A SPARSE GRID METHOD APPLIED TO STOCHASTIC FLUID-STRUCTURE INTERACTION

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Abstract. *The recent advances in computational power has enabled high-performance computing as an effective tool for analysis and design in Engineering and Applied Sciences. One major issue to be deeper understood and controlled is how uncertainties in the input data impacts the reliability of the results obtained through computer simulations. Specifically in the present work, the focus relies on hydro-ship dynamics in the context of floating offshore structures. Particular emphasis is placed on investigating uncertainty propagation in the nonlinear response of flow-structures interactions, (Xiu* et al. *2005), (Witteveen* et al. *2008). It is important to remind that waves and currents, major agents in the dynamics of the floating structures, are usually modeled as random processes. Therefore, stochastic modeling seems to offer an appropriate framework to tackle the external forces and uncertainties in the data, like, for instance, damping and boundary conditions. In the present work, a sparse grid stochastic collocation method, (Ganapathysubramanian* et al. *2007) is applied in a prototype problem of a single oscillator excited by means of an interaction force corresponding to the Morison formula, (Xiu* et al. *2002), containing the maritime wind velocity. Uncertainty in the system parameters are taken into account and the convergence of the method is analyzed along a number of numerical experiments.*

Keywords: *stochastic modeling, uncertainty propagation, flow-structures interactions*

1. INTRODUCTION

The increasing complexity involved in engineering systems has been, frequently, tackled with the use of sophisticated computational models. That, from the decision makers standpoint, requires the use of robust and reliable numerical simulators. Often, the reliability of those simulations is disrupted by the inexorable presence of uncertainty in the model data, such as inexact knowledge of system forcing, initial and boundary conditions, physical properties of the medium, as well as parameters in constitutive equations. These situations underscore the need for efficient uncertainty quantification (UQ) methods for the establishment of confidence intervals in computed predictions, the assessment of the suitability of model formulations, and/or the support of decision-making analysis, (Babuska *et al.* 2004).

The traditional statistical tool for uncertainty quantification within the realm of Engineering is the Monte Carlo method, (Elishkoff, 2003). This method requires, first, the generation of an ensemble of random realizations associated to the uncertain data, and then it employs deterministic solvers repetitively to obtain the ensemble of results. The ensemble results should be processed to estimate the mean and standard deviation of the final results. The implementation the Monte Carlo is straightforward, but its convergence rate is very slow (proportional to the inverse of the square root of the realization number) and often infeasible due the large CPU time needed to run the model in question.

Other technique that has been applied recently is the so called Stochastic Galerkin Method (SG), which employs Polinomial Chaos expansions to represent the solution and inputs to stochastic differential equations, (Babuska *et al.* 2004). A Galerkin projection minimizes the error of the truncated expansion and the resulting set of coupled equations is solved to obtain the expansion coefficients. SG methods are highly suited to dealing with ordinary and partial differential equations, even in the case of nonlinear dependence on the random data. The main drawback with SG relies on its need of solving a system of coupled equations that requires efficient and robust solvers and, most importantly, the modification of existing deterministic code. This last issue entails difficulties on using commercial or already in use codes. A non-intrusive method, referred to as Stochastic Collocation (SC), (Xiu *et al.* 2005), arises towards addressing this point. SC methods are built on the combination of interpolation methods and deterministic solvers, likely Monte Carlo. A deterministic problem is solved in each point of an abstract random space. Similarly to SG methods, SC methods achieve fast convergence when the solution posses sufficient smoothness in random space.

Specifically in the present work, the focus relies on hydro-ship dynamics in the context of floating offshore structures. Particular emphasis is placed on investigating uncertainty propagation in the nonlinear response of flow-structures interactions, (Xiu *et al.*2002), (Witteveen *et al.* 2008). It is important to remind that waves and currents, major agents in the dynamics of the floating structures, are usually modeled as random processes. Therefore, stochastic modeling seems to offer an appropriate framework to tackle the external forces and uncertainties in the data, like, for instance, damping and boundary conditions.

Here, the flow-structure interaction is modeled in a simple way focusing the assessment of an SC method as an effective tool for uncertainty quantification. The interaction is introduced by means the Morison's formula, which represents a challenge, despite the simplicity of the model itself, as far as the input is a nonlinear function of the random variables. Those variables represent the phase angle which inherent to the time series description of the wave induced motion.

2. ABSTRACT PROBLEM DEFINITION

To quantify the uncertainty in a system of differential equations we adopt a probabilistic approach and define a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Where Ω is the event space, $\mathcal{F} \subset 2^{\Omega}$ is the σ-algebra of subsets in Ω and $\mathcal{P} : \mathcal{F} \rightarrow [0, 1]$ is the probability measure. Utilizing this framework, the uncertainty in a model is introduced by representing the model input data as random field.

