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PROGRESSIVE DYNAMIC ANALYSIS OF SERIAL ROBOTS BASED ON SCREW THEORY

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Abstract. *Theory of screws applications are based on the combined representation of angular and linear velocity or, similarly, force and moment, as one element of a six-dimensional projective vector space. In a variety of areas of robotics, methods and formalisms based on screw theory have shown advantages over other techniques and have led to significant advances. These methods include the development of fast and efficient dynamics algorithms, discoveries in the nature of robot compliance and mechanism singularity, and the invention of numerous parallel mechanisms. One significant advantage of using screw theory is the possibility to reuse the partial model of the robot. It is possible to model two or more pieces of the robot independently and then combining these pieces forming a complete model. The standard approach (D-H) requires to remodel the whole manipulator. This paper presents the development of theory based on Lagrangian formalism and screw theory. As example of application a two DoF serial robot is employed to move a load holding it by a passive rotative joint. When the manipulator holds the load, the load and the manipulator form a three DoF serial robot with one passive joint, so the dynamic model is dramatically changed. The system formed by the manipulator and the payload becomes a completely new mechanism. Details of the impact in the dynamic model produced by the new link are presented with emphasis in the fact that the dynamic model doesn't need to be rebuilt from scrap. The authors believe this approach consists on a major advance with important applications in reconfigurable robots.*

Keywords: *dynamics; robotics, screw theory, robot manipulator dynamics*

1. INTRODUCTION

Dynamic analysis plays an important role in predicting the behavior of mechanical systems and achieving their best performance. In robotics there are two types of dynamical problems (Tsai, 1999): (i) the direct dynamics problem aims to find the response of a robot as a result of applied torques and/or forces; (ii) the inverse dynamics problem aims to find the actuator torques and/or forces required to generate a desired trajectory.

There are three main methods for dynamical equations formulation present in several textbooks of robotics: Newton-Euler's formulation, the principle of virtual work, and the Lagrangian formulation. The Newton-Euler's formulation requires the motion equations to be written for each body of a manipulator. It is an inherently recursive method and, consequently, it is computationally efficient (Tsai, 1999; Siciliano et al., 2009). However, the disadvantage of this method is the increase in analysis complexity with the increase in number of joints of the robot (Kelly et al., 2005). Moreover, the method is not suitable for parallel manipulators. The principle of virtual work method allows eliminating the constraint forces and moments at the joints from the motion equations (Wittenburg, 2008, p. 30). This method works fine with parallel manipulators. The Lagrangian formulation is systematic and of easy understanding, it provides the equations of motion in a compact analytical form advantageous for control design and it is effective if it is desired to include more complex mechanical effects such as flexible link deformation, suspended payload, etc.

Consider a robot that, during some task, needs to hold a payload by a movable union like a hook. The coupling between the hook and the payload could be modeled as a new passive joint added to the kinematic chain. So, the robot dynamic model is dramatically changed and the system formed by the manipulator and the payload becomes a completely new mechanism. Using the Newton Euler's formulation or the principle of virtual work, the dynamic model must be completely recalculated. That's why in this paper we introduce a new procedure for progressive dynamic analysis of robotic manipulators based on Lagrangian formulation and screw theory which are more flexible to introduce more complex mechanical effects. With this approach is possible to model several pieces of the robot independently, then combine the pieces to form a complete model. Using a more standard approach it is just no possible, or at least it not so directly; it is necessary to remodel the whole manipulator at once. To achieve the objectives, the concept of "joint-space inertia matrix" based on Ball's six principal screws of inertia is introduced.

2. PROGRESSIVE DYNAMIC ANALYSIS

The aim of this method is to build up a dynamic model of a manipulator in a progressive way. It means that each link and joint, from the base to the very last link (end-effector or payload), is analyzed at a time. To create the method we use concepts such: principal screws of inertia, generalized mass matrix, screw transformation matrix, kinetic energy, Jacobian, and Lagrangian formulation.

2.1 Joint-Space Inertia Matrix

The inertia matrices are fundamentals to describe the dynamic behavior of a rigid body. There are many text on the subject, see for example, (Craig, 1989, p. 205), (Tsai, 1999, p. 375), (Siciliano et al., 2009, p. 139), (Featherstone, 2008, p. 34), and (Kelly et al., 2005, p. 72, 95), but no author ever has approached the problem using screw theory. More details on screw theory can be found in (Ball, 1900; Hunt, 1978; Davidson and Hunt, 2004).

The kinetic energy[1] of each link is given by (Featherstone, 2008, p. 65):

$$\mathcal{T}_i = \frac{1}{2}\, {\$^{i}_{0,\,C_i}}^{T}\, \overline{\boldsymbol{M}}_i\, \$^{i}_{0,\,C_i} \tag{1}$$

where $\$^{i}_{0,\,C_i}$ is the twist of the link i center of mass relative to the base and expressed in a frame with axes parallel with the link frame i, and $\overline{\boldsymbol{M}}_i$ is the generalized mass matrix expressed in respect to the center of mass of link i in a frame parallel to frame i.

The computation of the generalized mass matrix is based on Ball's six principal screws of inertia that are paired such that two are aligned with each of the axes, with their pitches equal to the corresponding radii of gyration but of opposite sense (Ball, 1900; Tischler et al., 2000). The coordinates of the principal screws of inertia in axis-order (*i.e.* $[P,\, Q,\, R;\, L,\, M,\, N]$) are:

$$\$_1 = [a,\, 0,\, 0;\, 1,\, 0,\, 0] \tag{2}$$

$$\$_2 = [-a,\, 0,\, 0;\, 1,\, 0,\, 0] \quad \text{(or } [a,\, 0,\, 0;\, -1,\, 0,\, 0]\text{)} \tag{3}$$

$$\$_3 = [0,\, b,\, 0;\, 0,\, 1,\, 0] \tag{4}$$

$$\$_4 = [0,\, -b,\, 0;\, 0,\, 1,\, 0] \quad \text{(or } [0,\, b,\, 0;\, 0,\, -1,\, 0]\text{)} \tag{5}$$

$$\$_5 = [0,\, 0,\, c;\, 0,\, 0,\, 1] \tag{6}$$

$$\$_6 = [0,\, 0,\, -c;\, 0,\, 0,\, 1] \quad \text{(or } [0,\, 0,\, c;\, 0,\, 0,\, -1]\text{)} \tag{7}$$

where a, b and c are the radii of gyration of the body. An interesting property of these screws becomes apparent by forming a 6×6 matrix $\mathcal{S}_C$ with each row taking the coordinates of one of the screws in axis-order, then taking the product of this matrix and its transpose (Tischler et al., 2000):

$$\mathcal{S}_C^T\, \mathcal{S}_C = \begin{bmatrix} a & -a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -c \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 1 & 0 & 0 \\ -a & 0 & 0 & 1 & 0 & 0 \\ 0 & b & 0 & 0 & 1 & 0 \\ 0 & -b & 0 & 0 & 1 & 0 \\ 0 & 0 & c & 0 & 0 & 1 \\ 0 & 0 & -c & 0 & 0 & 1 \end{bmatrix} = 2 \begin{bmatrix} a^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

