MODEL REDUCTION BASED ON KARHUNEN-LOEVE AND SMOOTH KARHUNEN-LOEVE DECOMPOSITIONS FOR RANDOM MECHANICAL SYSTEMS

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Abstract. *In this paper we deal with the problem of constructing reduced-order models for linear and nonlinear discrete systems under random excitation. A Galerkin approximation is used to project the n-dimensional dynamics in a m-dimensional subspace generated by a set of Karhunen-Loève modes, or Smooth Karhunen-Loève modes. The resulting approximations are compared with each other and compared with the classical modal reduction technique based on the linear normal modes of the underlying linear system. Linear and nonlinear mechanical systems are considered with homogeneous and non-homogeneous mass distribution.*

Keywords: Karhunen-Loève theory, modal analysis, nonlinear random vibrations

1. INTRODUCTION

The Karhunen-Loève Decomposition (KLD) method has been extensively used as a tool for analyzing random fields, [Holmes et al., 1996]. The KLD reveals some coherent structures which have been advantageously used in different domains as, for example, the stochastic-finite-elements method, the simulation of random fields, the modal analysis of nonlinear systems, and construction of reduced-order models. Depending on the discipline and the properties of the random field under study, but also on the averaging operator used to build the KLD[Bellizzi and Sampaio, 2006, Bellizzi and Sampaio, 2007], this decomposition has been named principal component analysis (PCA), proper orthogonal decomposition (POD), or singular value decomposition (SVD). In structural vibration, the KLD has been principally applied to the displacement field but it can be applied to either the displacement or the velocity field and, also, the displacement-velocity field[Bellizzi and Sampaio, 2009].

Recently, a new multivariable data analysis method called Smooth Orthogonal Decomposition (SOD) has been proposed by [Chelidze and Zhou, 2006]. The SOD is defined from a maximization problem associated to a scalar time series of measurement but subject to a minimization constraint acting on the associated time derivative of the time series. The SOD can be used to extract normal modes and natural frequencies of multi-degree-of-freedom vibration systems. Free and forced sinusoidal responses have been considered in [Chelidze and Zhou, 2006] and randomly excited systems have been analyzed in [Farooq and Feeny, 2008]. The Smooth Orthogonal Decomposition has been formulated in term of a Smooth Karhunen-Loève Decomposition (SKLD) to analyze (time continuous) random fields in [Bellizzi and Sampaio, 2009b]. The SKLD is obtained solving a generalized eigenproblem defined from the covariance matrix of the random field and the covariance matrix of its time derivative.

In this paper we deal with the problem of constructing reduced-order models for linear and nonlinear discrete mechanical systems with preserve the second order structure. A Galerkin approximation is used to project the n-dimensional dynamics in a m-dimensional subspace. The efficiency of such an approach depends largely on whether the full solution dynamics are essentially contained in the subspace spanned by the basis vectors. The reduced order models are obtained using the set of Karhunen-Loève modes or smooth Karhunen-Loève modes. The resulting approximations are compared with each other and compared with the classical modal reduction technique based on the Linear Normal Modes (LNM) of the underlying linear system.

2. SMOOTH KARHUNEN-LOEVE DECOMPOSITION

Let $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ be a $\mathbb{R}^n$-valued random process indexed in $\mathbb{R}$. We assume that $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ is a second order stationary process and admits a time derivative process $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ which is also a second order stationary process. With these assumptions, the covariance matrices of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$, denoted $\mathbf{R_U} = \mathbb{E}(\mathbf{U}(t)^T\mathbf{U}(t))$ and $\mathbf{R_{\dot{U}}} = \mathbb{E}(\dot{\mathbf{U}}(t)^T\dot{\mathbf{U}}(t))$ respectively, do not depend on time. Without loss of generality, we will also assume that $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ is a zero-mean random process and that $\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$ are symmetric positive definite.