2.1 Governing Equations

Consider the general differential equation defined on a d-dimensional bounded domain $\mathcal{D} \subset \mathbb{R}^d$, $d = 1, 2, 3$ with boundary $\partial\mathcal{D}$. The problem consists on finding a stochastic function, $\mathbf{u} \equiv \mathbf{u}(\omega, \mathbf{x}) : \Omega \times \mathcal{D} \longrightarrow \mathbb{R}$, such that for $\mathcal{P}$-almost everywhere $\omega \in \Omega$, the following equation holds:

$$\mathcal{L}(\omega, \mathbf{x}; \mathbf{u}) = f(\omega, \mathbf{x}), \qquad \mathbf{x} \in \mathcal{D} \tag{1}$$

$$\mathcal{B}(\omega, \mathbf{x}; \mathbf{u}) = g(\omega, \mathbf{x}), \qquad \mathbf{x} \in \partial\mathcal{D} \tag{2}$$

where $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $d \geq 1$, are the space coordinates in $\mathbb{R}^d$, $\mathcal{L}$ is a linear or non linear differential operator and $\mathbf{u}(\omega) = (u_1(\omega), \ldots u_i(\omega)) \in \mathbb{R}^i$, $i \geq 1$, are unknown solutions, (Xiu *et al.* 2005).

To solve equations (1) and (2) sometimes it is necessary to reduce the infinite dimensional probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to a finite dimensional one. This can be accomplished by characterizing the probability space by a finite number of random variables. Thus, employing any truncated spectral expansion of the stochastic process in the probability space, one characterize the random inputs by a set of N random variables $y = (Y_1(\omega), \ldots, Y_N(\omega))$ and rewrite the random inputs as,

$$\mathcal{L}(\omega, \mathbf{x}; \mathbf{u}) = \mathcal{L}(Y^1(\omega), \ldots, Y^N(\omega), \mathbf{x}; \mathbf{u}), \qquad f(\omega, \mathbf{x}) = f(Y^1(\omega), \ldots, Y^N(\omega), \mathbf{x}), \tag{3}$$

Hence, following the Dob-Dynkin lemma, (Oskendal 2005), the solution of (1) and (2) can be described by the same set of random variables $\{Y^i(\omega)\}_{i=1}^N$, that reduce the infinite dimensional probability space to a N-dimensional space, i.e.,

$$\mathbf{u}(\omega, \mathbf{x}) = \mathbf{u}(Y^1(\omega), \ldots, Y^N(\omega), \mathbf{x}) \tag{4}$$

Now assuming that $\{Y^i\}_{i=1}^N$ are independent random variables with probability density functions $\rho_i : \Gamma^i \rightarrow \mathbb{R}^+$, and their images $\Gamma^i \equiv Y^i(\Omega)$ bounded intervals in $\mathbb{R}$ for $i = 1, \ldots, N$, the joint probability density of $y \equiv (Y^1, \ldots, Y^N)$ hold,

$$\rho(y) = \prod_{i=1}^{N} \rho_i(Y^i) \qquad \forall y \in \Gamma, \tag{5}$$

and the space support,

$$\Gamma \equiv \prod_{i=1}^{N} \Gamma^i \subset \mathbb{R}^N. \tag{6}$$

This allow us to rewrite (1) and (2) as a $(N + d)$ dimensional differential equation as following,

$$\mathcal{L}(y, \mathbf{x}; \mathbf{u}) = f(y, \mathbf{x}), \qquad (y, \mathbf{x}) \in \Gamma \times \mathcal{D} \tag{7}$$

$$\mathcal{B}(y, \mathbf{x}; \mathbf{u}) = g(y, \mathbf{x}), \qquad (y, \mathbf{x}) \in \Gamma \times \partial\mathcal{D} \tag{8}$$

with N dimensionality of the random space Γ and d the dimensionality of the physical space $\mathcal{D}$.

Now, we define a (finite dimensional) subspace $V_\Gamma \subset L^2_\rho(\Gamma)$, as the space of all square integrable function in Γ with respect to the measure $\rho(y)dy$. Hence, like others deterministic problems we can seek $\mathbf{u}_v(y, x) \in V_\Gamma(y)$ solving (7) and (8) for the following the weak formulation,

$$\int_\Gamma \rho(y)\mathcal{L}(y, \mathbf{x}, ; \mathbf{u}_v)v(y)dy = \int_\Gamma \rho(y)f(y, \mathbf{x})v(y)dy, \qquad \forall v(y) \in V_\Gamma, \mathbf{x} \in \mathcal{D} \tag{9}$$

$$\int_\Gamma \rho(y)\mathcal{B}(y, \mathbf{x}, ; \mathbf{u}_v)v(y)dy = \int_\Gamma \rho(y)g(y, \mathbf{x})v(y)dy, \qquad \forall v(y) \in V_\Gamma, \mathbf{x} \in \partial\mathcal{D}. \tag{10}$$

Thus becoming a deterministic problem in the physical domain $\mathcal{D}$ and can be solved by a common discretization technique as finite elements or finite volume method for example.