The resultant matrix contains the usual form of the inertia tensor $\boldsymbol{I}$ in the top-left corner with the exception of the factor $m/2$. The 6×6 matrix is related to the generalized mass matrix referred to a frame located at the center of mass and aligned with the principal axes. For a frame located at the center of mass, but not necessarily aligned with the principal axes, the generalized mass matrix of link i is given by:

$$\overline{\boldsymbol{M}}_i = \begin{bmatrix} \overline{\boldsymbol{I}}^{i} & 0 \\ 0 & m_i\, \boldsymbol{I}_3 \end{bmatrix} \tag{9}$$

where $\overline{\boldsymbol{I}}^{i}$ is the inertia tensor written in respect to a frame with origin coincident with the center of mass of link i and axes parallel to the link i frame axes, m_i is the mass of link i, and $\boldsymbol{I}_3$ is the identity matrix of order 3. If, and only if, frame i axes are parallel with the so called principal inertia axes of the body i, then $\overline{\boldsymbol{I}}^{i}$ is diagonal in the same form as Eq. (8) multiplied by a factor $m/2$. So, the product of matrices $\mathcal{S}_C^T\, \mathcal{S}_C$ is a tensor and, as a tensor, it can be transformed as

$$\overline{\boldsymbol{M}}_i = \frac{m_i}{2}\, \boldsymbol{T}^{i}_{C_i}\, \mathcal{S}_C^T\, \mathcal{S}_C\, \boldsymbol{T}^{C_i}_{i} \tag{10}$$

where $\boldsymbol{T}^{i}_{C_i}$ is a 6×6 screw transformation matrix given by

$$\boldsymbol{T}^{i}_{C_i} = \begin{bmatrix} \boldsymbol{R}^{i}_{C_i} & 0 \\ 0 & \boldsymbol{R}^{i}_{C_i} \end{bmatrix} \tag{11}$$

[1] Calligraphic typeset is used to avoid confusion with screw transformation matrix.

where $\boldsymbol{R}^i_{C_i}$ contains the principal moments of inertia axis versors given in respect to frame i. Note that $\boldsymbol{T}^{C_i}_i = \left(\boldsymbol{T}^i_{C_i}\right)^T$.

The confusing thing about twists (velocities) is that they are screws, consequently, their coordinates must be specified in some reference frame, but they measure the relative velocity of one body (one frame) in respect to another body (another frame). Hence, there are three frames related with a single screw quantity: the inertial frame in which the time derivatives are taken; the moving frame of which movement someone is interested in; and the reference frame in which the numbers, the coordinates, are expressed. Usually, the inertial frame and the reference frame are the same, but this is not mandatory. Therefore, the twists of the link i center of mass relative to the base and expressed in a frame with axes parallel with the link frame i is:

$$\$^i_{0,\,C_i} = \begin{bmatrix} \boldsymbol{\omega}^i_{0,\,i} \\ \boldsymbol{v}^i_{0,\,C_i} \end{bmatrix} \tag{12}$$

where $\boldsymbol{\omega}^i_{0,\,i}$ is the angular velocity of frame i in respect to a static frame instantaneously coincident with frame i, $\boldsymbol{v}^i_{0,\,C_i}$ is the linear velocity of the center of mass, C, of the link i in respect to a static frame instantaneously coincident with frame i.

The twist of the link i written in respect to the base frame, $\$^0_{0,\,i}$, can be transformed in the twist $\$^i_{0,\,C_i}$ by multiplying $\$^0_{0,\,i}$ by a 6×6 screw transformation matrix, $\boldsymbol{T}^{C_i}_0$, such that (Davidson and Hunt, 2004, p. 82), (Tsai, 1999, p. 206):

$$\$^i_{0,\,C_i} = \boldsymbol{T}^{C_i}_0 \, \$^0_{0,\,i} \tag{13}$$

$$\boldsymbol{T}^{C_i}_0 = \begin{bmatrix} \boldsymbol{R}^i_0 & 0 \\ -\boldsymbol{R}^i_0 \, S\left(\boldsymbol{p}^0_{C_i}\right) & \boldsymbol{R}^i_0 \end{bmatrix} \tag{14}$$

where $\boldsymbol{R}^i_0$ is the rotation matrix that describes the axes of frame i in respect to the base frame and depends on configuration, $\boldsymbol{p}^0_{C_i}$ is the position vector of the link i center of mass[2] given in respect to the base frame, and $S(\cdot)$ is the skew-symmetric operator (Siciliano et al., 2009, p. 81). The superscript C_i of $\boldsymbol{T}^{C_i}_0$ is used to indicate that the point implicated in the transformation is the link i center of mass. Therefore:

$$\mathcal{T}_i = \frac{1}{2}\left(\boldsymbol{T}^{C_i}_0 \, \$^0_{0,\,i}\right)^T \overline{\boldsymbol{M}}_i \, \boldsymbol{T}^{C_i}_0 \, \$^0_{0,\,i} = \frac{1}{2} \, {\$^0_{0,\,i}}^T \underbrace{{\boldsymbol{T}^{C_i}_0}^T \overline{\boldsymbol{M}}_i \, \boldsymbol{T}^{C_i}_0}_{\boldsymbol{M}^0_{C_i}} \, \$^0_{0,\,i} = \frac{1}{2} \, {\$^0_{0,\,i}}^T \boldsymbol{M}^0_{C_i} \, \$^0_{0,\,i} \,. \tag{15}$$

The twist $\$^0_{0,\,i}$ can be computed by (Siciliano et al., 2009, p. 80):

$$\$^0_{0,\,i} = \begin{bmatrix} \boldsymbol{\omega}^0_{0,\,i} \\ \boldsymbol{v}^0_{0,\,i} \end{bmatrix} = \boldsymbol{J}^0_i \, \dot{\boldsymbol{q}} \tag{16}$$

where $\boldsymbol{J}^0_i$ is the Jacobian that maps the vector of generalized variables speed, $\dot{\boldsymbol{q}}$, to the twist of the link i expressed in the base frame, $\$^0_{0,\,i}$. So

$$\mathcal{T}_i = \frac{1}{2} \, {\$^0_{0,\,i}}^T \boldsymbol{M}^0_{C_i} \, \$^0_{0,\,i} = \frac{1}{2}\left(\boldsymbol{J}^0_i(\boldsymbol{q}) \, \dot{\boldsymbol{q}}\right)^T \boldsymbol{M}^0_{C_i} \, \boldsymbol{J}^0_i(\boldsymbol{q}) \, \dot{\boldsymbol{q}} = \frac{1}{2} \, \dot{\boldsymbol{q}}^T \underbrace{{\boldsymbol{J}^0_i}^T \boldsymbol{M}^0_{C_i} \, \boldsymbol{J}^0_i}_{\boldsymbol{M}_i} \, \dot{\boldsymbol{q}} \tag{17}$$

$$\mathcal{T}_i = \frac{1}{2} \, \dot{\boldsymbol{q}}^T \boldsymbol{M}_i \, \dot{\boldsymbol{q}} \tag{18}$$

where $\boldsymbol{M}_i$ is the "joint-space inertia matrix" of the particular link i.

