# ESTIMATION OF BOUNDARY HEAT FLUX IN A THIN PLATE WITH THE CONJUGATE GRADIENT METHOD

**Farith M. Absi Salas, farith@cepel.br**
**Carolina Palma Naveira, cpncotta@hotmail.com**
**Henrique Massard da Fonseca, hmassard@gmail.com**
**Renato M. Cotta, cottarenato@terra.com.br**
**Christophe Pradère, christophe_predere@yahoo.fr**
**William Berk, willberk@globo.br**
**Helcio R. B. Orlande, helcio@mecanica,coppe.ufrj.br**
**Marcelo Cintra Bitencourt, m.bitencourt@petrobras.com.br**
**Paulo Henrique da Silva Moreira, phsmoreira@gmail.com.br**
Universidade Federal do Rio de Janeiro UFRJ/PEM/COPPE

**Abstract.** *In this paper we describe the solution of an inverse heat conduction problem dealing with the estimation of a boundary heat flux. Such heat flux is imposed on the surface of a thin metal plate and temperature measurements are considered to be taken over the non-heated surface. For the direct problem, a lumped formulation is used across the plate, so that the three-dimensional problem is formulated in two dimensions and in terms of an average transversal temperature. The inverse problem is solved with a function estimation approach based on the conjugate gradient method with adjoint problem. Simulated temperature measurements are used in the inverse analysis in order to show the capabilities of the proposed approach.*

***Keywords***: *Lumped Heat Equations, Inverse Problem, Conjugate Gradient Method, Function Estimation.*

## 1. INTRODUCTION

Inverse problems became a powerful and practical tool for analysis and design in engineering. In fact, inverse problems enable a much closer collaboration between experimental and theoretical researchers, in order to obtain the maximum of information regarding the physical problem under study.

In heat transfer, the classical inverse problem of estimating a boundary heat flux with temperature measurements taken inside a heat-conducting medium (Beck *et al,* 1985) has many practical applications. For example, the heat loads that the surface of space vehicles re-entering the atmosphere are subjected to, can be estimated through inverse analysis, by using temperature measurements taken within the thermal protection shield (Kanevce *et al* 1999, Oliveira and Orlande 2002, Oliveira and Orlande 2004, Mota *et al* 2004). If a technique that sequentially estimates such boundary heat flux is used, inverse analysis may allow for on-line trajectory corrections in order to reduce the heat load. An application of sequential inverse analysis of state estimation and control in relation with process tomography can be found in Ruuskanen *et al* (2006).

Inverse problems are ill-posed, that is, their solutions do not satisfy either one of the requirements of existence, uniqueness or stability (Hadamard, 1923). Therefore, the classical approach is to reformulate the problem as an approximate well-posed problem. A variety of techniques is nowadays available for the solution of inverse problems and there are a number of books and book chapters that cover various aspects of inverse problems (Tikhonov and Arsenin 1977, Beck and Arnold 1977, Alifanov 1994, Alifanov *et al* 1995, Dulikravich and Martin 1996, Ozisik and Orlande 2000, Kurspisz and Nowak 1995, Woodbury 2002, Murio 1993, Trujillo and Busby 1997, Kaipio and Somersalo 2004, Zabaras 2204, Calvetti and Somersalo 2007).

In this work we use the conjugate gradient method of function estimation with adjoint problem formulation (Alifanov 1994, Alifanov *et al* 1995, Ozisik and Orlande 2000), in order to solve the inverse problem of identifying a boundary heat flux in a heat conducting medium. The medium is a thin metallic plate, so that a partial lumped formulation is used in this work by neglecting temperature gradients across the plate. Simulated measured data is used in this analysis for functional forms containing discontinuities, which are the most difficult to be recovered with the solution of the inverse problem.

## 2. PHYSICAL PROBLEM AND MATHEMATICAL FORMULATION

The physical problem examined in this work is described in Fig. 2.1 and 2.2. These figures show a thin plate with area $L_xL_y$ and thickness $L_z$ which is heated on its surface with a circular electrical resistance. Such heater provides a uniform heat flux $q(\hat{x},\hat{y},\hat{t})$. The lateral surfaces were considered to be insulated, while the top boundary looses heat by convection and radiation to its surroundings. The simulated experimental temperature was supposed to be taken on the non-heated surface with a infrared camera.