3. MONTE-CARLO METHOD

Monte Carlo simulation (MCS) is a simple way to solve complex problems, for this reason is one of the most widely used methods for solving stochastic differential equations and validate new methods. Briefly, a typical simulation consists in carrying out K experiments, it is independent of the number of random variables $\{Y^i(\omega)\}_{i=1}^N$, and then performing a statistical analysis of the output, finding the means and variances for example. The main problem of this method is that their convergence rate is relatively slow, (Elishkoff *et al.* 2008), and many times are often infeasible due to the large CPU time needed to run a simulation.

The typical sequence of steps to implement the Monte Carlo method in a problem is the following:

- Identify the random parameters and their probabilities densities.
- Generate k random realizations.
- Solve the deterministic problem (7) and (8) with $y_j = (Y_j^1 \dots Y_j^N)$ as following,

for $j = 1 : k$

$$\{Y_j^i\}_{i=1}^N \equiv \{Y_j^i(\omega_i)\}_{i=1}^N \tag{11}$$
$$\mathcal{L}(y_j, \mathbf{x}; \mathbf{u}_j) = f(y_j, \mathbf{x}) \qquad (y_j, \mathbf{x}) \in \Gamma \times \mathcal{D} \tag{12}$$
$$\mathcal{B}(y_j, \mathbf{x}; \mathbf{u}_j) = g(y_j, \mathbf{x}) \qquad (y_j, \mathbf{x}) \in \Gamma \times \partial\mathcal{D} \tag{13}$$

end

- Post-process the solutions $\mathbf{u}_j \equiv \mathbf{u}(y_j, \mathbf{x})$ i.e. evaluate the solution statistics, i.e.

$$\langle \mathbf{u} \rangle \equiv \mathbb{E}[\mathbf{u}] = \frac{1}{M} \sum_{j=1}^{M} \mathbf{u}_j \tag{14}$$

Where $\mathbb{E}[\mathbf{u}] = \int_\Gamma \rho(y)\mathbf{u}(y)dy$ is the expectation, on the other hand higher order statical moments can be obtained, but they often, require accurate values or a larger set of outputs.

4. STOCHASTIC COLLOCATION METHODS

The basic idea of this method is to have a finite element approximation for the spatial domain and approximate the multi-dimensional stochastic space using interpolation functions on a set of collocations points $\{Y^i\}_{i=1}^N$ in the support space $\Gamma \subset \mathbb{R}^N$. Stochastic Collocation methods have arisen to address some limitations of Stochastic Galerkin (SG) method, (Babuska *et al.* 2004). The main constraints of the SG is the complexity that increases rapidly with the number of entries, and requires the solution of coupled systems of equations, sometimes this is only possible by modifying the current deterministic code.

The advantage of Stochastic Collocation method is that combines the strengths of non-intrusive sampling. Similarly to Monte Carlo methods, SC requires only the solution of a set of decoupled equations, allowing the model to be treated as a black box and solved it with existing deterministic solvers. For example in 1D, SC are built upon using Lagrange interpolation that relies on a finite number of points $y_1 \dots y_M$, some real constants $b_1 \dots b_M$, and a sub-space $V_I \in \prod_N$. Moreover, the following condition is enforced for $l \in V_I$

$$l(y_j) = b_j, \qquad j = 1, \dots M \tag{15}$$

Where, the points $y_1 \dots y_M$ are called interpolation nodes and V_I interpolation space. Lagrange interpolation of a smooth function, $f : \mathbb{R}^N \to \mathbb{R}$, find a polynomial $\mathcal{I}(f) \in V_I$ such that $\mathcal{I}(f)(y_i) = f(y_i), \forall i = 1, \dots, M$. The polynomial approximation $\mathcal{I}(f)$ can be expressed by using the Lagrange interpolation polynomials, i.e.

$$\mathcal{I}(f)(y) = \sum_{k=1}^{M} f(y_k) L_k(y_i), \tag{16}$$

where, $L_i(y) \in V_I$, $L_i(y_j) = \delta_{ij}$, with $1 \leq i, j \leq M$, are the Lagrange polynomials.