It's worthwhile to note that both $\boldsymbol{J}^0_i = \boldsymbol{J}^0_i(\boldsymbol{q})$ and $\boldsymbol{M}_i = \boldsymbol{M}_i(\boldsymbol{q})$ depend on the particular link i, but $\dot{\boldsymbol{q}}$ is unique. Hence, the total kinetic energy is given by:

$$\mathcal{T} = \sum_{i=1}^{n} \mathcal{T}_i = \sum_{i=1}^{n} \frac{1}{2} \, \dot{\boldsymbol{q}}^T \boldsymbol{M}_i \, \dot{\boldsymbol{q}} = \frac{1}{2} \, \dot{\boldsymbol{q}}^T \underbrace{\left(\sum_{i=1}^{n} \boldsymbol{M}_i\right)}_{\boldsymbol{M}} \dot{\boldsymbol{q}} \tag{19}$$

$$\mathcal{T} = \frac{1}{2} \, \dot{\boldsymbol{q}}^T \boldsymbol{M} \, \dot{\boldsymbol{q}} \tag{20}$$

where n is the number of joints.

[2] It is possible to consider many parts of one link instead of the whole link, *e.g.* , the structure of the link and the motor rotor. In this case, care should be taken to consider the transmission gain, if any.

Each term of the inertia matrix, $\boldsymbol{M}$, is calculated by:

$$\boldsymbol{M}_i = \boldsymbol{J}_i^0\left(\boldsymbol{q}\right)^T \boldsymbol{M}_{C_i}^0\, \boldsymbol{J}_i^0\left(\boldsymbol{q}\right) = \boldsymbol{J}_i^{0T}\, \boldsymbol{T}_0^{C_i\,T}\, \overline{\boldsymbol{M}}_i\, \boldsymbol{T}_0^{C_i}\, \boldsymbol{J}_i^0 = \left(\boldsymbol{T}_0^{C_i}\, \boldsymbol{J}_i^0\right)^T \overline{\boldsymbol{M}}_i\, \boldsymbol{T}_0^{C_i}\, \boldsymbol{J}_i^0 = \boldsymbol{J}_i^{C_i\,T}\, \overline{\boldsymbol{M}}_i\, \boldsymbol{J}_i^{C_i} \tag{21}$$

where $\boldsymbol{J}_i^{C_i} = \boldsymbol{T}_0^{C_i}\, \boldsymbol{J}_i^0$ is the Jacobian that maps the vector of generalized variables speed, $\dot{\boldsymbol{q}}$, to the twists of the link i center of mass relative to base expressed in a frame with axes parallel with the link i frame, $\$_{0,\,C_i}^{i}$.

The Jacobians ($\boldsymbol{J}_i^{C_i}$ and $\boldsymbol{J}_i^0$ for $i = 1, \ldots, n$, where n is the number of joints) have dimension $6\times i$. As a consequence, having in view Eq. (21), each matrix $\boldsymbol{M}_i$ has a different dimension. Hence, the summations symbols on Eq. (19) must be interpreted as a special kind of matrix sum of different dimension. In this special sum, a suitable number of null rows and columns are appended to each matrix in order to make all matrices of the same dimension. This is the main reason that inspired us to build up the manipulators dynamic model in a progressive fashion.

2.2 On Jacobian Calculation

There are three possible approaches to compute the $\boldsymbol{J}_i^{C_i}$ Jacobians. Assuming rotative joints only[3] and expressing all vectors in the base frame:

1. Using a generic Jacobian that maps the vector of generalized variables speed, $\dot{\boldsymbol{q}}$, to the twist of the link i expressed in the base frame, $\$_{0,\,i}^{0}$, and the screw transformation matrix that remaps to the center of mass twist (Tsai, 1999, p. 186-206):

$$\boldsymbol{T}_0^{C_i} = \begin{bmatrix} \boldsymbol{R}_0^i & 0 \\ -\boldsymbol{R}_0^i\, S\left(\boldsymbol{p}_{C_i}\right) & \boldsymbol{R}_0^i \end{bmatrix}, \tag{22}$$

$$\boldsymbol{J}_i^0 = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \cdots & \boldsymbol{z}_{i-1} \\ \boldsymbol{p}_0 \times \boldsymbol{z}_0 & \boldsymbol{p}_1 \times \boldsymbol{z}_1 & \boldsymbol{p}_2 \times \boldsymbol{z}_2 & \cdots & \boldsymbol{p}_{i-1} \times \boldsymbol{z}_{i-1} \end{bmatrix} \quad \text{and} \tag{23}$$

$$\boldsymbol{J}_i^{C_i} = \boldsymbol{T}_0^{C_i}\, \boldsymbol{J}_i^0\,. \tag{24}$$

From Eq. (22) to (24) is obtained:

$$\begin{aligned} \boldsymbol{J}_i^{C_i} &= \begin{bmatrix} \boldsymbol{R}_0^i & 0 \\ -\boldsymbol{R}_0^i\, S\left(\boldsymbol{p}_{C_i}\right) & \boldsymbol{R}_0^i \end{bmatrix} \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \cdots & \boldsymbol{z}_{i-1} \\ S\left(\boldsymbol{p}_0\right)\boldsymbol{z}_0 & S\left(\boldsymbol{p}_1\right)\boldsymbol{z}_1 & \cdots & S\left(\boldsymbol{p}_{i-1}\right)\boldsymbol{z}_{i-1} \end{bmatrix} && (25) \\ &= \begin{bmatrix} \boldsymbol{R}_0^i\, \boldsymbol{z}_0 & \boldsymbol{R}_0^i\, \boldsymbol{z}_1 & \cdots & \boldsymbol{R}_0^i\, \boldsymbol{z}_{i-1} \\ \boldsymbol{R}_0^i\, \boldsymbol{z}_0 \times \boldsymbol{R}_0^i\left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_0\right) & \boldsymbol{R}_0^i\, \boldsymbol{z}_1 \times \boldsymbol{R}_0^i\left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_1\right) & \cdots & \boldsymbol{R}_0^i\, \boldsymbol{z}_{i-1} \times \boldsymbol{R}_0^i\left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_{i-1}\right) \end{bmatrix} && (26) \\ &= \begin{bmatrix} \boldsymbol{R}_0^i\, \boldsymbol{z}_0 & \boldsymbol{R}_0^i\, \boldsymbol{z}_1 & \cdots & \boldsymbol{R}_0^i\, \boldsymbol{z}_{i-1} \\ \boldsymbol{R}_0^i\left(\boldsymbol{z}_0 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_0\right)\right) & \boldsymbol{R}_0^i\left(\boldsymbol{z}_1 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_1\right)\right) & \cdots & \boldsymbol{R}_0^i\left(\boldsymbol{z}_{i-1} \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_{i-1}\right)\right) \end{bmatrix}. && (27) \end{aligned}$$