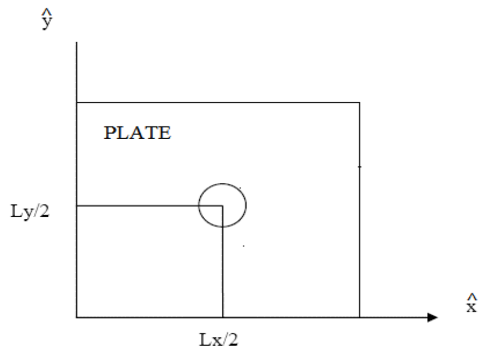


Figure 2.1 – Top view of the heated plate

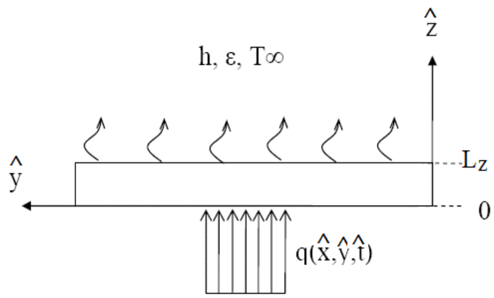


Figure 2.2 – Lateral view of the heated plate

The mathematical model describing the temperature distribution on the heated plate showed in Fig 2.1 and 2.2 can be expressed as:

$$\frac{1}{\hat{\alpha}}\frac{\partial T}{\partial \hat{t}} = \frac{\partial^2 T}{\partial \hat{x}^2} + \frac{\partial^2 T}{\partial \hat{y}^2} + \frac{\partial^2 T}{\partial \hat{z}^2} \quad \text{at} \quad 0 < \hat{x} < L_x, \quad 0 < \hat{y} < L_y, \quad 0 < \hat{z} < L_z \quad \text{and} \quad \hat{t} > 0 \tag{2.1.a}$$

with boundary and initial conditions, given by

$$\frac{\partial T}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0, \quad \hat{t} > 0 \tag{2.1.b}$$

$$\frac{\partial T}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = L_x, \quad \hat{t} > 0 \tag{2.1.c}$$

$$\frac{\partial T}{\partial \hat{y}} = 0 \quad \text{at} \quad \hat{y} = 0, \quad \hat{t} > 0 \tag{2.1.d}$$

$$\frac{\partial T}{\partial \hat{y}} = 0 \quad \text{at} \quad \hat{y} = L_y, \quad \hat{t} > 0 \tag{2.1.e}$$

$$-\hat{k}\frac{\partial T}{\partial \hat{z}} = q(\hat{x}, \hat{y}, \hat{t}) \quad \text{at} \quad \hat{z} = 0, \quad \hat{t} > 0 \tag{2.1.f}$$

$$-\hat{k}\frac{\partial T}{\partial \hat{z}} = \hat{h}(T - T_\infty) + \varepsilon\sigma(T^4 - T_\infty{}^4) \quad \text{at} \quad \hat{z} = L_z, \quad \hat{t} > 0 \tag{2.1.g}$$

$$T(\hat{x}, \hat{y}, \hat{z}, 0) = T_0 \quad \text{at} \quad \hat{t} = 0 \quad 0 < \hat{x} < L_x, \quad 0 < \hat{y} < L_y, \quad 0 < \hat{z} < L_z \tag{2.1.h}$$

Where $\hat{k}$ is the thermal conductivity, $\hat{\alpha}$ is the thermal diffusivity, $\hat{h}$ convective coefficient, $\varepsilon$ is the thermal emissivity and $\sigma$ Stefan-Boltzmann constant.

In a well-posed direct problem all physical properties, boundary and initial conditions are known. Thus, the temperature distribution at the plate can be obtained as a time and space function. Since we are dealing with a very thin plate, an accurate approximate formulation can be obtained by neglecting the temperature gradients along the $\hat{z}$ direction. Therefore we can solve the heat conduction problem in terms of an average temperature defined as

$$\overline{T}(\hat{x},\hat{y},\hat{t}) = \frac{1}{L_z}\int_{z'=0}^{z'=L_z} T(\hat{x},\hat{y},\hat{z},\hat{t})dz' \tag{2.2}$$

Hence, the three-dimensional PDE describing the physical problem defined above can be reduced to a two-dimensional problem. Such lumped system will be explored in dimensionless form in order to simplify the heat equation analysis. By assuming $L=L_x=L_y$ the following dimensionless groups were defined, as

$$\phi(x,y,t) = \frac{q(\hat{x},\hat{y},\hat{t})}{q_R} \qquad t = \frac{\hat{t}\alpha_R}{L^2}, \qquad x = \frac{\hat{x}}{L}, \qquad y = \frac{\hat{y}}{L}, \qquad \theta = \frac{\overline{T} - T_\infty}{q_R\, L/k_R}, \qquad k = \frac{\hat{k}}{k_R}, \tag{2.3}$$

to obtain the following dimensionless Lumped - 2D formulation for our heat conduction problem:

$$\frac{1}{\alpha}\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} - h(\theta)\theta + g(x,y,t) \qquad 0<x<1, \qquad 0<y<1, \qquad t>0 \tag{2.4.a}$$