By denoting,

$$\hat{\mathbf{u}}(y) \equiv \mathcal{I}\mathbf{u}(y) = \sum_{k=1}^{M} \mathbf{u}(y_k) L_k(y_i), \tag{17}$$

the collocation procedure to solve the stochastic equations (7) and (8) goes as follows,

$$R(\hat{\mathbf{u}}(y))_{y_k} = 0 \quad \forall k = 1 \dots M \tag{18}$$

where $R(\hat{\mathbf{u}}) = \mathcal{L}(\mathbf{u}) - f$ is the residual of (7). By using the property of Lagrange interpolation, we immediately obtain: for $k = 1 \dots M$

$$\mathcal{L}(y_k, \mathbf{x}; \mathbf{u}) = f(y_k, \mathbf{x}), \qquad (y_k, \mathbf{x}) \in \Gamma \times \mathcal{D} \tag{19}$$

$$\mathcal{B}(y_k, \mathbf{x}; \mathbf{u}) = g(y_k, \mathbf{x}), \qquad (y_k, \mathbf{x}) \in \Gamma \times \partial\mathcal{D} \tag{20}$$

Thus, the stochastic collocation method is equivalent to solving M deterministic problems (7) and (8), at each nodal point y_k, $k = 1, \dots, M$ in a given nodal set.

The computational complexity of the SC is solve M times a deterministic problem, where M is the total number of collocation points. For this reason we need to chose a nodal set with fewest possible number of points under prescribed accuracy requirement. Sparse Grids are presented as a effective option according to (Nobile *et al.* 2007). Hence, with a simple weighted sum of the value of the basis functions for all collocations points in the sparse grid. Essentially, a cubature of the interpolating function across the stochastic space, the integration is very straightforward because the weights corresponding to known nodal positions are computed a priori. The method allow us to obtain a visualization of the solution to the dependence on the random variables and also easily extract the mean and variance analytically as well its probability density function (PDF), leaving only the interpolation error.

As example, the mean can be evaluated by,

$$\mathbb{E}(\hat{\mathbf{u}})(\mathbf{x}) = \sum_{k=1}^{M} \mathbf{u}(y_k, x) \int_{\Gamma} L_k(y) \rho(y) dy \tag{21}$$

The evaluations of such expectations require explicit knowledge of the Lagrange interpolation polynomials $\{L_k(y)\}$, where for a given nodal set, the integrals of the polynomials can be determined numerically. In multivariate cases, such procedure can be cumbersome, but can be accomplished once for all at the pre-processing stage. The above described path has an straight extension to the multivariate case, in which the random space is no more 1D, by using tensor products of the one dimensional interpolations, (Xiu *et al.* 2005).

5. NUMERICAL EXAMPLE: FLOW-STRUCTURE INTERACTION

To illustrate the technics developed in the preceding sections, a simple application will be shown to follow. With this objective, a mass-spring-damper system with external parametric random excitation was employed. Then,

$$\ddot{y}(t) + c\dot{y}(t) + ky(t) = F(t, \theta_1, \theta_2) \tag{22}$$

this is a first approximation to a more complex model of structure with internal damping and stiffness to describe a flow-structure interactions. Where the effect of the flow on the system has been modeled via the force term given by the Morison formula:

$$F(t, \theta_1, \theta_2)) = \frac{1}{2} \rho C_D v(t, \theta_1, \theta_2) \mid v(t, \theta_1, \theta_2) \mid \tag{23}$$

where the parameters θ_1 θ_2 are assumed to be independent $[0, 2\pi]$ standard uniform random variables, C_D represent the drag coefficient, ρ the fluid density and $v(t)$ the velocity field of the fluid. Hence, this differential equation can be solved using a conventional Runge-Kutta method.

To calculate the velocity field, we used the Pierson-Moskovits (P-M) spectrum, given by

$$\mathcal{S}_{\eta\eta} = \frac{8.1 \times 10^{-3} g^2}{2\omega^5} exp\left(-0.74 \left(\frac{g}{U_{w,19.5}} \right)^4 \omega^{-4} \right) \tag{24}$$

where g is the gravity acceleration and $U_{w,19.5}$ is the wind speed at a height of 19.5m above the still water. This is the most extensively used spectrum for representing a fully developed sea and where the sea severity can be specified in terms

of the wind velocity. The ocean waves spectrum models are semi-empirical formulas, derived mathematically, but the formulation requires one or more experimentally determined parameters.