In this case it is necessary to employ one screw transformation matrix per body (link structure, motor rotor, etc.) and only one Jacobian per link. This approach is advantageous when the principal inertia axes are not parallel with link frame axes. Then $\boldsymbol{R}_0^i$ can compensate the alignment difference.

2. Building up each Jacobian directly:

$$\boldsymbol{J}_i^{C_i} = \begin{bmatrix} \boldsymbol{R}_0^i\, \boldsymbol{z}_0 & \boldsymbol{R}_0^i\, \boldsymbol{z}_1 & \cdots & \boldsymbol{R}_0^i\, \boldsymbol{z}_{i-1} \\ \boldsymbol{R}_0^i\left(\boldsymbol{z}_0 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_0\right)\right) & \boldsymbol{R}_0^i\left(\boldsymbol{z}_1 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_1\right)\right) & \cdots & \boldsymbol{R}_0^i\left(\boldsymbol{z}_{i-1} \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_{i-1}\right)\right) \end{bmatrix}. \tag{28}$$

In this case it is necessary to use one Jacobian per body.

3. Applying a Jacobian as presented by (Siciliano et al., 2009, p. 84-85):

$$\overline{\boldsymbol{T}}_0^{C_i} = \begin{bmatrix} \boldsymbol{R}_0^i & 0 \\ 0 & \boldsymbol{R}_0^i \end{bmatrix}, \tag{29}$$

$$\overline{\boldsymbol{J}}_i^0 = \begin{bmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 & \cdots & \boldsymbol{z}_{i-1} \\ \boldsymbol{z}_0 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_0\right) & \boldsymbol{z}_1 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_1\right) & \boldsymbol{z}_2 \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_2\right) & \cdots & \boldsymbol{z}_{i-1} \times \left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_{i-1}\right) \end{bmatrix} \tag{30}$$

$$\boldsymbol{J}_i^{C_i} = \overline{\boldsymbol{T}}_0^{C_i}\, \overline{\boldsymbol{J}}_i^0\,. \tag{31}$$

It is necessary to use a single screw transformation per link and one Jacobian per body (link structure, motor rotor, etc.) The Jacobian of Eq. (30) is also important to compute the gravitational vectors (see Eq. (36)).

[3] The adaptation for prismatic joints is straight forward.

Note that all vectors are given in base frame coordinates. So the position vector of the mass center of each body (link, motor, etc.), $\boldsymbol{p}_{C_i}$, are expressed in the base coordinated frame. The vector $\left(\boldsymbol{p}_{C_i} - \boldsymbol{p}_i\right)$ is the relative position of the mass center in respect to the link i frame (its magnitude is the distance from the mass center to the origin of frame i), but the coordinates are expressed in the base frame.

2.3 Lagrange's Equations of Motion Using the Joint-Space Inertia Matrix

The Lagrangian formulation of dynamic of robot is found in several textbooks of robotic (Kelly et al., 2005; Tsai, 1999; Siciliano et al., 2009) and it is presented here in a few lines for completeness. Lagrange's equations of motion is given by (Kelly et al., 2005, p. 95):

$$\boldsymbol{M}\left(\boldsymbol{q}\right)\,\ddot{\boldsymbol{q}} + \boldsymbol{C}\left(\boldsymbol{q},\,\dot{\boldsymbol{q}}\right)\,\dot{\boldsymbol{q}} + \boldsymbol{g}\left(\boldsymbol{q}\right) = \boldsymbol{\tau} \tag{32}$$

where

$$\boldsymbol{C}\left(\boldsymbol{q},\,\dot{\boldsymbol{q}}\right)\,\dot{\boldsymbol{q}} = \dot{\boldsymbol{M}}\left(\boldsymbol{q}\right)\,\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\,\left[\dot{\boldsymbol{q}}^T\,\boldsymbol{M}\left(\boldsymbol{q}\right)\,\dot{\boldsymbol{q}}\right] \tag{33}$$

and

$$\boldsymbol{g}\left(\boldsymbol{q}\right) = \frac{\partial \mathcal{U}\left(\boldsymbol{q}\right)}{\partial \boldsymbol{q}}. \tag{34}$$

Equation (32) is the dynamic equation for open chain robots of n-DoF. Observe that $\boldsymbol{C}\left(\boldsymbol{q},\,\dot{\boldsymbol{q}}\right)\,\dot{\boldsymbol{q}}$, in Eq. (32) and (33), is a vector of dimension n called the vector of centrifugal and Coriolis forces, $\boldsymbol{g}\left(\boldsymbol{q}\right)$, in Eq. (32) and (34), is a vector of dimension n of gravitational forces or torques and $\boldsymbol{\tau}$ is a vector of dimension n called the vector of external forces, which in general corresponds to the torques and forces applied by the actuators at the joints (Kelly et al., 2005). Usually, $\boldsymbol{C}\left(\boldsymbol{q},\,\dot{\boldsymbol{q}}\right)$ is calculated by (Kelly et al., 2005; Siciliano et al., 2009):

$$c_{i\,j} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial m_{i\,j}}{\partial q_k} + \frac{\partial m_{i\,k}}{\partial q_j} - \frac{\partial m_{j\,k}}{\partial q_i}\right)\dot{q}_k \tag{35}$$

where the terms between parentheses are called *Christoffel symbols of the first kind*. There are many other ways to compute matrix $\boldsymbol{C}\left(\boldsymbol{q},\,\dot{\boldsymbol{q}}\right)$, but this form makes easy the proof of some usable proprieties (Kelly et al., 2005; Siciliano et al., 2009).

