A 3D ANALYTICAL SOLUTION OF THE ADVECTION-DIFFUSION EQUATION APPLIED TO POLLUTANT DISPERSION IN ATMOSPHERE

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Abstract. *In this work we present a new three-dimensional analytical approach for the solution of the advection-diffusion equation to simulate the pollutant dispersion in the atmospheric boundary layer. This goal is reached applying the Generalized Integral Laplace Transform Technique considering variable eddy diffusivities and wind profiles in the considered equation. No approximation is made along the solution derivation so that is an exact solution except for the round-off error. The first simulations and comparisons with experimental data are presented. This new methodology is a promissing result because it may be used for quantitative and qualitative estimations of pollutant distribution.*

Keywords*: Analytical Solution, Integral Transform, Advection-Diffusion Equation, Atmospheric Dispersion*

1. INTRODUCTION

The advection-diffusion equation has been largely applied in the field of air pollution as well, heat and mass transfer problems. Exists a vast literature regarding the issue of numerical solution, but the analytical approaches are scarce and only for specialized problems of pollutant dispersion simulation in atmosphere, where strong assumptions regarding the eddy diffusivity coefficient and wind profile, except for some stationary problems. Among them we mention the works of Rounds (1955), Smith (1957), Scriven and Fischer (1975), Demuth (1978), van Ulden (1978), Nieuwstadt and de Haan (1981), Tagliazucca et al. (1985), Tirabassi (1989), Tirabassi and Rizza (1994), Sharan et al., (1996), Lin and Hildemann (1997), Tirabassi (2003).

In the last decade appeared in the literature an analytical solution for more realistic air pollution problems, valid for any wind profile and eddy diffusivity variable with the height, solving the 2D advection-diffusion equation by the Generalized Integral Laplace Transform Technique (GILTT) (Moreira et al., 2009). Recently was developed a semi-analytical solution for the 3D advection-diffusion equation combining the GILTT with the Advection-Diffusion Multilayer method (Costa et al., 2006). This solution is based on a discretization of the Atmospheric Boundary Layer (ABL) in N sub-layers where in each sub-layer the advection-diffusion equation is solved by the Laplace transform technique, considering an average value for the eddy diffusivity and wind speed profiles.

In this work, pursuing the task of searching analytical solutions, we solve analytically, the 3D advection-diffusion equation in Cartesian geometry using the GILTT method, in order to avoid the Atmospheric Boundary Layer discretization. For such, the pollutant concentration is expanded in a double series of eigenfunctions attained from an auxiliary Sturm-Liouville problem. Replacing this expansion in the original equation and taking moments we come out with linear second order matrix differential equations. Applying the order reduction and diagonalization techniques we solve the resulting first order linear matrix differential equation by the Laplace Transform method. Once the transformed problem is solved, the solution of the advection-diffusion equation is well determined by the mentioned double series expansion. Showing the existence of the solution, the Cauchy-Kowalesky theorem (Courant and Hilbert, 1989) guarantees the uniqueness. To our knowledge, analytical solution for this kind of problem doesn't exist in the literature.

Numerical results and comparison with the Copenhagen experimental data are also presented.

To reach this goal, we outline the paper as follows: in section 2, we report in detail the derivation of the GILTT solution for the three-dimensional advection-diffusion equation in Cartesian geometry. In section 3 the first numerical results and the comparison with the experimental data are presented, and finally in section 4, is discussed the principal aspects of this method and conclusions.

2. THE ANALYTICAL SOLUTION

The advection-diffusion equation of air pollution in the atmosphere is essentially a statement of conservation of the suspended material. The concentration turbulent fluxes are assumed to be proportional to the mean concentration gradient which is known as Fick-theory. This assumption, combined with the continuity equation, leads to the steady state advection-diffusion equation (Blackadar, 1997):

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}+w\frac{\partial c}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial c}{\partial z}\right) \quad (1)$$

where c denotes the average concentration, K_x, K_y, K_z and u, v, w are the Cartesian components of eddy diffusivity and wind profile, respectively and $0<z<h, 0<y<L_y,\ 0<x<L_x$, where h is the height of the ABL and L_x, L_y are far away from the source. Here we consider that the eddy diffusivities and wind profile have a continuous dependence on the z variable. The x-axis of the Cartesian coordinate system is aligned in the direction of the actual wind, the y-axis is oriented in the horizontal crosswind direction, and the z-axis is chosen vertically upwards.

