+1/2,j}^{R} = Q_{i+1,j} - \frac{\varepsilon}{2}\left(Q_{i+2,j} - Q_{i+1,j}\right), \text{ cell (i+1,j),} \quad (19)$$

leading to a second order fully one-sided scheme. If the first order scheme is defined by the numerical flux

$$F_{i+1/2,j} = F\left(Q_{i,j}, Q_{i+1,j}\right) \quad (20)$$

the second order space accurate numerical flux is obtained from

$$F_{i+1/2,j}^{(2)} = F\left(Q_{i+1/2,j}^{L}, Q_{i+1/2,j}^{R}\right). \quad (21)$$

Higher order flux vector splitting methods, such as those studied in this work, are obtained from:

$$F_{i+1/2,j}^{(2)} = F^{+}\left(Q_{i+1/2,j}^{L}\right) + F^{-}\left(Q_{i+1/2,j}^{R}\right). \quad (22)$$

All second order upwind schemes necessarily involve at least five mesh points or cells.

To reach high order solutions without oscillations around discontinuities, nonlinear limiters are employed, replacing the term ε in Eqs. (18) and (19) by these limiters evaluated at the left and at the right states of the flux interface. To define such limiters, it is necessary to calculate the ratio of consecutive variations of the conserved variables. These ratios are defined as follows:

$$r_{i-1/2,j}^{+} = \left(Q_{i+1,j} - Q_{i,j}\right)/\left(Q_{i,j} - Q_{i-1,j}\right) \quad \text{and} \quad r_{i+1/2,j}^{+} = \left(Q_{i+2,j} - Q_{i+1j}\right)/\left(Q_{i+1,j} - Q_{i,j}\right), \quad (23)$$

where the nonlinear limiters at the left and at the right states of the flux interface are defined by $\Psi^{L} = \Psi\left(r_{i-1/2,j}^{+}\right)$ and $\Psi^{R} = \Psi\left(1/r_{i+1/2,j}^{+}\right)$. In this work, five options of nonlinear limiters were considered to the numerical experiments. These limiters are defined as follows:

$$\Psi_{l}^{VL}(r_l) = \frac{r_l + |r_l|}{1 + r_l}, \text{ Van Leer (1974) limiter;} \quad (24)$$

$$\Psi_{l}^{VA}(r_l) = \frac{r_l + r_l^2}{1 + r_l^2}, \text{ Van Albada limiter;} \quad (25)$$

$$\Psi_{l}^{MIN}(r_l) = signal_l\, MAX\left(0, MIN\left(|r_l|, signal_l\right)\right), \text{ minmod limiter;} \quad (26)$$

$$\Psi_{l}^{SB}(r_l) = MAX\left(0, MIN\left(2r_l, 1\right), MIN\left(r_l, 2\right)\right), \text{ “Super Bee” limiter, due to Roe (1983);} \quad (27)$$

$$\Psi_{l}^{\beta-L}(r_l) = MAX\left(0, MIN\left(\beta r_l, 1\right), MIN\left(r_l, \beta\right)\right), \ \beta\text{-limiter,} \quad (28)$$

with “*l*” varying from 1 to 4 (two-dimensional space), $signal_l$ being equal to 1.0 if $r_l \geq 0.0$ and -1.0 otherwise, r_l is the ratio of consecutive variations of the *l*th conserved variable and β is a parameter assuming values between 1.0 and 2.0, being 1.5 the value assumed in this work.

With the implementation of the numerical flux vectors following this MUSCL procedure, second order spatial accuracy and TVD properties are incorporated in the algorithms.

5. IMPLICIT FORMULATION

All implicit schemes studied in this work used an ADI formulation to solve the algebraic nonlinear system of equations. Initially, the nonlinear system of equations is linearized considering the implicit operator evaluated at the time “*n*” and, posteriorly, the pentadiagonal system of linear algebraic equations is factored in two tridiagonal systems of linear algebraic equations, each one associated with a particular spatial direction. Thomas algorithm is employed to solve these two tridiagonal systems. The implicit schemes studied in this work were only applicable to the solution of the Euler equations, which implies that only the convective contributions were considered in the RHS ("Right Hand Side") operator.