Gravitational forces or torques due each link are:

$$\boldsymbol{g}_i = -m_i\,\overline{\boldsymbol{J}}_i^{0\,T}\,\$_g \tag{36}$$

where $\overline{\boldsymbol{J}}_i^0$ is the Jacobian given by Eq. (30) and $\$_g = \left[\begin{array}{cc} \boldsymbol{0}^T & \boldsymbol{g}_0^{0\,T} \end{array}\right]^T$ is acceleration screw in which $\boldsymbol{g}_0^0$ is the acceleration due gravity expressed in the base frame. Note that each vector $\boldsymbol{g}_i$ has size i, so care should be taken in adding then.

2.4 Method Systematization

Each term of joint-space inertia matrix, $\boldsymbol{M}_i$, is given by Eq. (21) and the Jacobian has dimension $6 \times i$. So, each $\boldsymbol{M}_i$ matrix has different dimension. Thus, the inertia matrix, $\boldsymbol{M}$, in Eq. (32) is given by:

$$\boldsymbol{M}\left(\boldsymbol{q}\right) = \boldsymbol{M}_1 \ddagger \boldsymbol{M}_2 \ddagger \cdots \ddagger \boldsymbol{M}_2$$

where the symbol ‡ is used to denote a special kind of matrix/vector addition in which a suitable number of null rows and columns are appended to make the matrices dimension compatible for addition. It could be considered $6 \times n$ Jacobians instead of $6 \times i$ and then all matrices $\boldsymbol{M}_i$ would have the same $n \times n$ dimension, but the purpose of this work is to consider the cumulative effects created by the addition of new link to the kinematic chain. So, n is not known *a priori* in this method because it can change during the task execution. Examples of this kind of application are found in liquid transportation tasks where it is possible to find a payload with passive joints used to allow the correct orientation of the recipient.

Since $\boldsymbol{C}$ depends on $\boldsymbol{M}$, in Eq. (35), in a linear fashion, it is possible to compute one matrix $\boldsymbol{C}_i$ per link applying Eq. (35) for each matrix $\boldsymbol{M}_i$. This allows us to build up a complete dynamic model of each sub-chain of the manipulator, from the base to the end-effector. So, the analysis is progressive in the meaning that it advances progressively from the base to the end-effector. The matrices $\boldsymbol{C}_i$ will be add up in the very same way than matrices $\boldsymbol{M}_i$ and so will be the vectors $\boldsymbol{g}_i$.

3. EXAMPLE

As case study is developed the dynamic model of a two-link planar arm, sometimes called pelican manipulator (Kelly et al., 2005, p. 113), in a progressive way. First the robot model is derived, then the payload is included. The payload is a crucible attached to the last link by a passive rotative joint in order to allow the crucible being kept in a horizontal orientation during the movement. Hence, the robot has two different models corresponding to two configurations: without the payload it is a two-link planar arm (see Fig. 1-a); and with the payload it is a three-link planar arm with one passive joint (see Fig. 1-b). These models are helpful to develope an anti-swing controller.

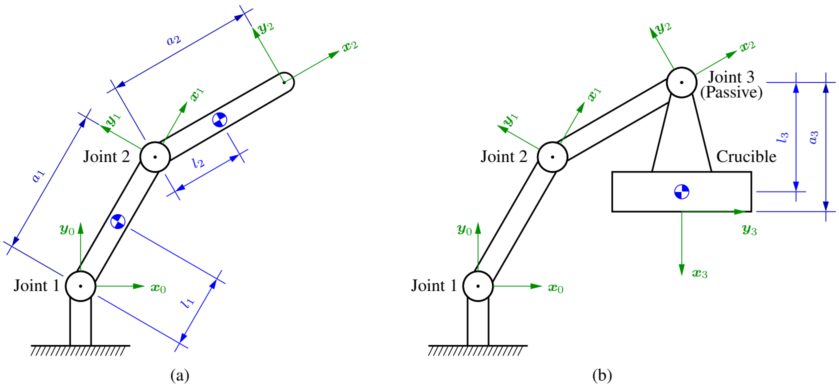

Figure 1. (a) The manipulator without the payload and (b) the manipulator with payload (crucible)

3.1 Model of 2-DoF Manipulator

The link lengths are a_1 and a_2, and the masses are m_1 and m_2 for links 1 and 2, respectively. The distance from the rotating axes to the centers of mass are denoted by l_1 and l_2 for links 1 and 2, respectively. Finally, I_{x_1}, I_{y_1}, and I_{z_1}, and I_{x_2}, I_{y_2}, and I_{z_2} denote the moments of inertia[4] of the links with respect to the axes that pass through the respective center of mass and are parallel the respective link frame axes. The products of inertia are neglected, but their inclusion is straight forward. Table 1 brings the arm parameters according to Denavit-Hartenberg convention (Siciliano et al., 2009, p. 42). The last row, $i = 2$, is respective to the payload. The joint variables are $\boldsymbol{q} = \begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix}^T$ for the manipulator only (without the payload). Joints 1 and 2 are actives and two actuators apply torques τ_1 and τ_2 to the respective joints. Therefore, the external torque vector is $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 & \tau_2 \end{bmatrix}^T$.

The joint-space inertia matrices, given by Eq. (9), are:

$$\overline{\boldsymbol{M}}_1 = \begin{bmatrix} I_{x_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{y_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{z_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_1 \end{bmatrix}, \text{ and } \overline{\boldsymbol{M}}_2 = \begin{bmatrix} I_{x_2} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{z_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m_2 \end{bmatrix}. \tag{37}$$

Using the first two rows of Tab. 1 it is possible to build the homogeneous transformations:

$$A_1^0 = \begin{bmatrix} \mathrm{c}_1 & -\mathrm{s}_1 & 0 & a_1\,\mathrm{c}_1 \\ \mathrm{s}_1 & \mathrm{c}_1 & 0 & a_1\,\mathrm{s}_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\ A_2^1 = \begin{bmatrix} \mathrm{c}_2 & -\mathrm{s}_2 & 0 & a_2\,\mathrm{c}_2 \\ \mathrm{s}_2 & \mathrm{c}_2 & 0 & a_2\,\mathrm{s}_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and} \tag{38}$$

$$A_2^0 = A_1^0\,A_2^1 = \begin{bmatrix} \mathrm{c}_{12} & -\mathrm{s}_{12} & 0 & a_1\,\mathrm{c}_1 + a_2\,\mathrm{c}_{12} \\ \mathrm{s}_{12} & \mathrm{c}_{12} & 0 & a_1\,\mathrm{s}_1 + a_2\,\mathrm{s}_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{39}$$

where $\mathrm{c}_i = \cos\theta_i$, $\mathrm{s}_i = \sin\theta_i$, $\mathrm{c}_{ij} = \cos(\theta_i + \theta_j)$, $\mathrm{s}_{ij} = \sin(\theta_i + \theta_j)$.