Hence, using the linear wave theory, also called Airy wave theory we can obtain the linear wave profile and velocity in terms of a spectral density, (Benaroya *et al.* 2005).

$$v_y(x,y,t) = \sum_{i=1}^{N} \omega_i \frac{coshk_i x}{sinhk_i d} cos(\omega_i t - k_i y - \theta_i)\sqrt{2\mathcal{S}_{\eta\eta}(\omega_i)\Delta\omega_i} \tag{25}$$

Assuming that the velocity field is composed by two terms corresponding to different waves periods t_1 and t_2, the equation holds,

$$v_y(y,t) = \frac{\omega_1}{sinh(k_1 d)} cos(\omega_1 t - k_1 y - \theta_1)\sqrt{2\mathcal{S}_{\eta\eta}(\omega_1)\Delta\omega_1} + \frac{\omega_2}{sinh(k_2 d)} cos(\omega_2 t - k_2 y - \theta_2)\sqrt{2\mathcal{S}_{\eta\eta}(\omega_2)\Delta\omega_2} \tag{26}$$

In the following Table 1 the values adopted are summarized,

Table 1. Simulations values

c	elasticity coefficient	80 N/m	c	damper coefficient	1000 Ns2/m
t_1	wave period1	5s	$U_{w,19.5}$	wind velocity	5 m/s
t_2	wave period2	2s	g	gravity acceleration	9.8 m/s2
ω_1	angular frequency1	1.25 rad/s	C_d	drag coefficient	1
ω_2	angular frequency2	3.14 rad/s	ρ	water density	1000kg/m3
k_1	wave number1	1 m^{-1}	A	front area	1m2
k_2	wave number2	1 m^{-1}	d	water depth	1m

As example, assuming θ_1 and θ_1 independent and uniformly distributed random number between 0 and 2π Figure 2 shown the response of $F(t,\theta_1,\theta_2))$ in $tspan = [0,30]$,

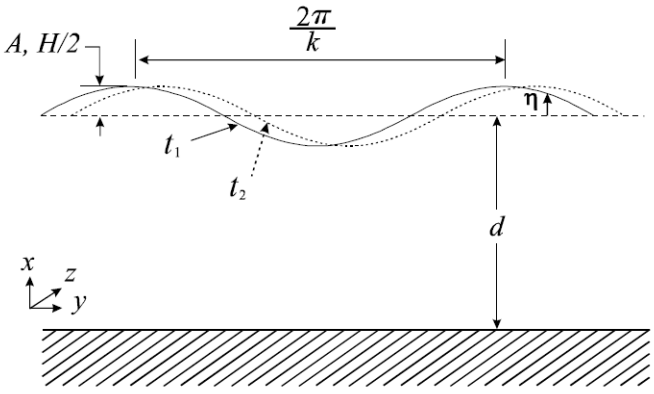


Figure 1. Wave scheme

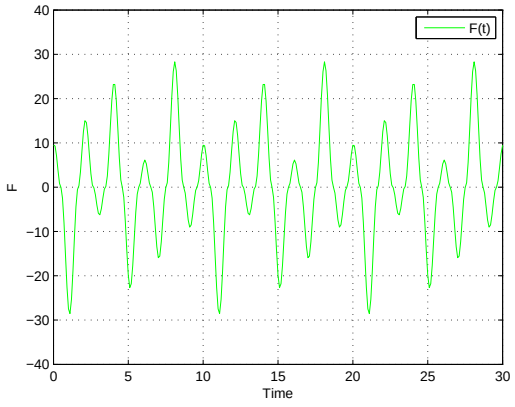


Figure 2. Response of excitation term for a random input

furthermore, integrating the Eq.(22) we obtain the followings responses:

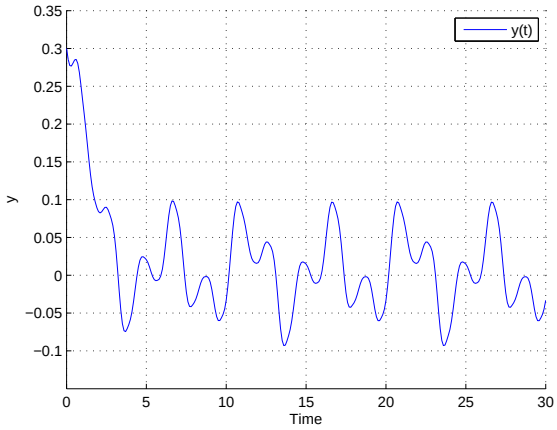


Figure 3. Solution with one random input

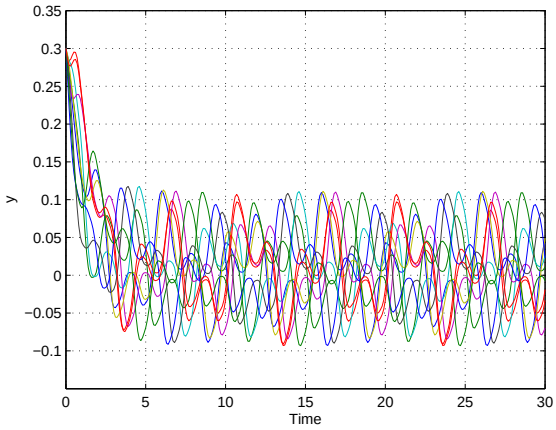


Figure 4. Solution with many random inputs

5.1 Monte carlo Method

Now, we analise the convergence of Monte-carlo performing eight experiments for increasing numbers of realizations, i.e. [10, 50, 100, 200, 500, 1000, 3000, 6000] and calculate the mean and variance of each experiment. The error is estimated calculating the norm between consecutive means, and likewise with the variance. Table 2 shown the results obtained at each experiment and the computational cost, as well the evolution of the error.

Table 2.

N	Experiments	Time(s)	Mean Error	Variancie Error
-	10	5.97	-	-
1	50	33.74	0.4649	0.0134
2	100	89.97	0.1669	0.0053
3	200	200.33	0.1025	0.0049
4	500	476.09	0.0589	0.0029
5	1000	1027.86	0.0336	0.0025
6	3000	2682.08	0.0109	0.0016
7	6000	5982.05	0.0051	0.0007

Figures (5) and (7) shown the mean and variance for different experiments with increasing number of realizations and Figures (6) and (8) the evolution of the error.