With boundary and initial conditions, given by:

$$\frac{\partial\theta}{\partial x} = 0 \qquad \text{at} \qquad x=0, \qquad t>0 \tag{2.4.b}$$

$$\frac{\partial\theta}{\partial x} = 0 \qquad \text{at} \qquad x=1, \qquad t>0 \tag{2.4.c}$$

$$\frac{\partial\theta}{\partial y} = 0 \qquad \text{at} \qquad y=0, \qquad t>0 \tag{2.4.d}$$

$$\frac{\partial\theta}{\partial y} = 0 \qquad \text{at} \qquad y=1, \qquad t>0 \tag{2.4.e}$$

$$\theta(x,y,0) = 0 \qquad \text{at} \qquad t=0, \qquad 0<x<1, \qquad 0<y<1, \tag{2.4.f}$$

where $h(\theta)$ is the linearized source term that takes into account the contributions of the boundary conditions at $\hat{z}=0$, and, $\hat{z}=L_z$ that is,

$$h(\theta) = \frac{L^2}{L_z k_R}[\hat{h} + 4\varepsilon\sigma(\theta q_R \frac{L}{k_R} + T_\infty)^3] \tag{2.5}$$

$$g(x,y,t) = \frac{L}{L_z}\phi(x,y,t) \tag{2.6}$$

and reference values of thermal conductivity, $k_R$, thermal diffusivity, $\alpha_R$, and heat flux, $q_R$, are also required.

The numerical method used to solve the problem given by Eqs. (2.4.a-f) is based on the finite element method (FEM). Triangular elements were used for the plate discretization, as presented by Fig. 2. The mesh surface plotted in this figure is composed of 1658 elements and 858 nodes. The time step used in time discretization was one second. Such number of elements and time steps were selected based on a grid convergence analysis. The weighted residuals approach, which provides a powerful approximate solution procedure, was used to formulate the problem in order to apply the finite element method.

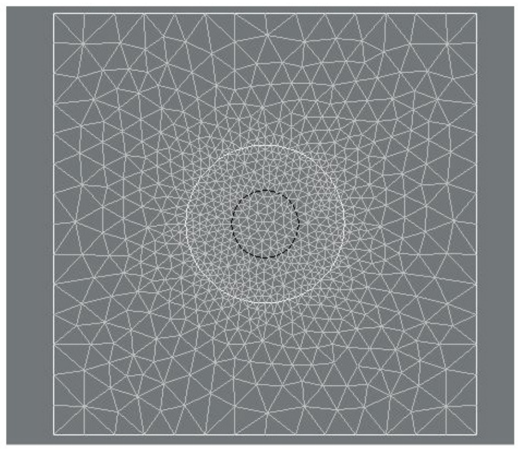

Figure 2. Mesh with triangular elements used for the FEM.

## 3. INVERSE PROBLEM AND SOLUTION METHODOLOGY

For the inverse problem considered in this work, the source-term in Eq. (2.4.a) resulting from the hat flux imposed by the electrical resistance, is regarded as unknown. For the estimation of such function we assume available temperature measurements at each of the finite elements over the non-heated surface. Such measurements can be taken with an infrared camera.

For the solution of such inverse problem, we use the conjugate gradient method of function estimation (Alifanov 1994, Alifanov *et al* 1995, Ozisik and Orlande 2000) and consider the minimization of the following functional:

$$S[g(x,y,t)] = \sum_{m=1}^{M}\int_{t=0}^{tf}[Y_m(t) - \theta(x_m, y_m, t; g(t))]^2\,dt \tag{3.1}$$

There is no prior information about *g(x,y,t)*, except that the function belongs to the domain called Hilbert's integrable quarter, denoted as $L_2$ in the domain of interest. The iterative procedure of the conjugate gradient method for the source term estimation is given by:

$$g^{k+1}(x,y,t) = g^k(x,y,t) - \beta^k d^k(x,y,t) \tag{3.2}$$

where the search direction $g^k(x,y,t)$ is obtained as:

$$d^k(x,y,t) = \nabla S[g^k(x,y,t)] + \gamma^k d^{k-1}(x,y,t) \tag{3.3}$$

The conjugation coefficient $\gamma^k$ is obtained through the Fletcher-Reeves expression, given by:

$$\gamma^k = \frac{\int_{t=0}^{tf}\int_{y0}^{yf}\int_{x0}^{xf}[\nabla S[g^k(x,y,t)]]^2\,dxdydt}{\int_{t=0}^{tf}\int_{y0}^{yf}\int_{x0}^{xf}[\nabla S[g^{k-1}(x,y,t)]]^2\,dxdydt} \tag{3.4}$$

with $\gamma^0 = 0$.

