INVERSE SIMULATION APPLIED TO NONLINEAR DUFFING EQUATION

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Abstract. *This paper presents the analysis of Inverse Simulation applied to nonlinear Duffing Equation, which represents several nonlinear systems dynamics. Simulations in different initial conditions are presented and the results are used with Inverse Simulation technique to obtain the excitation to the system. The phase-planes obtained from both cases are presented and requirements of convergence are discussed. The errors caused by the method and sample period are estimated, needed to verify real applicability of the technique. The Inverse Simulation approach assesses Newton-Raphson method and Jacobian Matrix to obtain the input time histories due to a specific output response. This analysis is very useful in aerospace systems in nonlinear control system design and validations.*

Keywords*: Duffing Equation, Inverse Simulation, Dynamic Systems*

1. INTRODUCTION

The modeling and simulation process consist of finding outputs time histories due to inputs applied to a representative model or dynamic system, for a given set of initial conditions. Identification processes find models that can represent dynamic systems, using input and output time histories, while Inverse Simulation finds inputs time histories that will produce a prescribed output. Several aerospace applications have been stated, in sense of finding answers to the presence of limit-cycle, continuous increasing and decreasing of amplitude during oscillations, and others. This nonlinear behavior can be observed on hybrid simulations where mathematical equations representing Dynamic Equations and real parts are present. The hybrid simulation to the Brazilian Satellite Launcher (VLS) is performed with real parts as hydraulic systems, actuators type movable nozzle and sensors, while the vehicle, filters and aerodynamic effects are represented by differential dynamic equations. In this way, we can obtain success in finding answers to some questions related to behavior due to uncontrollable and unobservable modes that can be present on typical complex linear and nonlinear dynamic systems.

The nonlinear effect, discussed through this work, can appear in terms of elastic potential energy as restoring forces or spring forces, and in this way, is presented an application of Inverse Simulation applied to the Duffing Equation, where nonlinear effects occur due to elastic potential energy.

2. THE INVERSE SIMULATION APPROACH

The mathematical formulation for Inverse Simulation is presented as follows, based on preliminary system model obtained from classical approach for modeling like Newton's Law, Lagrange Equations and Hamilton's Principle. The Forward Simulation consists of calculating the system response to a known input time history and in case of nonlinear dynamic systems, its mathematical model can be expressed by a matrix state-space model as follows.

$$\dot{x} = f(x,u) \qquad \dot{x}(0) = x_0 \tag{1}$$

The vectors x and u represent the state variables and inputs respectively, $f(x,u)$ are nonlinear functions describing the system's dynamic and $\dot{x}(0)$ the initial values. The output equation is

$$y = g(x) \tag{2}$$

The Inverse Simulation approach finds the input time histories that produced a specific output response obtained from the system dynamic, so we can differentiate Eq. (2) producing:

$$\dot{y} = \frac{dg}{dx}d\dot{x} = \frac{dg}{dx}\,f(x,u) \tag{3}$$

If Equation (3) is invertible so it is possible to write the equation as follows.

$$u = h(x, \dot{y}) \tag{4}$$

Substituting Equation (4) into Eq. (1) gives

$$\dot{x} = f(x, h(x, \dot{y})) = F(x, \dot{y}) \tag{5}$$

The Equations (4) and (5) provide a complete statement of Inverse Simulation, now with $\dot{y}$ as the input vector and u as the output vector. This inverse mathematical model is a new system dynamic and it is so quite different to original system. Murray-Smith (2000) and Thomson (1998) have shown applications of Inverse Simulation in many different areas based on different methods of inverse simulation, and according to the method and its features can be applied for specialized problems.

2.1. Integral and Differentiation-based Approach

Works from Murray-Smith (2000) show that the unknown values of x_n and u_n can be calculated using functions defined as follows

$$\begin{bmatrix} (x_n)_m \\ (u_n)_m \end{bmatrix} = \begin{bmatrix} (x_n)_{m-1} \\ (u_n)_{m-1} \end{bmatrix} - \begin{bmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial x} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_1}{\partial x} \end{bmatrix}^{-1} \begin{bmatrix} F_1((x_n)_{m-1}, (u_n)_{m-1}) \\ F_2((x_n)_{m-1}, (u_n)_{m-1}) \end{bmatrix} \tag{6}$$

where the matrix with partial derivatives is the Jacobian J. The method above is the differentiation-based approach and used in this work, applied to the Nonlinear Duffing Equation. The integration-based approach, from Hess (1991) is evaluated as follows.

$$(\dot{x}_{n-1})_m = f[(x_{n-1})_m, (u_{n-1})_m] \tag{7}$$

$$(x_n)_m = \int_{t_{n-1}}^{t_n} (\dot{x}_{n-1})_m \, dt + (x_{n-1})_m \tag{8}$$

$$(y_n)_m = g[(x_n)_m] \tag{9}$$

Defining the error function e_n as follows:

$$(e_n)_m = (y_n)_m - \bar{y}_n \tag{10}$$

computing errors and comparing with predefined threshold values, a Newton-Raphson algorithm is used to obtain:

$$(u_{n-1})_{m+1} = (u_{n-1})_m - [J]^{-1}(e_n)_m \tag{11}$$

This iterative process continues with m being incremented until all errors are under prescribed value error.