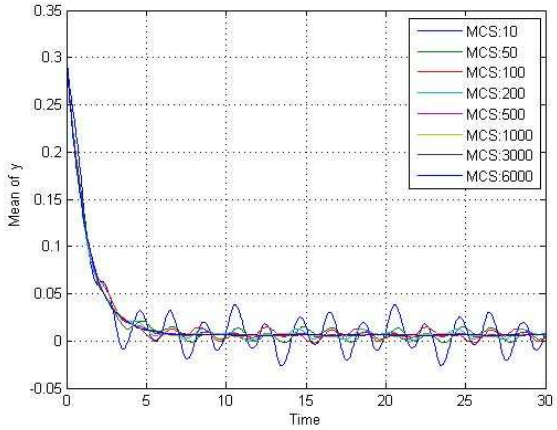


Figure 5. Mean of experiments for increasing realizations

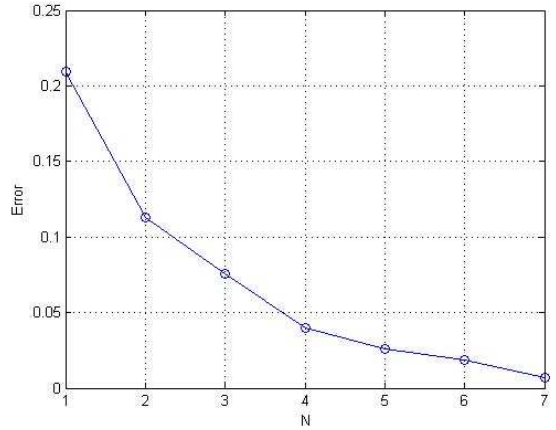


Figure 6. Error convergence of mean

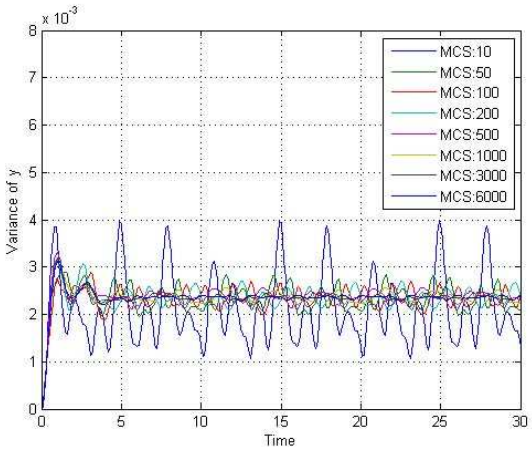


Figure 7. Variance of experiments for increasing realizations

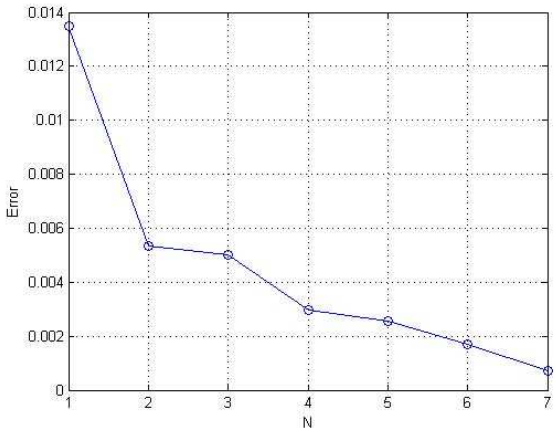
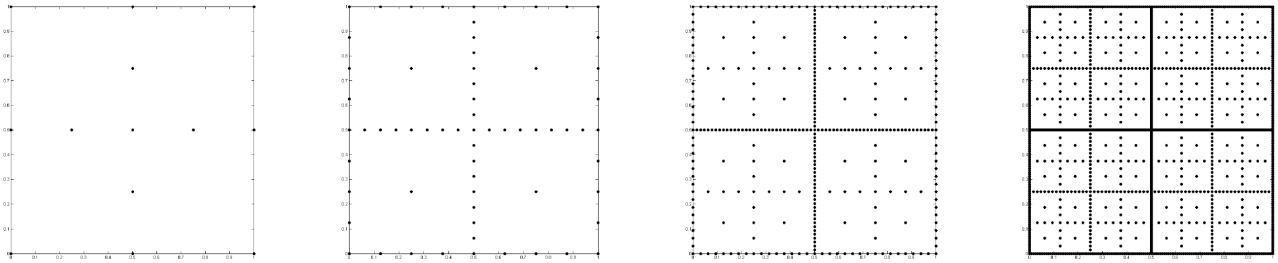


Figure 8. Error convergence of variance

5.2 Stochastic Collocation Method

In this section we showcase the collocation based strategy to solving Eq.(22) using a sparse grid to interpolate the stochastic solution. We analyse the method comparing the mean and variance solution for increasing levels of sparse grids interpolation, i.e. [2, 4, 6, 8] and a standard reference solution of Monte Carlo simulation computed for 6000 realizations. The error is estimated calculating the norm between the MCS solution, taken as reference, and the sparse grid solutions. Figure (9) show the increasing levels grids used in the analises, for two dimensional parameter space and isotropic Smolyak Clenshaw-Curtis abscissas. For the construction of the interpolation functions, was utilized the Sparse Grid interpolation toolbox of Matlab, developed by (Klimke 2008).



Level 2 Level 4 Level 6 Level 8

Figure 9. Sparse grid interpolation level

Table 3 shown the level of interpolation, the number of collocations points, the computational cost in seconds to calculate the moments respectively for each level, as well the evolution of the error comparing (SC) with (MCS).

Table 3.

N	Level	Grid points	Time(s)	Mean Error	Variance Error
1	2	13	21.43	0.1102	0.0139
2	4	65	78.23	0.0136	0.0019
3	6	321	265.63	0.0106	0.0009
4	8	1537	1020.26	0.0085	0.0006

Figures (10) and (12) shown the mean and variance for increasing levels of sparse grids interpolation and Figures (11) and (13) the evolution of the error. Finally, we present an estimation of the probability density function (PDF) function from histogram in two times. Finally, we present an estimation of the probability density function (PDF) function from histogram in two times $t = 1s$ and $t = 20s$.