The following expression for $\beta^k$ is obtained by minimizing $S[g^{k+1}(x,y,t)]$ along the direction $d^k(x,y,t)$ (Alifanov 1994, Alifanov *et al* 1995, Ozisik and Orlande 2000):

$$\beta^k = \frac{\sum_{m=1}^{M}\int_{t=0}^{tf}[Y_m(t)-\theta(x_m,y_m,t;g^k(t))]\Delta\theta(x_m,y_m,t;d^k)dt}{\sum_{m=1}^{M}\int_{t=0}^{tf}\Delta\theta(x_m,y_m,t;d^k)^2dt} \tag{3.5}$$

Note that $\beta^k$ in Eq. (5) depends on the function $\Delta\theta(x_m,y_m,t;d^k)$. This function is obtained by solving the sensitivity problem, as described below.

**3.1. Sensitivity problem**

The sensitivity problem is solved substituting $g(x,y,t)$ for $g(x,y,t)+\Delta g(x,y,t)$ in the direct problem, resulting on a temperature perturbation $\Delta\theta(x,y,t)$. The resulting equation is subtracted from the direct problem and, by neglecting second order terms, the following problem is obtained:

$$\frac{1}{\alpha}\frac{\partial\Delta\theta(x,y,t)}{\partial t} = \frac{\partial^2\Delta\theta}{\partial x^2}+\frac{\partial^2\Delta\theta}{\partial y^2}-H(\theta)\Delta\theta+\Delta G(x,y,t) \tag{3.6.a}$$

$$\frac{\partial\Delta\theta(x,y,t)}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{3.6.b}$$

$$\frac{\partial\Delta\theta(x,y,t)}{\partial x} = 0 \quad \text{at} \quad x = 1 \tag{3.6.c}$$

$$\frac{\partial\Delta\theta(x,y,t)}{\partial y} = 0 \quad \text{at} \quad y = 0 \tag{3.6.d}$$

$$\frac{\partial\Delta\theta(x,y,t)}{\partial y} = 0 \quad \text{at} \quad y = 1 \tag{3.6.e}$$

where

$$H(\theta) = \frac{L^2}{L_z k_R}\{h+4\varepsilon\sigma[\frac{6Lq_R}{k_R}T_\infty^2\theta+9(\frac{Lq_R}{k_R}{}_0)^2T_\infty\theta^2+4(\frac{Lq_R}{k_R}{}_0)^3\theta^3+T_\infty^3]\} \tag{3.7}$$

and

$$\Delta G(X,Y,T) = \frac{L}{L_z}\Delta g \tag{3.8}$$

**3.2. Adjoint problem**

In order to derive an expression for the gradient of the functional, we make use of another auxiliary problem which satisfies adjoint properties with respect to the sensitivity problem and is denoted as the adjoint problem. The adjoint problem is solved for the Lagrange multiplier $\lambda(x, y, t)$.

The adjoint problem is obtained by first writing the following extended functional:

$$S[g(x, y, t)] = \sum_{m=1}^{M} \int_{t=0}^{tf} [Y_m(t) - \theta(x_m, y_m, t)]^2 dt +$$
$$\int_{t=0}^{tf} \int_{y0}^{yf} \int_{x0}^{xf} \lambda(x, y, t) \left[ \frac{\partial \theta(x, y, t)}{\partial t} - \left( \frac{\partial^2 \theta(x, y, t)}{\partial x^2} \right) - \left( \frac{\partial^2 \theta(x, y, t)}{\partial y^2} \right) + h(\theta)\theta - g(x, y, t) \right] dxdydt \tag{3.9}$$

The directional derivative of the extended functional in the direction of the gradient or the source term is obtained by substituting in Eq. (3.9) $g(x, y, t)$ for $g(x, y, t) + \Delta g(x, y, t)$, $\theta(x, y, t)$ for $\theta(x, y, t) + \Delta\theta(x, y, t)$ and $S[g(x, y, t)]$ for $S[g(x, y, t)] + \Delta S[g(x, y, t)]$. By subtracting Eq. (3.9) from the resulting equation, we obtain:

$$\Delta S[g(x, y, t)] = \sum_{m=1}^{M} \int_{t=0}^{tf} ([Y_m(t) - (\theta(x_m, y_m, t) + \Delta\theta(x_m, y_m, t))]^2 - [Y_m(t) - \theta(x_m, y_m, t)]) dxdydt +$$
$$\int_{t=0}^{tf} \int_{y0}^{yf} \int_{x0}^{xf} \lambda(x, y, t) \left[ \frac{\partial \Delta\theta(x, y, t)}{\partial t} - \left( \frac{\partial^2 \Delta\theta(x, y, t)}{\partial x^2} \right) - \left( \frac{\partial^2 \Delta\theta(x, y, t)}{\partial y^2} \right) + \tilde{H}(\theta, \Delta\theta) - \Delta g(x, y, t) \right] dxdydt$$
$$\tag{3.10}$$