In order to solve the Eq. (1) we included the following boundary conditions:

$$\frac{\partial c}{\partial z}(x,y,0)=\frac{\partial c}{\partial z}(x,y,h)=\frac{\partial c}{\partial y}(x,0,z)=\frac{\partial c}{\partial y}(x,L_y,z)=\frac{\partial c}{\partial x}(L_x,y,z)=0 \quad (1a)$$

$$uc(0,y,z)=Q\delta(y-y_0)\delta(z-H_s) \quad (1b)$$

where δ is the Dirac delta function and H_s is the height source. A continuous point source of constant emission rate Q is assumed.

Now we are in position to solve problem (1), for the first time, by the GILTT approach. For this end, following the same procedure adopted for the two-dimensional problems presented in Moreira et al. (2009), we expand the concentration in the ensuing series:

$$c(x,y,z)=\sum_{m=0}^{M}\sum_{n=0}^{N}c_{m,n}(x)Y_n(y)Z_m(z) \quad (2)$$

At this point, it is relevant to recall that the eigenfunctions $Y_n(y)=\cos\lambda_n y$ and $Z_m(z)=\cos\lambda_m z$ are solutions of the well known Sturm-Liouville problem appearing in the y and z variables with respective eigenvalues $\lambda_n=\frac{n\pi}{L_y}$ and $\lambda_m=\frac{m\pi}{h}$ (n, m = 0,1,2,…).

To determine the unknown coefficient $c_{m,n}(x)$ we replace Eq. (2) in Eq. (1). This procedure leads to:

$$\begin{aligned}\sum_{m=0}^{M}\sum_{n=0}^{N}\Big(&-u\frac{\partial c_{m,n}(x)}{\partial x}Y_n(y)Z_m(z)-v\,c_{m,n}(x)\,Y_n{}'(y)Z_m(z)\;-w\,c_{m,n}(x)\,Y_n(y)Z_m{}'(z)\;+\\ &+K_x\frac{\partial^2 c_{m,n}(x)}{\partial x^2}Y_n(y)Z_m(z)+K_x{}'\frac{\partial c_{m,n}(x)}{\partial x}\,Y_n(y)Z_m(z)\;+\\ &+K_y\,c_{m,n}(x)\,Y_n{}''(y)Z_m(z)\;+K_y{}'\,c_{m,n}(x)\,Y_n{}'(y)Z_m(z)+\\ &+K_z\,c_{m,n}(x)\,Y_n(y)Z_m{}''(z)\;+K_z{}'\,c_{m,n}(x)\,Y_n(y)Z_m{}'(z)\Big)=0\end{aligned} \quad (3)$$

Here, prime and double prime means first and second derivative respectively.

Taking moments, we mean multiplying equation (3) by $Z_i(z)Y_j(y)$, integrating in the domain $0<z<h$ and $0<y<L_y$, using the orthogonality property of the eigenfunctions, we promptly obtain:

$$\begin{aligned}\sum_{m=0}^{M}\sum_{n=0}^{N}\Big[&\frac{\partial^2 c_{m,n}(x)}{\partial x^2}\int_0^h\int_0^{L_y}K_x\,Y_n(y)Y_j(y)Z_m(z)Z_i(z)dydz+\\ &+\frac{\partial c_{m,n}(x)}{\partial x}\int_0^h\int_0^{L_y}(K_x'-u)\,Y_n(y)Y_j(y)Z_m(z)Z_i(z)dydz+\\ &+c_{m,n}(x)\Big(\int_0^h\int_0^{L_y}(K_y'-v)\,Y_n{}'(y)Y_j(y)Z_m(z)Z_i(z)dydz+\\ &+\int_0^h\int_0^{L_y}(K_z'-w)\,Y_n(y)Y_j(y)Z_m{}'(z)Z_i(z)dydz+\\ &-\lambda_n^2\int_0^h\int_0^{L_y}K_y\,Y_n(y)Y_j(y)Z_m(z)Z_i(z)dydz+\\ &-\lambda_m^2\int_0^h\int_0^{L_y}K_z\,Y_n(y)Y_j(y)Z_m(z)Z_i(z)dydz\Big)\Big]=0\end{aligned} \quad (4)$$