3. NONLINEAR SYSTEMS AND DUFFING EQUATION

3.1. Nonlinear Systems

Mechanical and electrical systems can be modeled as nonlinear systems governed by the following differential equation

$$m\ddot{x} + \varphi(\dot{x}) + f(x) = B\cos\omega t \tag{12}$$

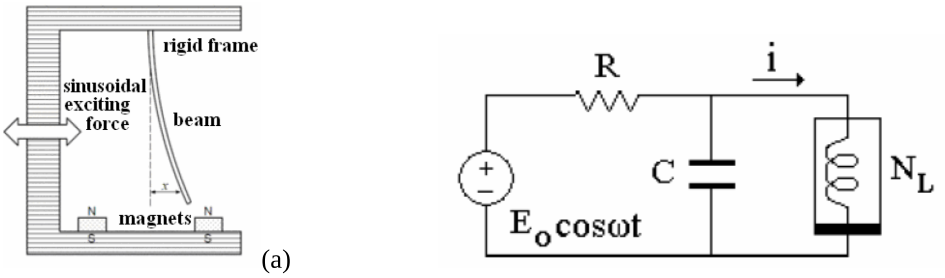
According to Stoker (1992) the term $m\ddot{x}$ is referred as the inertia forces, while the nonlinear term $\varphi(\dot{x})$ as the damping force, $f(x)$ as the *restoring forces* or *spring forces* and $B\cos\omega t$ as the external force or excitation. If the functions $\varphi(\dot{x})$ and $f(x)$ are linear under some simplification hypothesis or linearization process, it leads to classical methods to linear systems analysis. At the other hand, physical effects can lead to subclasses of nonlinear systems, e. g. nonlinear vibrations with $\varphi(\dot{x}) = -\dot{x} + \dot{x}^3/3$ known as *Rayleigh Equation* or *Van der Pol Equation*. In this case the nonlinear damping force tends to increasing on amplitude oscillation, while lower velocities tends to decrease the amplitude of oscillation.

3.1. The Duffing Equation

The Duffing Equation can be obtained from a damped forced nonlinear oscillator or from an electrical nonlinear oscillator, where nonlinearity is observed in its elastic potential energy associated to elastic behavior of a beam or magnetic effects of an inductor with a ferromagnetic core. Assuming $\varphi(\dot{x}) = kx$ and $f(x) = x^3$ in Eq. (1), we have the nonlinear differential equation to the Duffing Oscillator, as follows.

$$m\ddot{x} + k\dot{x} + x^3 = B\cos(\omega_0 t) \tag{13}$$

The Figure 1 shows typical mechanical and electrical system that can be modeled by Duffing Equation. According to Fig. 1(a) the motion of the steel beam is periodically forced and deflected toward two magnets, while in Fig. 1(b) the inductor produces nonlinear behavior through the ferromagnetic core.



(a) (b)

Figure 1. (a) Mechanical and (b) Electrical nonlinear oscillator

The Figure 2 shows the influences from values to k-B parameters and the regimes of the various long-term behaviors of Duffing Equation as mapped by Ueda (1980).

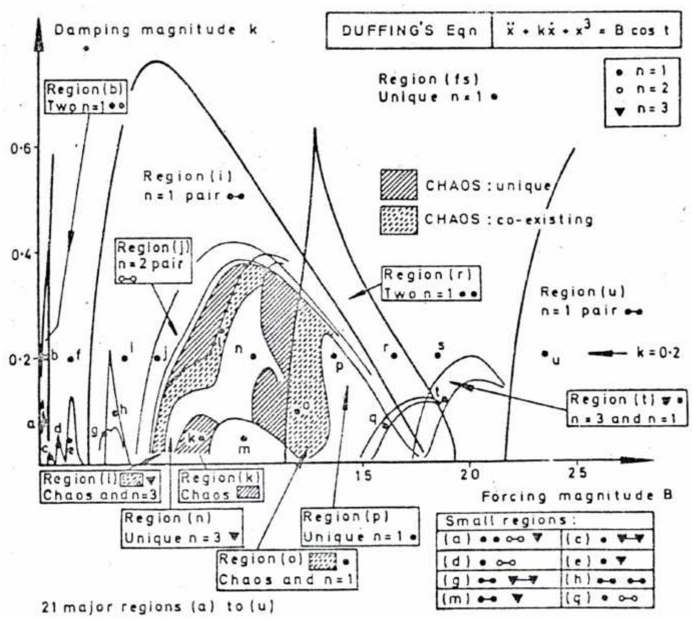


Figure 2. k-B mapping of Duffing Equation from Ueda (1980)