The positions of the centers of mass in respect to the base frame are calculated multiplying the respective homogeneous transformation by the position vector of the centers of mass in respect to the link frame. So, $\boldsymbol{p}_{C_1} = A_1^0\,\boldsymbol{p}_{C_1}^1$ and $\boldsymbol{p}_{C_2} =$

[4] The moments of inertia I_{x_1}, I_{y_1}, I_{x_2}, and I_{y_2} are unimportant because all axes are parallel the axis z and, consequently, they should be neglected.

Table 1. Arm parameters including the last link (payload)

i	A_i^{i-1}	a_i	α_i	d_i	θ_i
0	A_1^0	a_1	0	0	θ_1
1	A_2^1	a_2	0	0	θ_2
2	A_3^2	a_3	0	0	θ_3

$A_2^0 \boldsymbol{p}_{C_2}^2$ where the vector on both side of the equation were augmented be appending a 1 to the column matrices.

$$\boldsymbol{p}_{C_1}^1 = \begin{bmatrix} l_1 - a_1 \\ 0 \\ 0 \end{bmatrix}, \boldsymbol{p}_{C_2}^2 = \begin{bmatrix} l_2 - a_2 \\ 0 \\ 0 \end{bmatrix}, \Rightarrow \boldsymbol{p}_{C_1} = \begin{bmatrix} l_1 \mathrm{~c}_1 \\ l_1 \mathrm{~s}_1 \\ 0 \end{bmatrix}, \boldsymbol{p}_{C_2} = \begin{bmatrix} a_1 \mathrm{~c}_1 + l_2 \mathrm{~c}_{12} \\ a_1 \mathrm{~s}_1 + l_2 \mathrm{~s}_{12} \\ 0 \end{bmatrix}. \tag{40}$$

Remember that the origin of each link frame is located over the next axis (distal joint, see Fig. 1).

Using Eq. (14) with the rotation matrices obtained from the matrices of Eq. (38) and (39) and the position vectors of Eq. (40), the screw transformation matrices are written as:

$$\boldsymbol{T}_0^{C_1} = \begin{bmatrix} \mathrm{c}_1 & \mathrm{s}_1 & 0 & 0 & 0 & 0 \\ -\mathrm{s}_1 & \mathrm{c}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{c}_1 & \mathrm{s}_1 & 0 \\ 0 & 0 & l_1 & -\mathrm{s}_1 & \mathrm{c}_1 & 0 \\ l_1 \mathrm{~s}_1 & -l_1 \mathrm{~c}_1 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{41}$$

$$\boldsymbol{T}_0^{C_2} = \begin{bmatrix} \mathrm{c}_{12} & \mathrm{s}_{12} & 0 & 0 & 0 & 0 \\ -\mathrm{s}_{12} & \mathrm{c}_{12} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_1 \mathrm{~s}_2 & \mathrm{c}_{12} & \mathrm{s}_{12} & 0 \\ 0 & 0 & l_2 + a_1 \mathrm{~c}_2 & -\mathrm{s}_{12} & \mathrm{c}_{12} & 0 \\ a_1 \mathrm{~s}_1 + l_2 \mathrm{~s}_{12} & -a_1 \mathrm{~c}_1 - l_2 \mathrm{~c}_{12} & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

The z axis unit vector and the position vector of origin of the base frame in respect to base frame are:

$$\boldsymbol{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \boldsymbol{p}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{43}$$

The z axis unit vector and the position vector of origin of the frame 1 in respect to base frame are obtained from the matrix A_1^0 given by Eq. (38) and leads to:

$$\boldsymbol{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \boldsymbol{p}_1 = \begin{bmatrix} a_1 \mathrm{~c}_1 \\ a_1 \mathrm{~s}_1 \\ 0 \end{bmatrix}. \tag{44}$$

Equations (43) and (44) are used to calculated the Jacobians by Eq. (23) as:

$$\boldsymbol{J}_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{J}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & a_1 \mathrm{~s}_1 \\ 0 & -a_1 \mathrm{~c}_1 \\ 0 & 0 \end{bmatrix} \tag{45}$$

which, together with the screw transformation of Eq. (22), leads to

$$\boldsymbol{J}_1^{C_1} = \boldsymbol{T}_0^{C_1} \boldsymbol{J}_1^0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ l_1 \\ 0 \end{bmatrix}, \text{ and } \boldsymbol{J}_2^{C_2} = \boldsymbol{T}_0^{C_2} \boldsymbol{J}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ a_1 \mathrm{~s}_2 & 0 \\ l_2 + a_1 \mathrm{~c}_2 & l_2 \\ 0 & 0 \end{bmatrix} \tag{46}$$

by the Eq. (24).

The joint-space inertia matrix for each link is given by Eq. (21), which takes the form $\boldsymbol{M}_1 = \boldsymbol{J}_1^{C_1\,T}\,\overline{\boldsymbol{M}}_1\,\boldsymbol{J}_1^{C_1}$ and $\boldsymbol{M}_2 = \boldsymbol{J}_2^{C_2\,T}\,\overline{\boldsymbol{M}}_2\,\boldsymbol{J}_2^{C_2}$ for links 1 and 2, respectively, as:

$$\boldsymbol{M}_1 = \left[m_1\,l_1{}^2 + I_{z_1}\right]\ ,\ \text{and}\ \boldsymbol{M}_2 = \begin{bmatrix} m_2\ \left(a_1{}^2 + l_2{}^2 + 2\,a_1\,l_2\ \mathrm{c}_2\right) + I_{z_2} & m_2\,l_2\ (l_2 + a_1\ \mathrm{c}_2) + I_{z_2} \\ m_2\,l_2\ (l_2 + a_1\ \mathrm{c}_2) + I_{z_2} & m_2\,l_2{}^2 + I_{z_2} \end{bmatrix} \tag{47}$$

Applying Eq. (35) for matrix $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ leads to:

$$\boldsymbol{C}_1 = [0]\ ,\ \text{and}\ \boldsymbol{C}_2 = \begin{bmatrix} -m_2\,a_1\,l_2\ \mathrm{s}_2\ \dot{\theta}_2 & -m_2\,a_1\,l_2\ \mathrm{s}_2\ \left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ m_2\,a_1\,l_2\ \mathrm{s}_2\ \dot{\theta}_1 & 0 \end{bmatrix}. \tag{48}$$