The ADI form of the Steger and Warming (1981) and of Radespiel and Kroll (1995) TVD schemes is defined by the following two step algorithm:

$$\left\{I + \Delta t_{i,j}\Delta_\xi^- A_{i+1/2,j}^+ + \Delta t_{i,j}\Delta_\xi^+ A_{i+1/2,j}^-\right\}\Delta Q_{i,j}^* = [RHS]_{i,j}^n \text{, to the } \xi \text{ direction;} \quad (29)$$

$$\left\{I + \Delta t_{i,j}\Delta_\eta^- B_{i,j+1/2}^+ + \Delta t_{i,j}\Delta_\eta^+ B_{i,j+1/2}^-\right\}\Delta Q_{i,j}^{n+1} = \Delta Q_{i,j}^* \text{, to the } \eta \text{ direction;} \quad (30)$$

$$Q_{i,j}^{n+1} = Q_{i,j}^n + \Delta Q_{i,j}^{n+1}, \quad (31)$$

where the matrices $A^\pm$ and $B^\pm$ are defined as:

$$A_{i\pm1/2,j}^\pm = [T]_{i\pm1/2,j}^n \Omega_{i\pm1/2,j}^\pm \left[T^{-1}\right]_{i\pm1/2,j}^n;\ B_{i,j\pm1/2}^\pm = [T]_{i,j\pm1/2}^n \Phi_{i,j\pm1/2}^\pm \left[T^{-1}\right]_{i,j\pm1/2}^n; \quad (32)$$

$$\Omega_{i\pm1/2,j}^\pm = diag\left[\left(\lambda_\xi^l\right)^\pm\right]_{i\pm1/2,j}^n;\ \Phi_{i,j\pm1/2}^\pm = diag\left[\left(\lambda_\eta^l\right)^\pm\right]_{i,j\pm1/2}^n; \quad (33)$$

$$\left(\lambda_\xi^l\right)^\pm = 0.5\left(\lambda_\xi^l \pm \left|\lambda_\xi^l\right|\right),\ \left(\lambda_\eta^l\right)^\pm = 0.5\left(\lambda_\eta^l \pm \left|\lambda_\eta^l\right|\right),\ \Delta_\xi^- = (\cdot)_{i,j} - (\cdot)_{i-1,j},\ \Delta_\xi^+ = (\cdot)_{i+1,j} - (\cdot)_{i,j}; \quad (34)$$

$$\Delta_\eta^- = (\cdot)_{i,j} - (\cdot)_{i,j-1},\ \Delta_\eta^+ = (\cdot)_{i,j+1} - (\cdot)_{i,j}. \quad (35)$$

with the λ_ξ and λ_η being the eigenvalues of the Euler equations, defined in Maciel (2006, 2008b,c); *diag* [·] is a diagonal matrix; and the similarity transformation matrices are defined by:

$$T = \begin{bmatrix} 1 & 0 & \alpha & \alpha \\ u_{int} & h_y' \rho_{int} & \alpha\left(u_{int} + h_x' a_{int}\right) & \alpha\left(u_{int} - h_x' a_{int}\right) \\ v_{int} & -h_x' \rho_{int} & \alpha\left(v_{int} + h_y' a_{int}\right) & \alpha\left(v_{int} - h_y' a_{int}\right) \\ \frac{\phi^2}{\gamma - 1} & \rho_{int}\left(h_y' u_{int} - h_x' v_{int}\right) & \alpha\left(\frac{\phi^2 + a_{int}^2}{\gamma - 1} + a_{int}\tilde{\theta}\right) & \alpha\left(\frac{\phi^2 + a_{int}^2}{\gamma - 1} - a_{int}\tilde{\theta}\right) \end{bmatrix}; \quad (36)$$

$$\alpha = \rho_{int} / \left(\sqrt{2} a_{int}\right),\ \beta = 1 / \left(\sqrt{2}\rho_{int} a_{int}\right),\ \phi^2 = (\gamma - 1)\frac{u_{int}^2 + v_{int}^2}{2},\ \tilde{\theta} = h_x' u_{int} + h_y' v_{int}; \quad (37)$$