In this work, is chosen five periodic attractors leading to synchronous oscillations and sub-harmonics, so the parameters are m = 1 (mass), B = 10.2 (forced term amplitude) and k = 0.08 (damping term magnitude). The angular velocity to the forced term is ω_0 = 1 rd/s and initial conditions to the simulations (five attractors A1 to A5) are presented in Table 1.

Table 1. Initial conditions to attractors A1 to A5.

Attractor	$x(0) = x_1(0) =$	$\dot{x}(0) = x_2(0) =$
A1	-0.21	0.02
A2	1.05	0.77
A3	-0.67	0.02
A4	-0.46	0.30
A5	-0.43	0.12

The Figure 3 shows the model used to forward simulations.

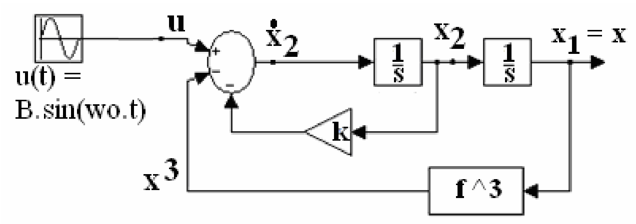
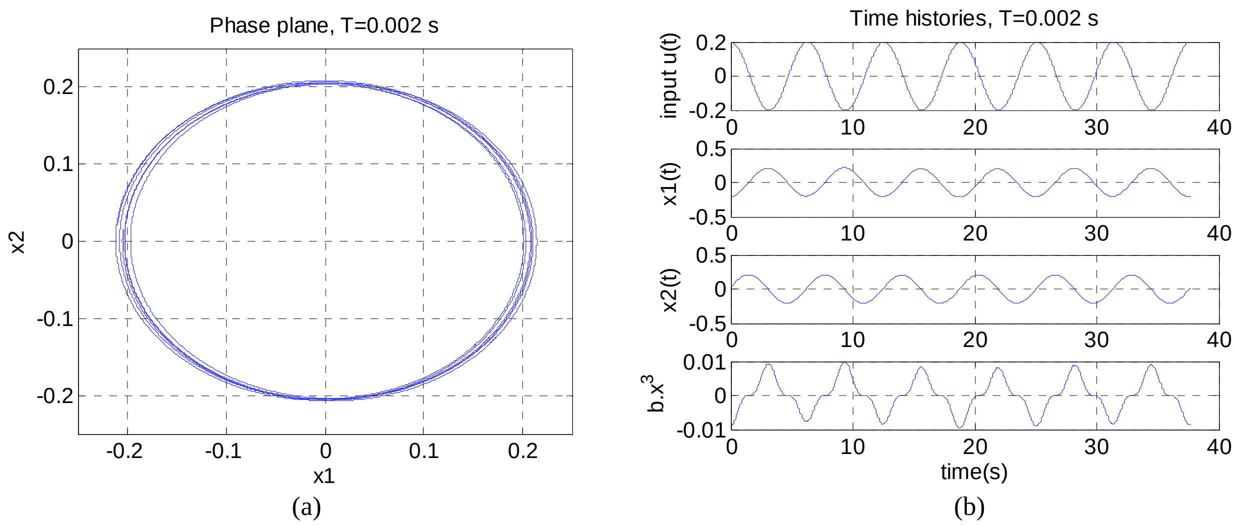