where $\tilde{H}(\theta, \Delta\theta) = h(\theta + \Delta\theta)[\theta + \Delta\theta] - h(\theta)\theta$ and the function $h(\theta)$ is defined in Eq. (2.5). By neglecting the second order terms, this equation becomes:

$$\Delta S[g(x, y, t)] = \sum_{m=1}^{M} \int_{t=0}^{tf} \int_{y=0}^{1} \int_{x=0}^{1} 2\theta(x, y, t)[\theta(x, y, t) - Y(x, y, t)] \delta(x - x_m, y - y_m) dxdydt +$$
$$\int_{t=0}^{tf} \int_{y=0}^{1} \int_{x=0}^{1} \lambda(x, y, t) \left[ \frac{\partial \Delta\theta(x, y, t)}{\partial t} - \left( \frac{\partial^2 \Delta\theta(x, y, t)}{\partial x^2} \right) - \left( \frac{\partial^2 \Delta\theta(x, y, t)}{\partial y^2} \right) + \tilde{H}(\theta, \Delta\theta) - \Delta g(x, y, t) \right] dxdydt$$
$$\tag{3.11}$$

Equation (3.11) is then integrated by parts, resulting on

$$\Delta S[g(x, y, t)] = \sum_{m=1}^{M} \int_{t=0}^{tf} \int_{y=0}^{1} \int_{x=0}^{1} 2\theta(x, y, t)[\theta(x, y, t) - Y(x, y, t)] \delta(x - x_m, y - y_m) dxdydt +$$
$$\int_{t=0}^{tf} \int_{y=0}^{1} \int_{x=0}^{1} \frac{\partial \lambda(x, y, t)}{\partial t} - \left( \frac{\partial^2 \lambda(x, y, t)}{\partial x^2} \right) - \left( \frac{\partial^2 \lambda(x, y, t)}{\partial y^2} \right) +$$
$$H(\theta)\lambda(x, y, t) + 2(\theta(x, y, t) - Y(x, y, t))\delta(x - x_m, y - y_m)] \Delta\theta dxdydt +$$
$$\int_{t=0}^{tf} \int_{y=0}^{1} [\Delta\theta(1, y, t) \frac{\partial \lambda(x, y, t)}{\partial x} |_{x=1}] dydt - \int_{t=0}^{tf} \int_{y=0}^{1} [\Delta\theta(0, y, t) \frac{\partial \lambda(x, y, t)}{\partial x} |_{x=0}] dydt +$$
$$\int_{t=0}^{tf} \int_{x=0}^{1} [\Delta\theta(x, 1, t) \frac{\partial \lambda(x, y, t)}{\partial x} |_{y=1}] dxdt - \int_{t=0}^{tf} \int_{x=0}^{1} [\Delta\theta(x, 0, t) \frac{\partial \lambda(x, y, t)}{\partial x} |_{y=0}] dxdt +$$
$$\int_{y=0}^{yf} \int_{x=0}^{1} [\lambda(x, y, tf) \Delta\theta(x, y, tf)] dxdy - \int_{t=0}^{tf} \int_{y=0}^{1} \int_{x=0}^{1} [\lambda(x, y, t) \Delta g(x, y, t)] dxdydt \tag{3.12}$$

By assuming that the temperature perturbation $\Delta\theta(x, y, t)$ tends to zero, the terms of Eq. (3.12) may be rearranged for the formulation of the adjoint problem as:

$$-\frac{\partial\lambda(x, y,t)}{\partial t}=\left(\frac{\partial^2\lambda(x, y,t)}{\partial x^2}\right)-\left(\frac{\partial^2\lambda(x, y,t)}{\partial y^2}\right)-H(\theta)\lambda(x, y,t)-$$
$$2\sum_{m=1}^{M}(\theta(x, y,t)-Y(x, y,t))\delta(x-x_m, y-y_m) \tag{3.13.a}$$

$$\frac{\partial\lambda(x, y,t)}{\partial x}=0 \text{ at x = 0} \tag{3.13.b}$$

$$\frac{\partial\lambda(x, y,t)}{\partial x}=0 \text{ at x = 1} \tag{3.13.c}$$

$$\frac{\partial\lambda(x, y,t)}{\partial y}=0 \text{ at y = 0} \tag{3.13.d}$$

$$\frac{\partial\lambda(x, y,t)}{\partial y}=0 \text{ at y = 1} \tag{3.13.e}$$

$$\lambda(x, y,tf)=0 \text{ for t = } t_f \tag{3.13.f}$$

In Eqs. (3.13a-f) we have a final value problem, once the function $\lambda(x, y,t)$ is known at $t = t_f$. A new time variable is defined as $\tau=t_f-t$, so that $\frac{\partial}{\partial t}=-\frac{\partial}{\partial\tau}$ and the adjoint problem becomes an initial value problem.