which can be recast in matrix form like:

$$B_1 Y''(x) + B_2 Y'(x) + B_3 Y(x) = 0 \tag{5}$$

Here *Y(x)* is the vector whose components are { $c_{m,n}(x)$ } and B_1, B_2, B_3 are the matrices whose entries are respectively:

$$(b_1)_{m,n,i,j} = \int_0^h \int_0^{L_y} K_x\, Y_n(y) Y_j(y) Z_m(z) Z_i(z) dydz$$

$$(b_2)_{m,n,i,j} = \int_0^h \int_0^{L_y} (K_x' - u)\, Y_n(y) Y_j(y) Z_m(z) Z_i(z) dydz$$

$$(b_3)_{m,n,i,j} = \int_0^h \int_0^{L_y} (K_y' - v)\, Y_n^{'(y)Y_j(y)Z_m(z)Z_i(z)} dydz \; + \int_0^h \int_0^{L_y} (K_z' - w)\, Y_n(y) Y_j(y) Z_m{}'(z) Z_i(z) dydz + \\ -\lambda_n^2 \int_0^h \int_0^{L_y} K_y\, Y_n(y) Y_j(y) Z_m(z) Z_i(z) dydz - \lambda_m^2 \int_0^h \int_0^{L_y} K_z\, Y_n(y) Y_j(y) Z_m(z) Z_i(z) dydz$$

Equation (5) can be rewritten as:

$$Y''(x) + FY'(x) + GY(x) = 0 \tag{6}$$

where the matrix *F* is defined as $F = B_1^{-1} B_2$ and matrix *G* as $G = B_1^{-1} B_3$. The integrals appearing in B_1, B_2 and B_3 are solved numerically via Gauss Legendre Quadrature.

Similar procedure leads to the boundary conditions: $Y(0) = c_{m,n}(0) = QZ_i(H_s)Y_j(y_0)A^{-1}$ and $Y'(L_x) = c_{m,n}{}'(L_x) = 0$, where A^{-1} is the inverse of matrix A having the entry:

$$(a)_{m,n,i,j} = \int_0^h \int_0^{L_y} u\, Y_n(y) Y_j(y) Z_m(z) Z_i(z) dydz. \tag{6a}$$

Applying the standard procedure of order reduction to Eq. (6) we come out with the result:

$$Z^{'}(x) + HZ(x) = 0\,, \tag{7}$$

Here *Z*(*x*) is the vector $Z(x) = col(Z_1(x), Z_2(x))$ and the matrix *H* has the block form $H = \begin{bmatrix} 0 & -I \\ G & F \end{bmatrix}$, subjected to the respectively boundary conditions for the vector components:

$$Z_1(0) = QZ_i(H_s)Y_j(y_0)A^{-1} \text{ and } Z_2(L_x) = 0\,. \tag{7a}$$

The transformed problem (7) is solved applying the Laplace Transform technique and diagonalization (Buske et al., 2007; Moreira et al., 2009). Applying the Laplace transform to Eq. (7), we obtain:

$$s\overline{Z}(s) + H\overline{Z}(s) = Z(0)\,, \tag{8}$$

where $\overline{Z}(s)$ denotes the Laplace Transform of the vector $Z(x)$. Observing that the matrix *H* has distinct eigenvalues, we can write:

$$H = X\, D\, X^{-1}. \tag{9}$$

Here *D* is the diagonal matrix of eigenvalues of the matrix *H*, *X* is the matrix of the respective eigenfunctions and X^{-1} it is the inverse. Indeed, replacing Eq. (9) in Eq. (8) and using standard algebraic operations we obtain:

$$\overline{Z}(s) = X\left(sI + D\right)^{-1} X^{-1} Z(0)\,. \tag{10}$$

The elements of the matrix *(sI + D)* have the form *{s +* d_i*}* where d_i are the eigenvalues of the matrix H given in Eq. (7). It is known that the inverse of a diagonal matrix is the inverse of their elements, in other words, the elements of *(sI*