2.1 Decomposition principle

The Smooth Karhunen-Loève Decomposition of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ aims at obtaining the most characteristic constant vectors Γ in the sense that they maximize the ratio of the ensemble average of the inner product between $\mathbf{U}(t)$ and Γ and

the inner product between $\dot{\mathbf{U}}(t)$ and Γ

$$\max_{\Gamma \in \mathbb{R}^n} \frac{\mathbb{E}(<\mathbf{U}(t), \Gamma>^2)}{\mathbb{E}(<\dot{\mathbf{U}}(t), \Gamma>^2)} \tag{1}$$

where $<,>$ denotes the inner product in $\mathbb{R}^n$.

Due to the stationary property, the objective function (1) reads as

$$\max_{\Gamma \in \mathbb{R}^n} \frac{\Gamma^T \mathbf{R_U} \Gamma}{\Gamma^T \mathbf{R_{\dot{U}}} \Gamma} \tag{2}$$

showing that the cost function is time independent and only depends on the covariance matrices of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$. The vectors which yields the maximum are solutions of the eigenproblem

$$\mathbf{R_U} \mathbf{\Gamma}_k = \sigma_k \mathbf{R_{\dot{U}}} \mathbf{\Gamma}_k. \tag{3}$$

The Smooth Karhunen-Loève Decomposition (SKLD) of the random field will then be given by

$$\mathbf{U}(t) = \sum_{k=1}^{n} \zeta_k(t) \mathbf{\Gamma}_k \tag{4}$$

where the vectors $\mathbf{\Gamma}_k$ solve the generalized eigenproblem (3) and the scalar random processes, $\zeta_k(t)$ are given by

$$\zeta_k(t) = \frac{\mathbf{\Gamma}_k^T \mathbf{R_U} \mathbf{U}(t)}{\mathbf{\Gamma}_k^T \mathbf{R_U} \mathbf{\Gamma}_k} = \frac{\mathbf{\Gamma}_k^T \mathbf{R_{\dot{U}}} \mathbf{U}(t)}{\mathbf{\Gamma}_k^T \mathbf{R_{\dot{U}}} \mathbf{\Gamma}_k}. \tag{5}$$

Note that the scalar processes $\{\zeta_k(t),\ t \in \mathbb{R}\}$ can be defined from either $\mathbf{R_U}$ or $\mathbf{R_{\dot{U}}}$, that is they do not depend on which one of this two covariance matrices is used.

For this definition, the following notation is used: the eigenvalues σ_k are called the Smooth Karhunen-Loève Values (SKLVs) ($\mathbf{\Sigma} = \mathrm{diag}(\sigma_k)$), the eigenvectors $\mathbf{\Gamma}_k$ are called the Smooth Karhunen-Loève Modes (SKLMs) ($\mathbf{\Gamma} = [\mathbf{\Gamma}_1 \mathbf{\Gamma}_1 \cdots \mathbf{\Gamma}_n]$), and the scalar random processes $\{\zeta_k(t)\}$ are called the Smooth Karhunen-Loève Components (SKLCs).

The generalized eigenproblem (3) is a *statistical* version (for *continuous-time* random process), of the generalized eigenvalue problem introduced in [Chelidze and Zhou, 2006] to characterize the SOD, what constitutes a major difference. In the definition (3) only the covariance matrices $\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$ are used, no other operator is necessary. The idea comes from [Bellizzi and Sampaio, 2009]. The results are, of course, similar to the ones presented in [Chelidze and Zhou, 2006, Farooq and Feeny, 2008], but now, since one relies on the covariance matrices, one has a powerful computation tool, not available before.

The objective function used to define the SKLD differs significantly from that used to define the classical Karhunen-Loève Decomposition, [Bellizzi and Sampaio, 2006]. Here the denominator of the objective function takes the covariance matrix of the time-derivative process $\{\dot{\mathbf{U}}(t),\ t \in \mathbb{R}\}$ into account (which justifies the name *smooth* KLD).