**3.3. Gradient equation**

In the derivation of the adjoint problem, the last term of Eq. (3.12) left over. Such term is used to find the functional gradient $\nabla S[g(x, y,t)]$, that is,

$$\Delta S[g(x, y,t)]=-\int_{t=0}^{tf}\int_{y=0}^{1}\int_{x=0}^{1}[\lambda(x, y,t)\Delta g(x, y,t)]dxdydt \tag{3.14}$$

From the hypothesis that $g(x, y,t)$ belongs to $L_2$ space in the spatial and time domains, the directional derivative of $\Delta S[g(x, y,t)]$ in the direction of $\Delta g(x, y,t)$ may be written as (Alifanov 1994, Alifanov *et al* 1995, Ozisik and Orlande 2000):

$$\Delta S[g(x, y,t)]=\int_{t=0}^{tf}\int_{y=0}^{1}\int_{x=0}^{1}[\nabla S[g(x, y,t)]\Delta g(x, y,t)]dxdydt \tag{3.15}$$

Comparing Eq. (3.14) and (3.15) we obtain:

$$\nabla S[g(x, y,t)]=-\lambda(x, y,t) \tag{3.16}$$

Knowing $\nabla S[g(x, y,t)]$, $\gamma^k$ is obtained from Eq. (3.4) and the search direction can be obtained by Eq. (3.3). Thus, the iterative process given by Eq. (3.1) may be applied. The iterative procedure of the conjugate gradient method is stopped by using the Discrepancy Principle (Alifanov 1994, Alifanov *et al* 1995, Ozisik and Orlande 2000).

## 4. RESULTS AND DISCUSSIONS

In this section we present the estimated heat flux function $\phi(x, y, t)$ obtained from the inverse problem approach with the Conjugate Gradient method which was described in the previous section. In such inverse process the estimated heat flux function is obtained from temperature measurements over the non-heated surface of the thin plate. Such measurements can be taken, for example, with an infrared camera. In this work the temperature measurements are simulated from the expression:

$$Y_m = T_m^{exact} + \omega\upsilon \tag{4.1}$$

where ω is a random number with normal distribution and unitary standard deviation and $\upsilon = 0.01\, T_m^{\max}$ that is 1% of the maximum exact temperature obtained from the solution of the direct problem (2.4.a-f) with $q_R$.

The total time of heat flux application was considered as 120 s and the measurements are taken in one second interval over 800 points on the plate surface. The heat flux function $\phi(x, y, t)$ is estimated for a brass thin plate, with L=0.16m, $L_z$ = 0.001m, $\hat{\alpha}$ = 3.41 x $10^{-5}$ $m^2$/s, $\hat{k}$ = 111 W/(m K) , an effective thermal convective coefficient $\hat{h}$ = 10 W/($m^2$ K), $T_0$=$T_\infty$= 291 K and $\varepsilon$ =0.97. The circular electrical resistance with a diameter D = 0.0254m provided a uniform heat flux, taken as the reference value $q_R$, $q(\hat{x}, \hat{y}, \hat{t}) = 18255$ W/( $m^2$ K). The reference parameters are defined such as $k_R = \hat{k}$ , $\alpha_R = \hat{\alpha}$ , and $q_R$ = 18255 W/$m^2$K.

Figure 3 shows the objective function $S[g(x_i,y_i,t_i)]$ with respect to the number of iterations. This figure present the minimization process of the objective function expressed in Eq. (3.1) for several values of initial guess $q_0$ for the unknown flux, for the case involving errorless measurements. Note that for low values of $q_0$ (0 to 0.2) the conjugate gradient method does not converge and the function $S[g(x_i,y_i,t_i)]$ reach constant high values. However, for $q_0$ between 0.3 and 1.0 the method converges and the objective function tends to zero in most cases with around 25 iterations. It is clear from Fig 3 that the minimization process is dependent of the choice of $q_0$. However the conjugate gradient with adjoint problem method is effective and converges for a wide range of $q_0$.