Figure 3. Duffing Equation Forward Simulation

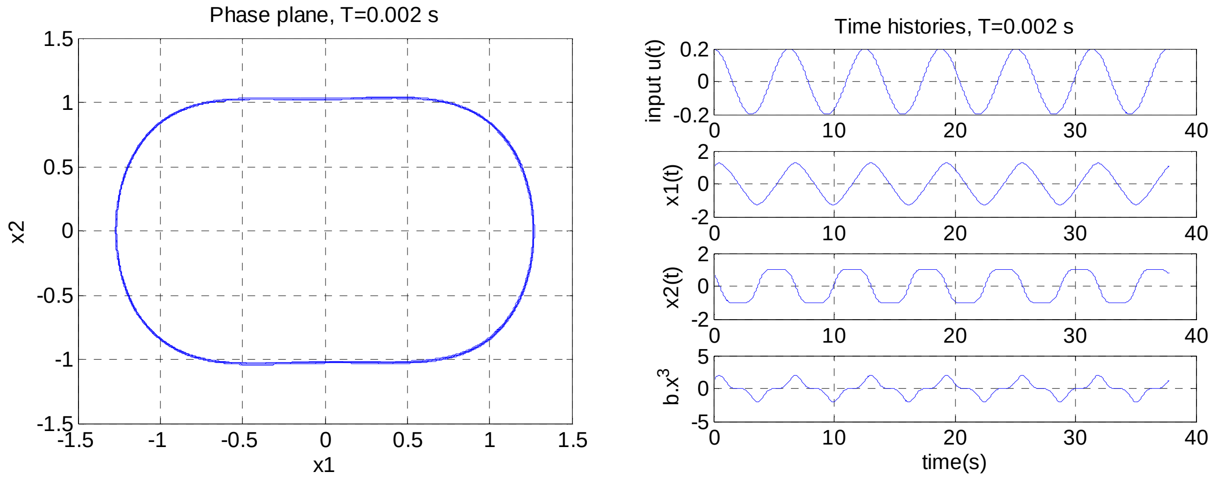
The figures as follow show results from forward simulations, assuming a sinusoidal input and sample period T=0.002 s. The Fig. 4(a) shows the phase plane and Fig. 4(b) the state variables time histories due to initial conditions of attractor A1.



(a) (b)

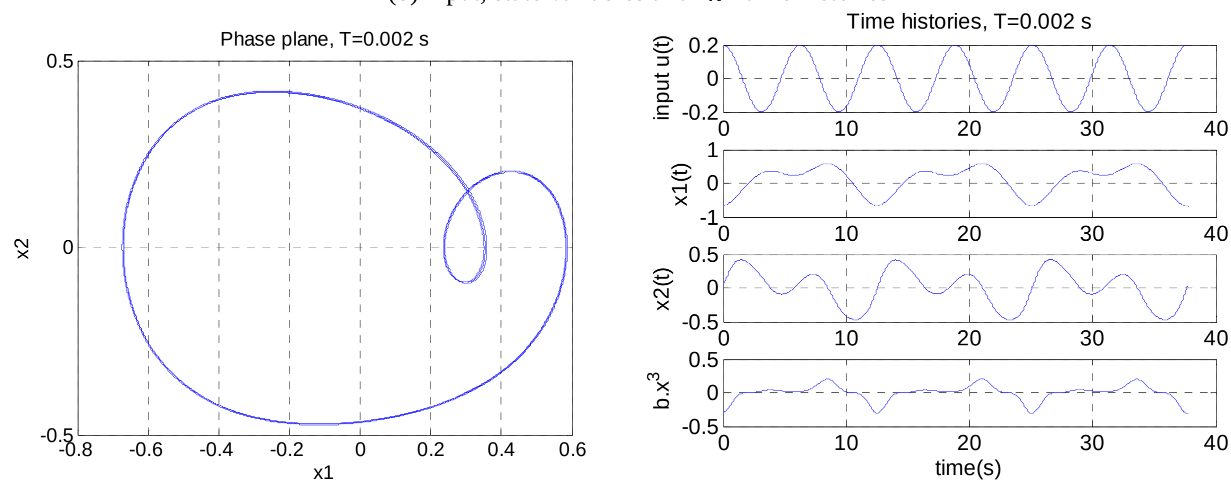
Figure 4. Forward simulation to Duffing Equation, Attractor A1
(a) phase plane (b) input, state variables and x^3 time histories

The Figure 5(a) shows the phase plane and Fig. 5(b) the state variables time histories due to initial conditions to attractor A2, while Fig. 6 due to attractor A3.



(a) (b)

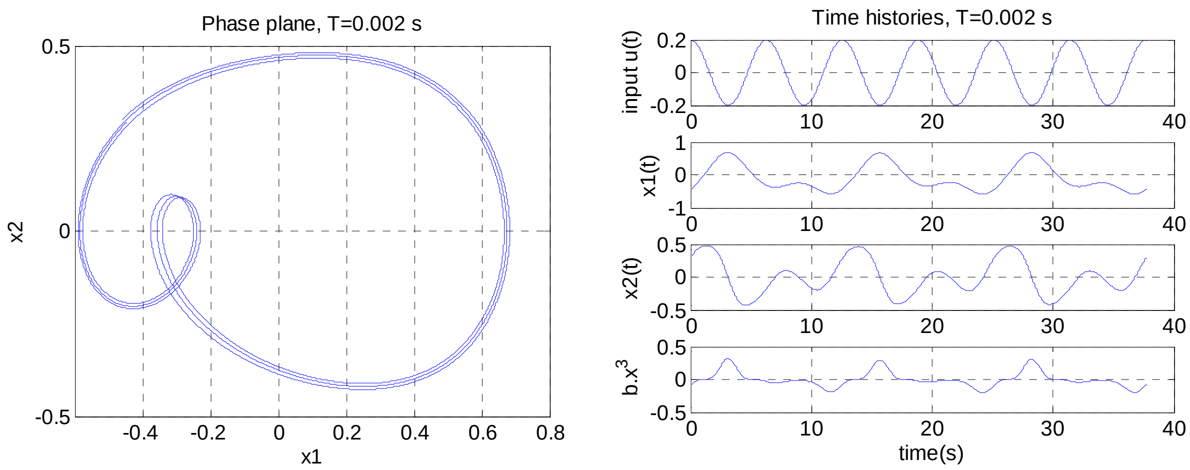
Figure 5. Forward simulation to Duffing Equation, Attractor A2 (a) phase plane
(b) input, state variables and x^3 time histories