$+ D)^{-1}$ are $\frac{1}{s+d_i}$ whose transformed inverse of Laplace is $e^{-d_i x}$. $\exp(Dx)$ being the diagonal matrix with elements $e^{-d_i x}$ the final solution is given by:

$$Z(x) = X\exp(Dx)X^{-1}Z(0) = M(x)\xi, \tag{11}$$

where $M(x) = X\exp(Dx)$ and $\xi = X^{-1}Z(0)$. By the choice of a new arbitrary constant vector ξ, we avoid the inversion of *X*. To this point we must notice that all components of the arbitrary constant vector ξ are unknown.

As mentioned, to construct the solution of problem (7) we need to apply the condition $Z_2(L_x) = 0$ and also $Z_1(0) = QZ_i(H_s)Y_j(y_0)A^{-1}$ because the new constant vector definition (ξ). For such we recast the solution given by Eq. (11) like:

$$\begin{pmatrix} Z_1(x) \\ Z_2(x) \end{pmatrix} = \begin{pmatrix} M_{11}(x) & M_{12}(x) \\ M_{21}(x) & M_{22}(x) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}. \tag{12}$$

To determine the unknown vector ξ, we solve the following linear system resulting from the application of the boundary conditions to the solution appearing in Eq. (12), namely

$$\begin{pmatrix} M_{11}(0) & M_{12}(0) \\ M_{21}(L_x) & M_{22}(L_x) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} Z_1(0) \\ Z_2(L_x) \end{pmatrix}. \tag{13}$$

Once the constant vector is obtained, the solution for the pollutant concentration is well determined and given by the classical result:

$$c(x,y,z) = \sum_{m=0}^{M}\sum_{n=0}^{N} c_{m,n}(x)Y_n(y)Z_m(z) \tag{14}$$

where $Y_n(y)$ and $Z_m(z)$ are obtained from the solution of the Sturm-Liouville problems ($Y_n(y) = \cos\lambda_n y$ and $Z_m(z) = \cos\lambda_m z$; $\lambda_n = \frac{n\pi}{L_y}$ and $\lambda_m = \frac{m\pi}{h}$ $(n,m = 0,1,2,\ldots)$.) and $c_{m,n}(x)$ comes from the solution of the transformed problem (Eq. 7).

At this point it is important to stress that the crosswind integration of the three-dimensional advection-diffusion equation leads to the well known solution (Wortmann et al., 2005; Buske et al. 2007; Tirabassi et al., 2009; Moreira et al., 2009):

$$c^y(x,z) = \sum_{m=0}^{M} c_m(x)Z_m(z) \tag{15}$$

3. PERFORMANCE AGAINST EXPERIMENTAL DATA

The advection-diffusion equation has been widely applied in operational atmospheric dispersion models to predict the mean concentrations of contaminants in the ABL. In principle, it is possible to obtain from this equation a theoretical model of dispersion from a continuous point source, given appropriate physical boundary and initial conditions, plus a knowledge of the mean wind velocity and concentration turbulent fluxes. This last aspect could be interpreted as the physical connection with the actual atmosphere. In fact the reliability of each model strongly depends on the way as turbulent parameters are calculated and related to the current understanding of the ABL.