$$T^{-1} = \begin{bmatrix} 1 - \frac{\phi^2}{a_{int}^2} & (\gamma - 1)\frac{u_{int}}{a_{int}^2} & (\gamma - 1)\frac{v_{int}}{a_{int}^2} & -\frac{\gamma - 1}{a_{int}^2} \\ -\frac{h_y' u_{int} - h_x' v_{int}}{\rho_{int}} & \frac{h_y'}{\rho_{int}} & -\frac{h_x'}{\rho_{int}} & 0 \\ \beta\left(\phi^2 - a_{int}\tilde{\theta}\right) & \beta\left[h_x' a_{int} - (\gamma - 1)u_{int}\right] & \beta\left[h_y' a_{int} - (\gamma - 1)v_{int}\right] & \beta(\gamma - 1) \\ \beta\left(\phi^2 + a_{int}\tilde{\theta}\right) & -\beta\left[h_x' a_{int} + (\gamma - 1)u_{int}\right] & -\beta\left[h_y' a_{int} + (\gamma - 1)v_{int}\right] & \beta(\gamma - 1) \end{bmatrix}. \quad (38)$$

The properties defined at interface are calculated by arithmetical average and h_x' and h_y' defined according to Maciel (2006, 2008b,c). The *RHS* operator required in Eq. (29) is defined to the Steger and Warming (1981) TVD scheme as:

$$\left[RHS_{(SW)}\right]_{i,j}^n = -\Delta t_{i,j} / V_{i,j}\left(\tilde{F}_{i+1/2,j}^{(SW)} + \tilde{F}_{i-1/2,j}^{(SW)} + \tilde{F}_{i,j+1/2}^{(SW)} + \tilde{F}_{i,j-1/2}^{(SW)}\right)^n, \quad (39)$$

with $\tilde{F}_{i+1/2,j}^{(SW)}$ calculated according to Eq. (8). The *RHS* operator defined to the Radespiel and Kroll (1995) TVD scheme is defined as:

$$\left[RHS_{(RK)}\right]_{i,j}^n = -\Delta t_{i,j} / V_{i,j}\left(R_{i+1/2,j}^{e(RK)} - R_{i-1/2,j}^{e(RK)} + R_{i,j+1/2}^{e(RK)} - R_{i,j-1/2}^{e(RK)}\right)^n, \quad (40)$$

with $R_{i+1/2,j}^{e(RK)}$ calculated according to Eq. (10). The cell volume, $V_{i,j}$, is calculated according to Maciel (2006, 2008c).

6. SPATIALLY VARIABLE TIME STEP

The basic idea of this procedure consists in keeping constant the CFL number in all calculation domain, allowing, hence, the use of appropriated time steps to each specific mesh region during the convergence process. Hence, according to the definition of the CFL number, it is possible to write:

$$\Delta t_{i,j} = CFL(\Delta s)_{i,j} / c_{i,j} \ , \tag{41}$$

where CFL is the "Courant-Friedrichs-Lewy" number to provide numerical stability to the scheme; $c_{i,j} = \left[\left(u^2 + v^2\right)^{0.5} + a\right]_{i,j}$ is the maximum characteristic speed of information propagation in the calculation domain; and $(\Delta s)_{i,j}$ is a characteristic length of information transport. On a finite volume context, $(\Delta s)_{i,j}$ is chosen as the minor value found between the minor centroid distance, involving the (i,j) cell and a neighbor, and the minor cell side length.

7. INITIAL AND BOUNDARY CONDITIONS

7.1 Initial Conditions

Values of freestream flow are adopted for all properties as initial condition, in the whole calculation domain, to the studied problem in this work (Jameson and Mavriplis, 1986, and Maciel, 2002):

$$Q_\infty = \left\{1 \quad M_\infty \cos\theta \quad M_\infty \sin\theta \quad \left[1/\left[\gamma(\gamma-1)\right] + 0.5 M_\infty^2\right]\right\}^T , \tag{42}$$

where M_∞ represents the freestream Mach number and θ is the flow attack angle.