(a) (b)

Figure 6. Forward simulation Duffing Equation, Attractor A3 (a) phase plane and (b) $u(t)$, x_1, x_2 and x^3 time histories

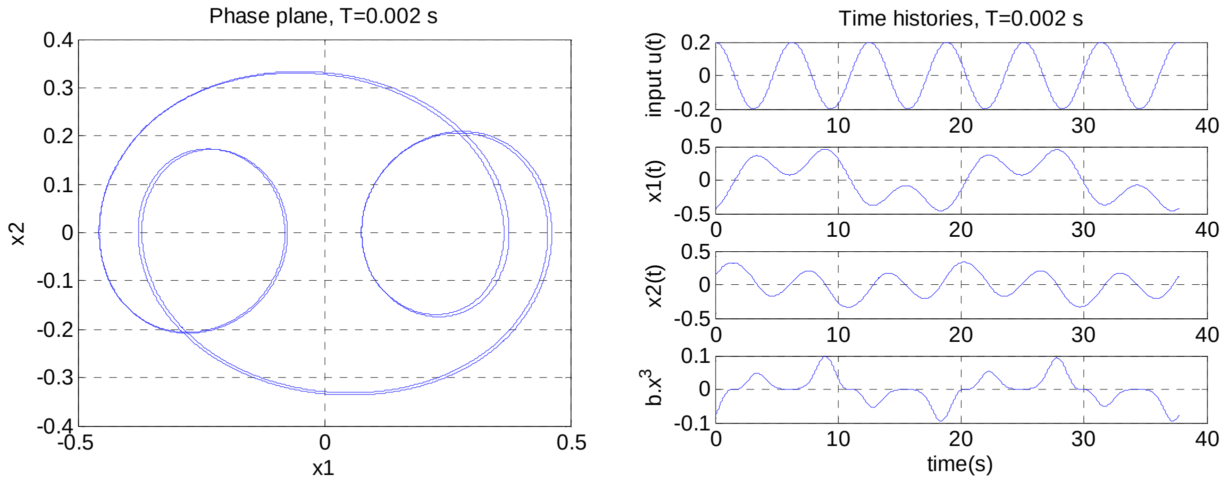
The Figure 7(a) shows the phase plane and Fig. 7(b) the state variables time histories due to initial conditions to attractor A4.



(a) (b)

Figure 7. Forward simulation to Duffing Equation, Attractor A4 (a) phase plane
(b) input, state variables and x^3 time histories

The Figure 8(a) shows the phase plane to Duffing Equation, attractor A5 and Fig. 8(b) the input, state variables and $\dot{x}^3$ time histories.



(a) (b)

Figure 8. Forward simulation to Duffing Equation, Attractor A5 (a) phase plane (b) input, state variables and x^3 time histories

The time histories due to state variables and output were applied to the Inverse Simulation and the results are presented in sections as follow.

4. INVERSE SIMULATIONS

This section presents the application of Inverse Simulation based on differentiation-based approach according to Equations (6) to (13). The energy and state variables are defined as follow

$x_1 = x$ (Displacement) $x_2 = \dot{x}$ (Velocity)

$\dot{x}_1 = \dot{x}$ (Velocity) $\dot{x}_2 = \ddot{x}$ (Acceleration)

Resulting the nonlinear state-space equation:

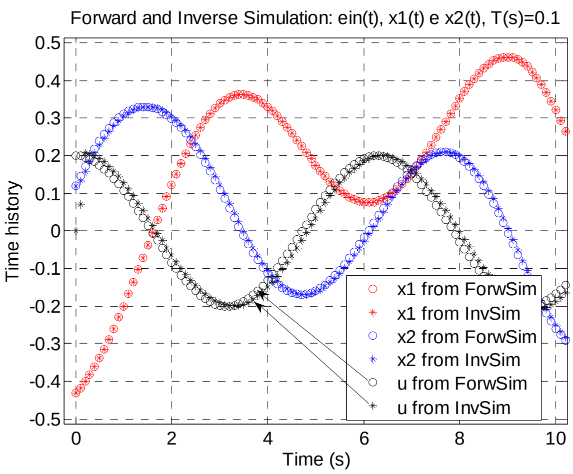
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{m}x_2 - \dfrac{1}{m}x_1^3 + \dfrac{1}{m}u \\ y = x_1 \end{cases} \tag{14}$$

Obtaining:

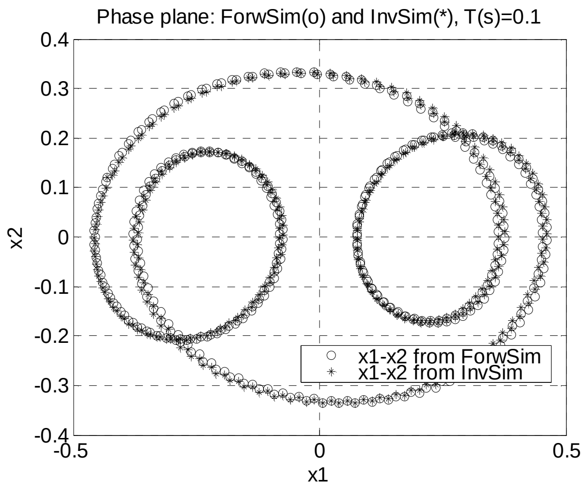
$$\begin{cases} x_2 - \dfrac{x_1^n - x_1^{n-1}}{T} = 0 \\ -\dfrac{k}{m}x_2 - \dfrac{1}{m}x_1^3 + \dfrac{1}{m}u - \dfrac{x_2^n - x_2^{n-1}}{T} = 0 \\ x_1 - y_m = 0 \end{cases} \tag{15}$$

4.1. Inverse Simulation - Attractor A4 and A5

The results from application of differentiation-based approach to the Inverse Simulation (InvSim) are presented in the figures as follow, based on time histories obtained from Forward Simulations (ForwSim) with attractors A4 and A5. The Newton-Raphson algorithm was used to determine values of input and state variables time histories, in what according to Murray-Smith (2000) is appropriate for a problem of this kind.

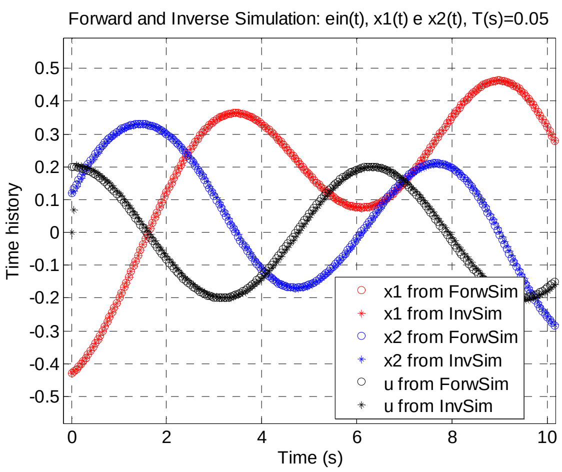


(a)

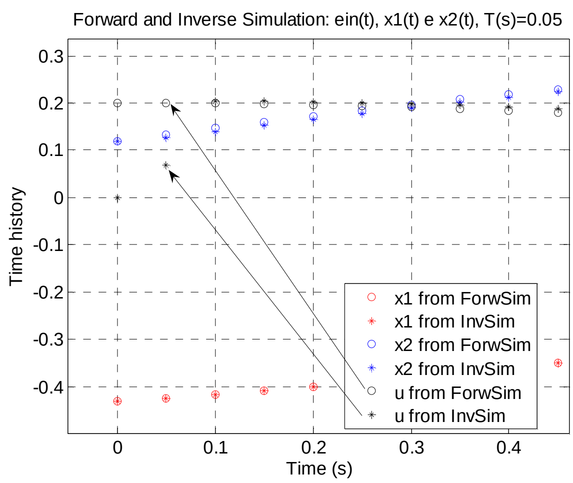


(b)

Figure 9. Results from Inverse Simulation, attractor A5, T = 100ms (a) Time histories (b) Phase plane comparison