The dynamic model is completed by the gravitational vector, $\boldsymbol{g}$, calculated term by term using Eq. (36) in with $\$_g = \begin{bmatrix} 0 & 0 & 0 & 0 & -g & 0 \end{bmatrix}^T$ and Eq. (30) by:

$$\overline{\boldsymbol{J}}_1^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ -l_1\ \mathrm{s}_1 & 0 \\ l_1\ \mathrm{c}_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{49}$$

$$\overline{\boldsymbol{J}}_2^0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -a_1\ \mathrm{s}_1 - l_2\ \mathrm{s}_{12} & -l_2\ \mathrm{s}_{12} \\ a_1\ \mathrm{c}_1 + l_2\ \mathrm{c}_{12} & l_2\ \mathrm{c}_{12} \\ 0 & 0 \end{bmatrix} \tag{50}$$

$$\boldsymbol{g}_1 = -m_1\,\overline{\boldsymbol{J}}_1^{0\,T}\,\$_g = [m_1\,g\,l_1\ \mathrm{c}_1] \tag{51}$$

$$\boldsymbol{g}_2 = -m_2\,\overline{\boldsymbol{J}}_2^{0\,T}\,\$_g = \begin{bmatrix} m_2\,g\ (a_1\ \mathrm{c}_1 + l_2\ \mathrm{c}_{12}) \\ m_2\,g\,l_2\ \mathrm{c}_{12} \end{bmatrix} \tag{52}$$

where g is the acceleration due to gravity.

The progressive model is obtained summing the matrices and vectors as: $\boldsymbol{M} = \boldsymbol{M}_1 \ddagger \boldsymbol{M}_2$, $\boldsymbol{C} = \boldsymbol{C}_1 \ddagger \boldsymbol{C}_2$, and $\boldsymbol{g} = \boldsymbol{g}_1 \ddagger \boldsymbol{g}_2$ that must be applied in Eq. (32). The symbol $\ddagger$ is used to denote a special kind of matrix/vector addition (see Section 2).

3.2 Payload Inclusion

The effects due the payload are taken into account considering a new link attached to the last link by a passive rotative joint. Note that the model of the manipulator (two axes) is completed and now a extra like will be attached to the model. Therefore, the joint variables are $\boldsymbol{q} = \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 \end{bmatrix}^T$. The external torque vector is $\boldsymbol{\tau} = \begin{bmatrix} \tau_1 & \tau_2 & 0 \end{bmatrix}^T$ since joint 3 is passive; no external torque is applied to it. The link mass is m_3 and center of mass is located at a distance l_3 from the joint 3 axis (see Fig. 1). The moments of inertia of the links with respect to the axes that pass through the respective center of mass and are parallel the respective link frame axes are denoted by I_{x_3}, I_{y_3}, and I_{z_3} and the products of inertia are neglected. Equation (53) brings the homogeneous transformations computed with the data available in Tab. 1 and, together with Eq. (39), gives the transformations in Eq. (54).

$$A_3^2 = \begin{bmatrix} \mathrm{c}_3 & -\mathrm{s}_3 & 0 & a_3\ \mathrm{c}_3 \\ \mathrm{s}_3 & \mathrm{c}_3 & 0 & a_3\ \mathrm{s}_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \tag{53}$$

$$A_3^0 = A_2^0\,A_3^2 = \begin{bmatrix} \mathrm{c}_{123} & -\mathrm{s}_{123} & 0 & a_1\ \mathrm{c}_1 + a_2\ \mathrm{c}_{12} + a_3\ \mathrm{c}_{123} \\ \mathrm{s}_{123} & \mathrm{c}_{123} & 0 & a_1\ \mathrm{s}_1 + a_2\ \mathrm{s}_{12} + a_3\ \mathrm{s}_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{54}$$

The position vector of the center of mass of payload is given by $\boldsymbol{p}_{C_3} = A_3^0\,\boldsymbol{p}_{C_3}^3$ where

$$\boldsymbol{p}_{C_3}^3 = \begin{bmatrix} l_3 - a_3 \\ 0 \\ 0 \end{bmatrix}\ ,\ \boldsymbol{p}_{C_3} = \begin{bmatrix} a_1\ \mathrm{c}_1 + a_2\ \mathrm{c}_{12} + l_3\ \mathrm{c}_{123} \\ a_1\ \mathrm{s}_1 + a_2\ \mathrm{s}_{12} + l_3\ \mathrm{s}_{123} \\ 0 \end{bmatrix}, \tag{55}$$

Using, again, Eq. (14) with the rotation matrices obtained from the matrices of Eq. (54) and the position vectors of Eq. (55), the screw transformation matrix is written as:

$$\boldsymbol{T}_0^{C_3} = \begin{bmatrix} \mathrm{c}_{123} & \mathrm{s}_{123} & 0 & 0 & 0 & 0 \\ -\mathrm{s}_{123} & \mathrm{c}_{123} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23} & \mathrm{c}_{123} & \mathrm{s}_{123} & 0 \\ 0 & 0 & l_3 + a_2\ \mathrm{c}_3 + a_1\ \mathrm{c}_{23} & -\mathrm{s}_{123} & \mathrm{c}_{123} & 0 \\ a_1\ \mathrm{s}_1 + a_2\ \mathrm{s}_{12} + l_3\ \mathrm{s}_{123} & -a_1\ \mathrm{c}_1 - a_2\ \mathrm{c}_{12} - l_3\ \mathrm{c}_{123} & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{56}$$

Extracting $\boldsymbol{z}_2$ and $\boldsymbol{p}_2$ from Eq. (39) and applying the Eq. (23) and (22), gives

$$\boldsymbol{J}_3^{C_3} = \boldsymbol{T}_0^{C_3}\, \boldsymbol{J}_3^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23} & a_2\ \mathrm{s}_3 & 0 \\ l_3 + a_2\ \mathrm{c}_3 + a_1\ \mathrm{c}_{23} & l_3 + a_2\ \mathrm{c}_3 & l_3 \\ 0 & 0 & 0 \end{bmatrix} \tag{57}$$