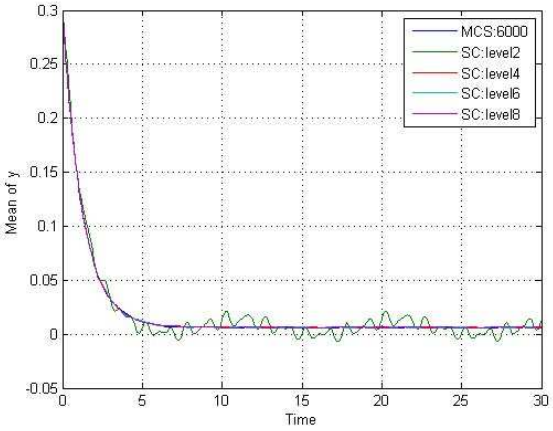


Figure 10. Mean for increasing level

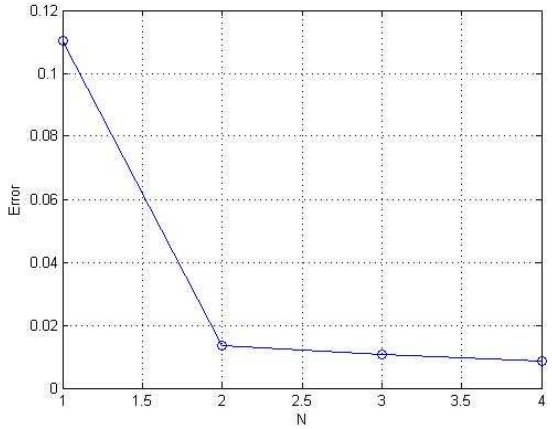


Figure 11. Error convergence of mean

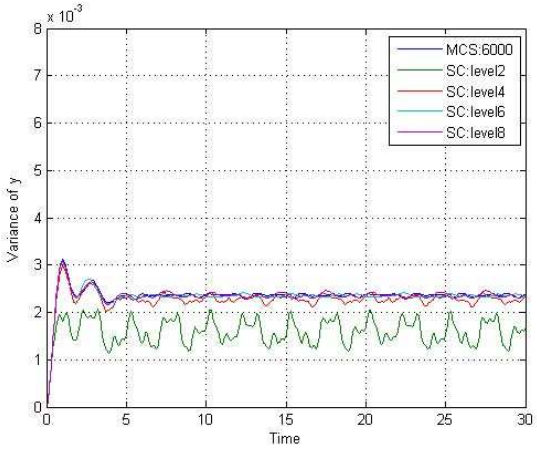


Figure 12. Variance for increasing level

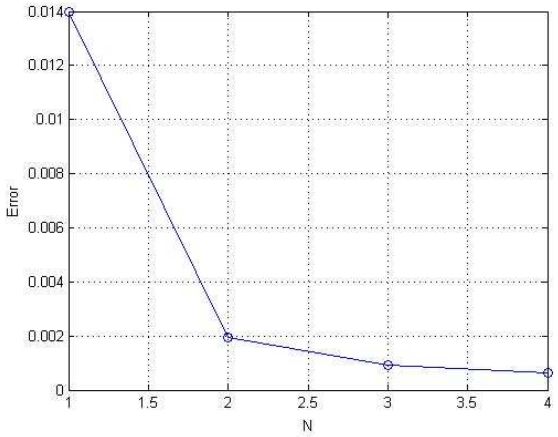


Figure 13. Error convergence of variance

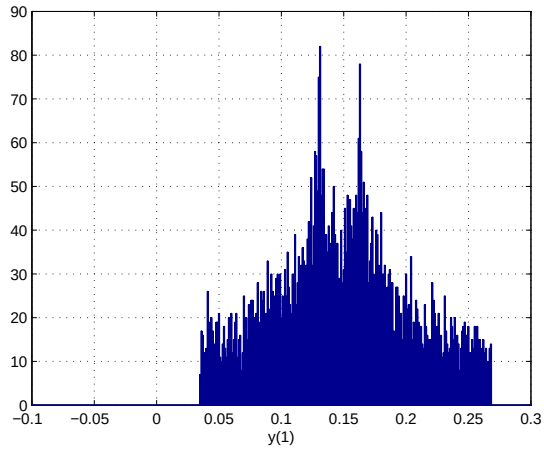


Figure 14. Histogram t=1s

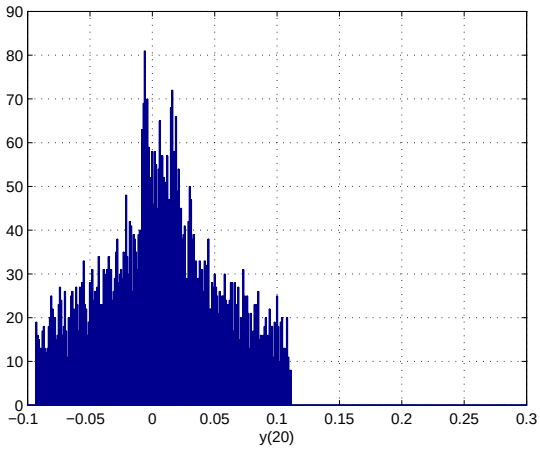


Figure 15. Histogram in t=20s

6. CONCLUSIONS

In this work we present the Monte Carlo Simulation method and Sparse Grid Stochastic Collocation method for solving a partial differential equations with random input data. Like Monte Carlo method, Sparse Grid Stochastic Collocation method leads to the solution of uncoupled deterministic problems and, as such, it is simple to implement. These non-intrusive methods, allows convert any deterministic code into a code that solves the corresponding stochastic problem.

However, compared with Monte Carlo Simulation method, Sparse Grid Stochastic Collocation method, present significative reduction in the number of experiments required to achieve the same level of accuracy. This was presented in Section 5, where utilizing the error estimated, we compare both methods in terms of computational work to achieve the same accuracy and (SGC) was much more efficient than (MCS).

Future work of this research will include the study of complex systems and the analysis of anisotropic and adaptative version of the Sparse Grid Stochastic Collocation method. Many problems vary rapidly in only some dimensions, remaining much smoother in other dimensions, consequently, it is important increase the level of accuracy only in certain dimension. This can improve the results obtained by reducing the curse of dimensionality, i.e. the problem caused by the exponential increase in volume associated with adding extra dimensions to a mathematical space.

7. ACKNOWLEDGEMENTS

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