7.2 Boundary Conditions

The boundary conditions are basically of three types: solid wall, entrance and exit. These conditions are implemented in special cells named ghost cells.

(a) Wall condition: This condition imposes the flow tangency at the solid wall. This condition is satisfied considering the wall tangent velocity component of the ghost volume as equals to the respective velocity component of its real neighbor cell. At the same way, the wall normal velocity component of the ghost cell is equaled in value, but with opposite signal, to the respective velocity component of the real neighbor cell.

The pressure gradient normal to the wall is assumed be equal to zero, following an inviscid formulation. The same hypothesis is applied to the temperature gradient normal to the wall, considering adiabatic wall. The ghost volume density and pressure are extrapolated from the respective values of the real neighbor volume (zero order extrapolation), with these two conditions. The total energy is obtained by the state equation of a perfect gas.

(b) Entrance condition:
(b.1) Subsonic flow: Three properties are specified and one is extrapolated, based on analysis of information propagation along characteristic directions in the calculation domain (Maciel, 2002). In other words, three characteristic directions of information propagation point inward the computational domain and should be specified. Only the characteristic direction associated to the "(q_{normal}-a)" velocity can not be specified and should be determined by interior information of the calculation domain. Pressure was the extrapolated variable to the present problem. Density and velocity components had their values determined by the freestream flow properties. The total energy per unity fluid volume is determined by the state equation of a perfect gas.
(b.2) Supersonic flow: All variables are fixed with their freestream flow values.

(c) Exit condition:
(c.1) Subsonic flow: Three characteristic directions of information propagation point outward the computational domain and should be extrapolated from interior information (Maciel, 2002). The characteristic direction associated to the "(q_{normal}-a)" velocity should be specified because it penetrates the calculation domain. In this case, the ghost volume's pressure is specified by its freestream value. Density and velocity components are extrapolated and the total energy is obtained by the state equation of a perfect gas.
(c.2) Supersonic flow: All variables are extrapolated from the interior domain due to the fact that all four characteristic directions of information propagation of the Euler equations point outward the calculation domain and, with it, nothing can be fixed.

8. CONCLUSIONS

In the present work, first part of this study, the theories involving the extension of the first order versions of the numerical schemes of Steger and Warming (1981) and of Radespiel and Kroll (1995) to second order, incorporating hence TVD properties through a MUSCL approach, and the implicit numerical implementation of these second order TVD schemes are detailed. The schemes are implemented on a finite volume context, using a structured spatial discretization. First order time integrations like ADI approximate factorization are programmed. The Euler equations in conservation and integral forms, in two-dimensions, are solved. The steady state physical problem of the moderate supersonic flow along a compression corner is studied and compared with theoretical results. A spatially variable time step procedure is also implemented aiming to accelerate the convergence to the steady solution. The gains in convergence with this procedure were demonstrated in Maciel (2005, 2008a). The results have demonstrated that the most accurate solutions are obtained with the Steger and Warming (1981) TVD scheme using Van Leer and minmod nonlinear limiters, when implemented in its explicit version.

9. REFERENCES

Beam, R. M., and Warming, R. F., 1978, “An Implicit Factored Scheme for the Compressible Navier-Stokes Equations”, AIAA Journal, Vol. 16, No. 4, pp. 393-402.

Douglas, J., 1955, “On the Numerical Integration of $u_{xx}+u_{yy}=u_t$ by Implicit Methods”, Journal of the Society of Industrial and Applied Mathematics, Vol. 3, pp. 42-65.

Douglas, J., and Gunn, J. E., 1964, “A General Formulation of Alternating Direction Methods”, Numerische Mathematiks, Vol. 6, pp. 428-453.

Godunov, S. K., 1959, “A Difference Scheme for Numerical Computation of Discontinuous Solution of Hydrodynamic Equations”, Math. Sbornik, Vol. 47, pp. 271-306.

Hirsch, C., 1990, “Numerical Computation of Internal and External Flows – Computational Methods for Inviscid and Viscous Flows”, John Wiley & Sons Ltd, 691p.