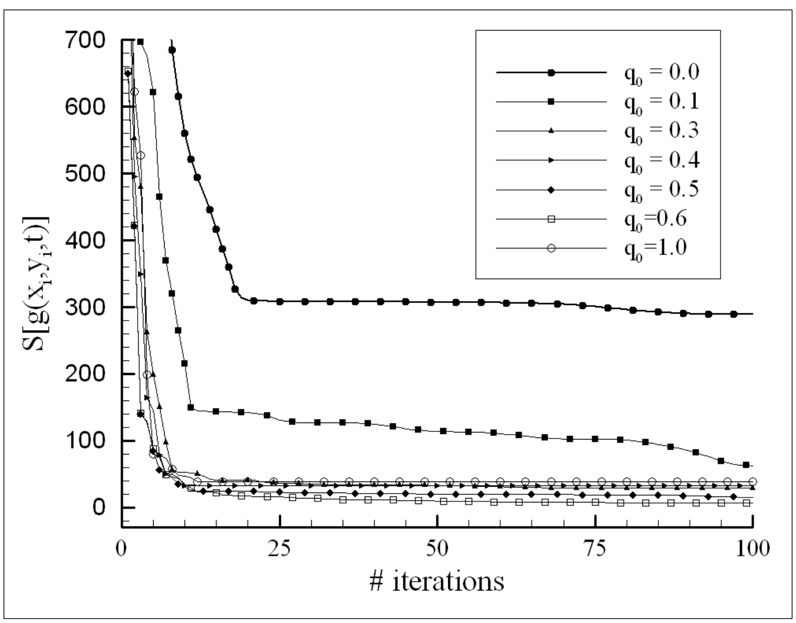


Figure 3. Objective function minimization for several values of $q_0$

In Fig. 4 we show the estimated heat flux $\phi(x_i, y_i, t) = q(\hat{x}_i, \hat{y}_i, \hat{t}) / q_R$ for different values of start guess $q_0$. The estimated heat flux plotted in Fig. 4 represents the temporal evolution of $\phi(x_i, y_i, t)$ in the middle point of the plate $\hat{x}_i = \hat{y}_i = L/2$ . This figure shows the important role of the start guess $q_0$ in the convergence of the estimation method. In this case, the heat flux is almost perfectly estimated for $q_0$ = 0.6.

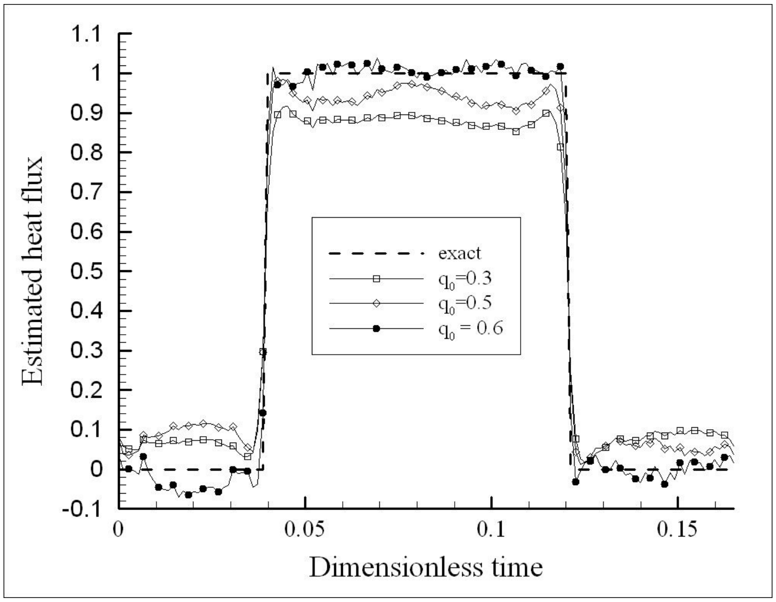


Figure 4. Estimated heat flux for three values of $q_0$ as a function of dimensionless time.

Figure 5 illustrate a qualitative representation of the estimated heat flux over the plate surface at $\hat{t}$ = 60 s. This figure shows the spatial variation of the dimensionless heat flux over the plate area with $q_0$ = 0.6. The estimated heat flux $\phi(x, y, t)$ takes unity values on the electrical resistance area (red circle). The heat flux is negligible in most of the plate surface, as represented by the white and yellow colors in Fig. 5.

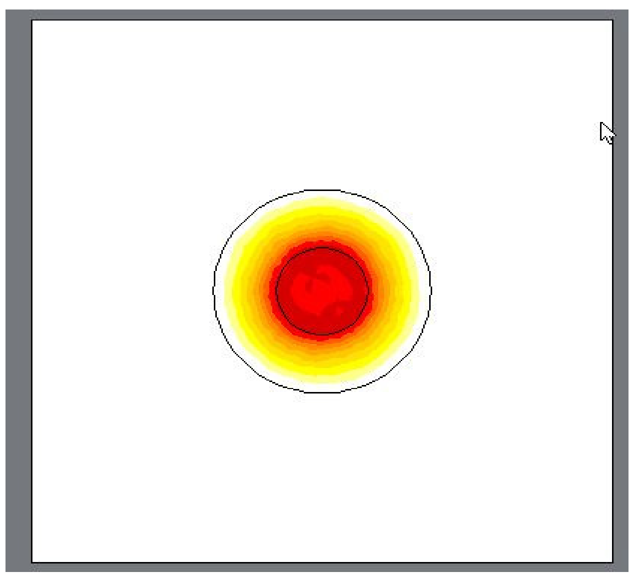

Figure 5. Illustration of the estimated heat flux over the plate surface with $q_0$=0.6, $\hat{t}$ = 60s.