The present work considers the turbulence parameterisation scheme suggested by Degrazia et al. (2002). In terms of the convective scaling parameters, the eddy diffusivity can be formulated as:

$$K_\alpha = \frac{0.583 w_* h c_i \psi^{2/3}(z/h)^{4/3} X^* \left[0.55(z/h)^{2/3} + 1.03 c_i^{1/2}\psi^{1/3}\left(f_m^*\right)_i^{2/3} X^*\right]}{\left[0.55(z/h)^{2/3}\left(f_m^*\right)_i^{1/3} + 2.06 c_i^{1/2}\psi^{1/3}\left(f_m^*\right)_i X^*\right]^2} \tag{16}$$

where X^* is the adimensional distance, $c_{v,w}$=0.36, c_u=0.3, $(f_m^*)_i$ is the normalized frequency of the spectral peak. According to Kaimal et al. (1976) and Caughey (1982):

$$(f_m^*)_i = \frac{z}{(\lambda_m)_i} \tag{17}$$

with $(\lambda_m)_u = (\lambda_m)_v = 1.5h$ and $(\lambda_m)_w = 1.8h[1-\exp(-4z/h) - 0.0003\exp(8z/h)]$ where $(\lambda_m)_i$ is the peak wavelength of the turbulent velocity spectra. The dissipation function used is the mean value $\psi = 0.4$ (Caughey, 1982).

The expression used for evaluating mean wind is the power law profile (Panofsky and Dutton, 1984):

$$\frac{\bar{u}_z}{\bar{u}_1} = \left(\frac{z}{z_1}\right)^n \tag{18}$$

where $\bar{u}_z$ and $\bar{u}_1$ are the mean wind velocity at the heights z and z_1, while n is an exponent that is related to the intensity of turbulence (Irwin, 1979).

In order to illustrate the aptness of the discussed formulation to simulate contaminant dispersion in the ABL, we evaluate the performance of the discussed solutions against experimental ground-level concentration using the Copenhagen dispersion experiment. This experiment was carried out in the northern part of Copenhagen and described by Gryning and Lyck (1984). It consisted of tracer released without buoyancy from a tower at a height of 115 m, and collection of tracer sampling units at the ground-level positions at the maximum of three crosswind arcs. The sampling units were positioned at two to six kilometers from the point of release. The site was mainly residential with a roughness length of the 0.6 m.

The simulations presented here assume that $v = w = 0$ and that the wind is higher than 2m/s. Table 1 shows the statistical analysis of the presented model compared with the data from experiments of Copenhagen for crosswind integrated and centerline concentrations considering that in the y direction the plume has a Gaussian distribution. The statistical indices are defined as (Hanna, 1989):

NMSE (normalized mean square error) = $\overline{(C_o - C_p)^2} / \overline{C_p}\,\overline{C_o}$,

FA2 = fraction of data (%, normalized to 1) for $0.5 \le (C_p / C_o) \le 2$,

COR (correlation coefficient) = $\overline{(C_o - \overline{C_o})(C_p - \overline{C_p}} / \sigma_o \sigma_p$,

FB (fractional bias) = $\overline{C_o} - \overline{C_p} / 0.5(\overline{C_o} + \overline{C_p})$,

FS (fractional standard deviations) = $(\sigma_o - \sigma_p) / 0.5(\sigma_o + \sigma_p)$,

where the subscripts o and p refer to observed and predicted quantities, respectively, and the overbar indicates an averaged value. The statistical index FB says if the predicted quantities underestimate or overestimate the observed ones. The statistical index NMSE represents the model values dispersion in respect to data dispersion. The best results are expected to have values near to zero for the indices NMSE, FB and FS, and near to 1 in the indices COR and FA2.

Table 1. Statistical evaluation of model results for the approximated steady-state, three-dimensional solution for Fickian flows, Copenhagen experiment (centerline concentrations.

Model	NMSE	COR	FA2	FB	FS
GILTT 2D	0.06	0.92	1.00	-0.14	-0.02
GILTT 3D	0.33	0.80	0.87	0.28	0.09

Analysing the statistical indices we notice that the model simulate satisfactorily the observed concentrations, regarding the NMSE, FB and FS values relatively near to zero and COR relatively near to 1.