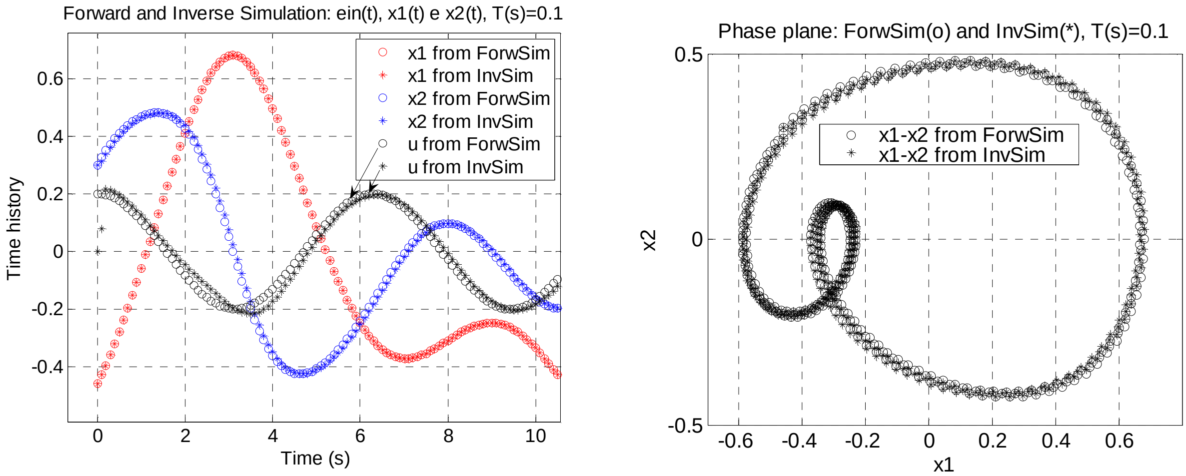
(a)



(b)

Figure 10. Results from Inverse Simulation, attractor A5, T=50ms: (a) Time histories (b) Zoom

The figure 11 shows the results obtained from application of Inverse Simulation considering Attractor A4.



(a) (b)

Figure 11. Results from Inverse Simulation for attractor A4, T = 100ms: (a) time histories and (b) Phase plane comparison

The root mean squared error (RMSE) associated to Inverse simulation considering attractors A4 and A5 are presented in Table 2 as follows.

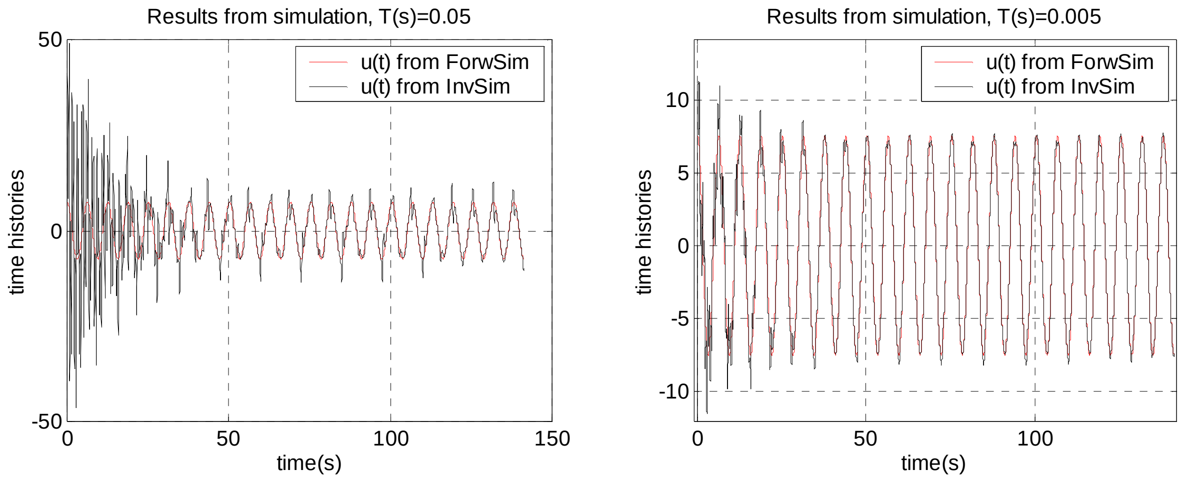
Table 2. Root Mean Square Error (RMSE).

	$T(s)$	u	x_1	x_2
RMSE – Attractor A4	0.1	0.02100	0.00000	0.00850
RMSE – Attractor A5	0.1	0.02588	0.00000	0.01131
RMSE – Attractor A4	0.01	0.00455	0.00000	0.00113
RMSE – Attractor A5	0.01	0.00428	0.00000	0.00085

It can be observed from Table 2, low values RMSE to displacement x_2 and relative RMSE to the input time history u obtained from the Inverse Simulation.