## 5. CONCLUSIONS

In this work we applied the conjugate gradient method of function estimation for the identification of a boundary heat flux applied on a thin plate. The formulation of the physical problem is given in terms of a partial lumping approach, by neglecting temperature gradients across the thickness of the plate.

Simulated temperature measurements containing random errors, supposedly taken with an infrared camera over the non-heated surface, were used in the inverse analysis. The results obtained in this paper reveals that the present solution methodology is capable of providing accurate estimates for the spatial and time variation of the unknown heat flux.

## 6. ACKNOWLEDGEMENTS

This work was supported by CNPq, CAPES and FAPERJ.

## 7. REFERENCES

Alifanov, O. M., Inverse Heat Transfer Problems, Springer-Verlag, New York, 1994.

Alifanov, O. M., Artyukhin, E. and Rumyantsev, A., Extreme Methods for Solving Ill-Posed Problems with Applications to Inverse Heat Transfer Problems, Begell House, New York, 1995.

Beck, J. V. and Arnold, K. J., Parameter Estimation in Engineering and Science, Wiley Interscience, New York, 1977.

Beck, J. V., Blackwell, B. and St. Clair, C. R., Inverse Heat Conduction: Ill-Posed Problems, Wiley Interscience, New York, 1985.

Calvetti, D. and Somersalo, E., Introduction to Bayesian Scientific Computing. Ten Lectures on Subjective Computing, Springer, Maryland, MD, 2007.

Dulikravich, G. S. and Martin, T. J., "Inverse Shape and Boundary Condition Problems and Optimization in Heat Conduction", Chapter 10 in Advances in Numerical Heat Transfer, 1, 381-426, Minkowycz, W. J. and Sparrow, E. M. (eds.), Taylor and Francis, New York, 1996.

Hadamard, J., Lectures on Cauchy's Problem in Linear Differential Equations, Yale University Press, New Haven, CT, 1923.

Kaipio, J. and Somersalo, E., Statistical and Computational Inverse Problems, Applied Mathematical Sciences 160, Springer-Verlag, Berlin, Deutschland, 2004.

Kanevce, L., Kanevce, G. and Angelevski, "Comparison of Two-kinds of Experiments for Estimation of Thermal Properties of Ablative Composite", Paper EXP01, 3rd Int. Conf. on Inverse Problems in Engineering: Theory and Practice, pp. 473-480, Port Ludlow, WA, 1999.

Kurpisz, K. and Nowak, A. J., Inverse Thermal Problems, WIT Press, Southampton, UK, 1995.

Mota, C. A. A., Mikhailov, M. D., Orlande, H. R. B. and Cotta, R. M., Identification of Heat Flux Imposed by an Oxyacetylene Torch, 10th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference, Albany, New York, August 29 – September 2. , 2004

Murio, D. A., The Mollification Method and the Numerical Solution of Ill-Posed Problems, Wiley Interscience, New York, 1993.

Oliveira, A. P. and Orlande, H. R. B., "Estimation of the heat flux at the surface of ablating materials", Inverse Problems in Engineering Mechanics III, M. Tanaka and G. S.Dulikravich (editors), Elsevier, Amsterdam, Netherlands, pp. 39-48, 2002.

Oliveira, A. P. and Orlande, H. R. B., "Estimation Of The Heat Flux At The Surface Of Ablating Materials By Using Temperature And Surface Position Measurements", Inverse Problems In Science And Engineering, v. 12, n. 5, pp. 563-577, 2004.

Ozisik, M. N. and Orlande, H. R. B., Inverse Heat Transfer: Fundamentals and Applications, Taylor and Francis, New York, 2000.

Ruuskanen, A.R., Seppänen, A., Duncan, S., Somersalo, E., Kaipio, J.P., Using process tomography as a sensor for optimal control, Applied Numerical Mathematics, v. 56, pp. 37-54, 2006.

Tikhonov, A. N. and Arsenin, V. Y., Solution of Ill-Posed Problems, Winston & Sons, Washington, DC, 1977.

Trujillo, D. M. and Busby, H. R., Practical Inverse Analysis in Engineering, CRC Press, Boca Raton, Florida, 1997.

Woodbury, K., Inverse Engineering Handbook, CRC Press, Boca Raton, Florida, 2002.

Zabaras, N., Inverse Problems in Heat Transfer, chapter 17, in the Handbook of Numerical Heat Transfer, 2nd Edt., John Wiley & Sons, (W.J. Minkowycz, E.M. Sparrow, J. Y. Murthy, editors.), New York, New York, 2004.