Figure 1 show the observed and predicted scatter diagram of centerline ground-level concentrations. In this respect, it is important to note that the model reproduced fairly well the observed concentration for the Copenhagen experiment.

4. CONCLUSIONS

We presented an analytical approach to solve the three-dimensional advection-diffusion equation using integral transform technique. Moreover the Cauchy-Kowaleski theorem guarantees the existence and uniqueness of the solution, because no approximation is made along the solution derivation except for the series truncation of the solution.

Analytical solutions of equations are of fundamental importance in understanding and describing physical phenomena, since they are able to take into account all the parameters of a problem, and investigate their influence. Moreover, when using models, while they are rather sophisticated instruments that ultimately reflect the current state of knowledge on turbulent transport in the atmosphere, the results they provide are subject to a considerable margin of error. This is due to various factors, including in particular the uncertainty of the intrinsic variability of the atmosphere. Models, in fact, provide values expressed as an average, i.e., a mean value obtained by the repeated performance of many experiments, while the measured concentrations are a single value of the sample to which the ensemble average provided by models refer. This is a general characteristic of the theory of atmospheric turbulence and is a consequence of the statistical approach used in attempting to parameterize the chaotic character of the measured data. An analytical solution can be useful in evaluating the performances of numerical model (that solve numerically the advection diffusion equation) that could compare their results, not only against experimental data but, in an easier way, with the solution itself in order to check numerical errors without the uncertainties presented above.

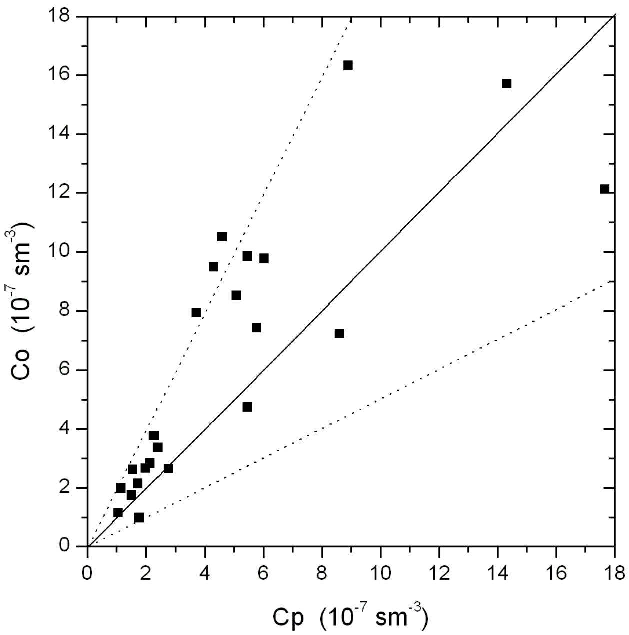

Figure 1. Scatter diagram for the solution (13) together with Eq. (38): Observed (Co) and predicted (Cp) centerline ground-level concentration normalised with emission rate (*c/Q*). Data between lines correspond to ratio Co/Cp $\in [0.5,2]$.

We will step forward checking the new model to other stability conditions, apply to different parameterizations and compare the results with other experimental data sets.

5. ACKNOWLEDGEMENTS

The authors thank to CNPq and CAPES for the financial support of this work.

6. REFERENCES

Blackadar, A.K., 1997, Turbulence and diffusion in the atmosphere: lectures in Environmental Sciences, Springer-Verlag, 185pp.

Buske, D., Vilhena, M.T., Moreira, D.M. and Tirabassi, T., 2007. Simulation of pollutant dispersion for low wind conditions in stable and convective planetary boundary layer. Atmos. Environ. 41, 5496-5501.

Caughey, S.J., 1982. Observed characteristics of the atmospheric boundary layer. Atmospheric turbulence and air pollution modeling, Edited by F.T.M Nieustadt and H. van Dop. Reidel, Boston.

Costa, C.P., Vilhena, M.T., Moreira, D.M. and Tirabassi, T., 2006. Semi-analytical solution of the steady three-dimensional advection-diffusion equation in the planetary boundary layer. Atmos. Environ. 40 (29), 5659-5669.