4.2. Inverse Simulation - chaotic movement

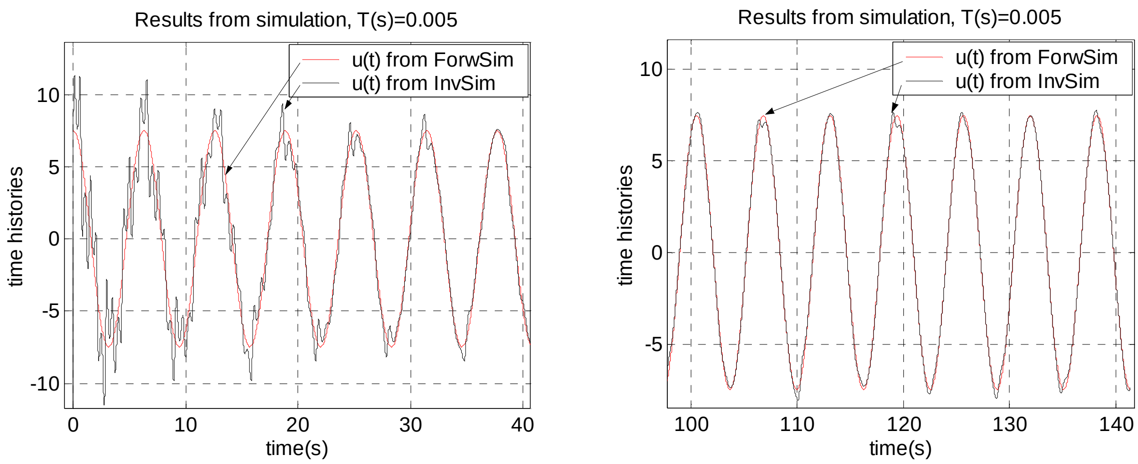
The results from the application of differentiation-based approach to the Inverse Simulation are presented in the figures as follow, based on time histories from forward simulations with the chaotic movement case. The parameters are m = 1, k = 0.05, B = 7.5, $x(0) = -5$ and $\dot{x}(0) = 0$. The Fig. 10 (a) shows results using T=50ms and can be observed greater errors in 0-25s during the transient period, while lower errors to the steady-state period. In Fig. 12(b) the errors are better using sample period T=5ms.



(a) (b)

Figure 12. Results from Inverse Simulation: (a) T = 50ms (b) T = 5ms

The Figure 13 shows details from Fig. 12(b).



(a) (b)

Figure 13. Zoom in Figure 12(b): (a) Transient regime and (b) steady-state regime

The Table 3 compiles the errors obtained from Inverse Simulation comparing results with T=50ms and T=5ms.

Table 3. Root Mean Square Error (RMSE) – Transients setting to steady-state chaos.

$T(s)$	u	x_1	x_2
0.05	6.31747	0.00000	0.54541
0.005	0.84730	0.00000	0.05473

The attractor due to chaotic movement produces signals with harmonics tuned on excitation frequency and other component type narrow-band white noise.

5. CONCLUSIONS

The main objective of this paper is to present Inverse Simulation Approach applied to nonlinear system making use of Duffing Equation. In this way, results in finding input and states time histories to a specific and desired output show coherence and low values on errors RMSE. The convergence on inverse simulation process was obtained although effects from sampling time, initial conditions, amplitude of excitation, nonlinear models, chaos on transient and steady-state regimes. The interest on input time histories to a specific output is to find what efforts has to be applied from pilots or control systems to vehicles, e. g., VLS, Unmanned Aerial Vehicles (UAVs), submarines and airplanes. So this work has widely application in civil and military areas.

In this work only effects from nonlinearities in potential elastic energy were considered and it can be extended the interest on other effects from nonlinearities in kinetics energy expression. This analysis is very useful in aerospace systems on nonlinear control system design and validation, because some results due to typical behavior of nonlinear systems have been observed in VLS Hybrid Simulations (Hardware-in-the-Loop simulations). So results from analysis of inverse simulations are of great interest mainly in find answers to some questions in nonlinear systems concerning the VLS hybrid simulation.

Dynamic Systems with several modes of vibration requires short sample periods but requires long time to the inverse simulation. In these cases is recommended simulations with different sample periods according to the dynamic (time constants) respectively to the faster and to the slower mode. It is recommended to analyze the Power Spectral Density function on results from Forward Simulation before performing Inverse Simulation, to verify the presence of frequencies due to slow and fast modes and to different modes of vibrations.

The presence of chaotic behavior due to system dynamics, external disturbance, excitation amplitudes and initial conditions are very important to Inverse Simulations, mainly to choice preliminary models, sample periods and stability analysis.

The reason for error values RMSE=0 for x_1 is due to output equation $y = x_1$, but in many other cases the output can be a combination of variables. It can be observed the direct influence of sample period on RMSE values, where fast sample periods leads on low RMSE. The Forward Simulation with different initial conditions affects the nonlinear behavior of the movement but do not affect the results from Inverse Simulations.

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