Jameson, A., and Mavriplis, D. J., 1986, “Finite Volume Solution of the Two-Dimensional Euler Equations on a Regular Triangular Mesh”, AIAA Journal, Vol. 24, No. 4, pp. 611-618.

Kutler, P., 1975, “Computation of Three-Dimensional, Inviscid Supersonic Flows”, Lecture Notes in Physics, Vol. 41, pp. 287-374.

Liou, M., and Steffen Jr., C. J., 1993, “A New Flux Splitting Scheme”, Journal of Computational Physics, Vol. 107, pp. 23-39.

Maciel, E. S. G., 2002, “Simulação Numérica de Escoamentos Supersônicos e Hipersônicos Utilizando Técnicas de Dinâmica dos Fluidos Computacional”, PhD thesis, ITA, CTA, São José dos Campos, SP, Brazil, 258 p.

Maciel, E. S. G., 2005, “Analysis of Convergence Acceleration Techniques Used in Unstructured Algorithms in the Solution of Aeronautical Problems – Part I”, Proceedings of the XVIII International Congress of Mechanical Engineering (XVIII COBEM), Ouro Preto, MG, Brazil.

Maciel, E. S. G., 2006, “Comparison Between the Yee, Warming and Harten and the Hughson and Beran High Resolution Algorithms in the Solution of the Euler Equations in Two-Dimensions – Theory”, Proceedings of the XXVII Iberian Latin-American Congress on Computational Methods in Engineering (XXVII CILAMCE), Belém, PA, Brazil.

Maciel, E. S. G., 2008a, “Analysis of Convergence Acceleration Techniques Used in Unstructured Algorithms in the Solution of Aerospace Problems – Part II”, Proceedings of the XII Brazilian Congress of Thermal Engineering and Sciences (XII ENCIT), Belo Horizonte, MG, Brazil.

Maciel, E. S. G., 2008b, “Comparison Among the First Order Upwind Algorithms of Roe, of Steger and Warming, of Van Leer and of Chakravarthy and Osher in the Solution of the Euler Equations in 2D – Theory”, Proceedings of the 8th Symposium of Computational Mechanics (VIII SIMMEC), Belo Horizonte, MG, Brazil.

Maciel, E. S. G., 2008c, “Comparação Entre Algoritmos de Separação de Vetores de Fluxo e de Diferenças de Fluxo de Primeira Ordem na Solução das Equações de Euler em Duas Dimensões – Teoria”, Proceedings of the Primer Congreso Argentino de Ingeniería Mecánica (I CAIM),Bahía Blanca, Argentina.

Peaceman, D. W., and Rachford, H. H., 1955, “The Numerical Solution of Parabolic and Elliptic Differential Equations”, Journal of the Society of Industrial and Applied Mathematics, Vol. 3, pp. 28-41.

Radespiel, R., and Kroll, N., 1995, “Accurate Flux Vector Splitting for Shocks and Shear Layers”, Journal of Computational Physics, Vol. 121, pp. 66-78.

Roe, P. L., 1983, In Proceedings of the AMS-SIAM Summer Seminar on Large-Scale Computation in Fluid Mechanics, Edited by B. E. Engquist *et al*, Lectures in Applied Mathematics, Vol. 22, p. 163.

Steger, J. L., and Warming, R. F., 1981, “Flux Vector Splitting of the Inviscid Gasdynamic Equations with Application to Finite-Difference Methods”, Journal of Computational Physics, Vol. 40, pp. 263-293.

Van Leer, B., 1974, “Towards the Ultimate Conservative Difference Scheme. II. Monotonicity and Conservation Combined in a Second-Order Scheme”, Journal of Computational Physics, Vol. 14, pp. 361-370.

Van Leer, B., 1982, “Flux-Vector Splitting for the Euler Equations”, Proceedings of the 8th International Conference on Numerical Methods in Fluid Dynamics, E. Krause, Editor, Lecture Notes in Physics, Vol. 170, pp. 507-512,

Springer-Verlag, Berlin.
Yanenko, N. N., 1971, “The Method of Fractional Steps”, Springer Verlag, N.Y., EUA.

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