Courant, R. and Hilbert, D., 1989. Methods of Mathematical Physics. John Wiley & Sons.

Degrazia, G. A., Moreira, D. M., Campos, C. R. J., Carvalho, J. C. and Vilhena, M. T., 2002. Comparison between an integral and algebraic formulation for the eddy diffusivity using the Copenhagen experimental dataset. Il Nuovo Cimento 25C, 207–218.

Demuth, C., 1978. A contribution to the analytical steady solution of the diffusion equation for line sources. Atmos. Environ. 12, 1255-1258.

Gryning, S.E. and Lyck, E. 1984. Atmospheric dispersion from elevated source in an urban area: comparison between tracer experiments and model calculations. J. Appl. Meteor. 23, 651-654.

Hanna, S.R., 1989. Confidence limit for air quality models as estimated by bootstrap and jacknife resampling methods. Atmos. Environ. 23, 1385-1395.

Irwin, J.S., 1979. A theoretical variation of the wind profile power-low exponent as a function of surface roughness and stability, Atm. Environ. 13, 191-194.

Kaimal, J.C., Wyngaard, J.C., Haugen, D.A., Coté, O.R., Izumi, Y., Caughey, S.J. and Readings, C.J., 1976. Turbulence structure in the convective boundary layer. J. Atmos. Sci. 33, 2152-2169.

Lin, J.S. and Hildemann, L.M., 1997. A generalised mathematical scheme to analytically solve the atmospheric diffusion equation with dry deposition. Atmos. Environ. 31, 59-71.

Moreira, D.M., Vilhena, M.T., Buske, D. and Tirabassi, T.,2009 . The state-of-art of the GILTT method to simulate pollutant dispersion in the atmosphere. Atmos. Research 92, 1-17.

Nieuwstadt F.T.M. and de Haan B.J., 1981. An analytical solution of one-dimensional diffusion equation in a non.stationary boundary layer with an application to inversion rise fumigation. Atmos. Environ. 15, 845-851.

Panofsky, H.A. and Dutton, J.A., 1984. Atmospheric Turbulence. John Wiley & Sons, New York.

Rounds W., 1955. Solutions of the two -dimensional diffusion equation. Trans. Am. Geophys. Union 36, 395-405.

Scriven R.A. and Fisher B.A., 1975. The long range transport of airborne material and its removal by deposition and washout-II. The effect of turbulent diffusion. Atmos. Environ. 9, 59-69.

Sharan, M., Singh, M.P. and Yadav, A.K., 1996. A mathematical model for the atmospheric dispersion in low winds with eddy diffusivities as linear function of downwind distance. Atmos. Environ. 30, 1137-1145.

Smith F.B., 1957. The diffusion of smoke from a continuous elevated poinr source into a turbulent atmosphere. J. Fluid Mech. 2, 49-76.

Tagliazucca, M., Nanni, T. and Tirabassi, T., 1985. An analytical dispersion model for sources in the surface layer. Nuovo Cimento 8C, 771-781.

Tirabassi, T., 1989. Analytical air pollution and diffusion models. Water, Air and Soil Pollution 47, 19-24.

Tirabassi T. and Rizza U. 1994. Applied dispersion modelling for ground-level concentrations from elevated sources. Atmos. Environ. 28, 611-615.

Tirabassi, T., 2003. Operational advanced air pollution modeling. PAGEOPH 160 (1-2), 5-16.

Tirabassi, T., Tiesi, A., Buske, D., Moreira, D.M. and Vilhena, M.T., 2009. Some characteristics of a plume from a point source based on analytical solution of the two-dimensional advection-diffusion equation. Atmos. Environ. 43, 2221-2227.

van Ulden, A.P., 1978. Simple estimates for vertical diffusion from sources near the ground. Atmos. Environ. 12, 2125-2129.

Wortmann, S., Vilhena, M.T., Moreira, D.M. and Buske, D., 2005. A new analytical approach to simulate the pollutant dispersion in the PBL. Atmos. Environ. 39, 2171-2178.

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