from which Eq. (21) leads to

$$\begin{aligned}
\boldsymbol{M}_3 &= \boldsymbol{J}_3^{C_3\,T}\, \overline{\boldsymbol{M}}_3\, \boldsymbol{J}_3^{C_3} = \begin{bmatrix} m_{3_{11}} & m_{3_{12}} & m_{3_{13}} \\ m_{3_{21}} & m_{3_{22}} & m_{3_{23}} \\ m_{3_{31}} & m_{3_{32}} & m_{3_{33}} \end{bmatrix}, \text{ where} \qquad (58) \\
m_{3_{11}} &= m_3\ (l_3 + a_2\ \mathrm{c}_3 + a_1\ \mathrm{c}_{23})^2 + m_3\ (a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23})^2\ m_3 + I_{z_3} \\
m_{3_{12}} &= m_3\ \left(a_2{}^2 + l_3{}^2 + a_1\, a_2\ \mathrm{c}_2 + 2\, a_2\, l_3\ \mathrm{c}_3 + a_1\, l_3\ \mathrm{c}_{23}\right) + I_{z_3} \\
m_{3_{13}} &= l_3\, m_3\ (l_3 + a_2\ \mathrm{c}_3 + a_1\ \mathrm{c}_{23}) + I_{z_3} \\
m_{3_{21}} &= m_3\ \left(a_2{}^2 + l_3{}^2 + a_1\, a_2\ \mathrm{c}_2 + 2\, a_2\, l_3\ \mathrm{c}_3 + a_1\, l_3\ \mathrm{c}_{23}\right) + I_{z_3} \\
m_{3_{22}} &= m_3\ \left(a_2{}^2 + l_3{}^2 + 2\, a_2\, l_3\ \mathrm{c}_3\right) + I_{z_3} \\
m_{3_{23}} &= m_3\, l_3\ (l_3 + a_2\ \mathrm{c}_3) + I_{z_3} \\
m_{3_{31}} &= l_3\, m_3\ (l_3 + a_2\ \mathrm{c}_3 + a_1\ \mathrm{c}_{23}) + I_{z_3} \\
m_{3_{32}} &= m_3\, l_3\ (l_3 + a_2\ \mathrm{c}_3) + I_{z_3} \\
m_{3_{33}} &= m_3\, l_3{}^2 + I_{z_3}
\end{aligned}$$

and using Eq. (35), gives:

$$\begin{aligned}
\boldsymbol{C}_3 &= \begin{bmatrix} c_{3_{11}} & c_{3_{12}} & c_{3_{13}} \\ c_{3_{21}} & c_{3_{22}} & c_{3_{23}} \\ c_{3_{31}} & c_{3_{32}} & c_{3_{33}} \end{bmatrix}, \text{ where} \qquad (59) \\
c_{3_{11}} &= -m_3\ \left(a_1\ (a_2\ \mathrm{s}_2 + l_3\ \mathrm{s}_{23})\ \dot{\theta}_2 + l_3\ (a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23})\ \dot{\theta}_3\right) \\
c_{3_{12}} &= -m_3\ \left((a_2\ \mathrm{s}_2 + l_3\ \mathrm{s}_{23})\ \left(\dot{\theta}_1 + \dot{\theta}_2\right) + l_3\ (a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23})\ \dot{\theta}_3\right) \\
c_{3_{13}} &= -m_3\, l_3\ (a_2\ \mathrm{s}_3 + a_1\ \mathrm{s}_{23})\ \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right) \\
c_{3_{21}} &= m_3\ \left(a_1\ (a_2\ \mathrm{s}_2 + l_3\ \mathrm{s}_{23})\ \dot{\theta}_1 - a_2\, l_3\ \mathrm{s}_3\ \dot{\theta}_3\right) \\
c_{3_{22}} &= -m_3\, l_3\, a_2\ \mathrm{s}_3\ \dot{\theta}_3 \\
c_{3_{23}} &= -m_3\, l_3\, a_2\ \mathrm{s}_3\ \left(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3\right) \\
c_{3_{31}} &= m_3\, l_3\ \left(a_2\ \mathrm{s}_3\ \left(\dot{\theta}_1 + \dot{\theta}_2\right) + a_1\ \mathrm{s}_{23}\ \dot{\theta}_1\right) \\
c_{3_{32}} &= m_3\, l_3\, a_2\ \mathrm{s}_3\ \left(\dot{\theta}_1 + \dot{\theta}_2\right) \\
c_{3_{33}} &= 0
\end{aligned}$$

The gravitational vector is calculated by Eq. (36) and the Jacobian of Eq. (30):

$$\overline{\boldsymbol{J}}_3^0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -a_1\ \mathrm{s}_1 - a_2\ \mathrm{s}_{12} - l_3\ \mathrm{s}_{123} & -a_2\ \mathrm{s}_{12} - l_3\ \mathrm{s}_{123} & -l_3\ \mathrm{s}_{123} \\ a_1\ \mathrm{c}_1 + a_2\ \mathrm{c}_{12} + l_3\ \mathrm{c}_{123} & a_2\ \mathrm{c}_{12} + l_3\ \mathrm{c}_{123} & l_3\ \mathrm{c}_{123} \\ 0 & 0 & 0 \end{bmatrix} \tag{60}$$

$$\boldsymbol{g}_3 = -m_3\, \overline{\boldsymbol{J}}_3^{0\,T}\ \$_g = \begin{bmatrix} m_3\, g\ (a_1\ \mathrm{c}_1 + a_2\ \mathrm{c}_{12} + l_3\ \mathrm{c}_{123}) \\ m_3\, g\ (a_2\ \mathrm{c}_{12} + l_3\ \mathrm{c}_{123}) \\ m_3\, g\, l_3\ \mathrm{c}_{123} \end{bmatrix}. \tag{61}$$

The final and complete model is given by Eq. (32) in which matrices $\boldsymbol{M}$ and $\boldsymbol{C}$ and the vector $\boldsymbol{g}$ are given by the summations:

$$\boldsymbol{M} = \boldsymbol{M}_1 \ddagger \boldsymbol{M}_2 \ddagger \boldsymbol{M}_3 \tag{62}$$

$$\boldsymbol{C} = \boldsymbol{C}_1 \ddagger \boldsymbol{C}_2 \ddagger \boldsymbol{C}_3 \tag{63}$$

$$\boldsymbol{g} = \boldsymbol{g}_1 \ddagger \boldsymbol{g}_2 \ddagger \boldsymbol{g}_3 \tag{64}$$

where, again, the symbol $\ddagger$ is used to denote a special kind of matrix/vector addition (see Section 2). Matrices $\boldsymbol{M}_1$, $\boldsymbol{M}_2$, $\boldsymbol{M}_3$, $\boldsymbol{C}_1$, $\boldsymbol{C}_2$, and $\boldsymbol{C}_3$ are given by Eq. (47), (58), (48), and (59), respectively. Vectors $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, and $\boldsymbol{g}_3$, are given by Eq. (51), (52), and (61), respectively.

4. CONCLUSIONS

This work presented some advances in the theory applied to the field of computational dynamics of multibody systems. Particularly, it shows how the dynamics effect of a serial kinematic chain, or more generally a tree like kinematic chain, can be summed. The Jacobians employed in the $\boldsymbol{M}_i$ computations are simpler than the counterpart employed in the literature (Siciliano et al., 2009, p. 150). Screw approach can be used to reduce the efforts in computation if the dimension of the screw system, λ, is less than 6. If instead of 6, it was considered in the example that $\lambda = 3$, the Jacobians would be reduced to dimension $3 \times i$ and that could save some manipulation effort. The presented approach is quite helpful in the growing field reconfigurable robots. In futures works, closed kinematic chains should be addressed. The changes necessary to apply the method in kinematic chains with prismatic joint and to model joint motor and transmission are really straight forward.

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