2.2 Some properties of the SKLD

2.2.1 Properties of SKLV, SKLM, and SKLC

The matrices $\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$ being symmetric positive definite, all the SKLVs (eigenvalues) ν_k are strictly positive and the set of the vectors $\mathbf{\Gamma}_k$ (the SKLMs) constitutes a basis which is orthogonal with respect to both covariance matrices $\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$. Note that the SKLM are unique up to a scaling constant.

The scalar processes $\{\zeta_k(t),\ t \in \mathbb{R}\}$ are correlated

$$\mathbb{E}(\zeta_k(t)\zeta_l(t)) = \frac{\mathbf{\Gamma}_k^T \mathbf{R_U} \mathbf{R_U} \mathbf{R_U} \mathbf{\Gamma}_l}{\mathbf{\Gamma}_k^T \mathbf{R_U} \mathbf{\Gamma}_k \mathbf{\Gamma}_l^T \mathbf{R_U} \mathbf{\Gamma}_l} = \frac{\mathbf{\Gamma}_k^T \mathbf{R_{\dot{U}}} \mathbf{R_U} \mathbf{R_{\dot{U}}} \mathbf{\Gamma}_l}{\mathbf{\Gamma}_k^T \mathbf{R_{\dot{U}}} \mathbf{\Gamma}_k \mathbf{\Gamma}_l^T \mathbf{R_{\dot{U}}} \mathbf{\Gamma}_l}. \tag{6}$$

So, the SKLVs are not related to energy distribution and, of course, the SKLD does not satisfy the standard optimality relationship. So, properly speaking, the SKLV is not a Karhunen-Loève decomposition. The introduction of regularity has then its drawbacks.

2.2.2 Linear transformation of the SKLD

Let $\{\mathbf{V}(t),\ t \in \mathbb{R}\}$ be a $\mathbb{R}^n$-valued random process defined as

$$\mathbf{V}(t) = \mathbf{A}\mathbf{U}(t) \tag{7}$$

where $\mathbf{A}$ is a square invertible matrix.

From the relationships

$$\mathbf{R_V} = \mathbf{A}\mathbf{R_U}\mathbf{A}^T \text{ and } \mathbf{R_{\dot V}} = \mathbf{A}\mathbf{R_{\dot U}}\mathbf{A}^T$$

it can be shown that the SKLVs of $\{\mathbf{V}(t),\ t \in \mathbb{R}\}$ coincide with those of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ and the sets of the SKLMs satisfy the condition

$$\mathbf{\Gamma}_k(V) = \mathbf{A}^{-T}\mathbf{\Gamma}_k(U) \tag{8}$$

where $\mathbf{\Gamma}_k(U)$ (respectively $\mathbf{\Gamma}_k(V)$) denotes a SKLM of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ (respectively, of $\{\mathbf{V}(t),\ t \in \mathbb{R}\}$). Finally, following (5), the SKLCs are invariant with respect to the linear change of variables if and only if $\mathbf{A}\mathbf{A}^T = \mathbf{I}$.

3. MECHANICAL INTERPRETATION OF THE SKLD

Consider a discrete linear mechanical system with n degrees of freedom governing by the equations of motion

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) = \mathbf{F}(t) \tag{9}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n \times n$ symmetric square matrices and the excitation vector, $\{\mathbf{F}(t), t \in \mathbb{R}\}$, is a zero-mean white-noise random excitation (i.e., $\mathbf{R}_F(\tau) = \mathbb{E}(\mathbf{F}(t+\tau)\mathbf{F}^T(t)) = \mathbf{S}_F\delta(\tau)$, where the intensity $\mathbf{S}_F$ is a symmetric constant matrix).

Let $\mathbf{\Phi} = [\mathbf{\Phi}_1 \cdots \mathbf{\Phi}_i \cdots \mathbf{\Phi}_n]$ the modal matrix defined from the Linear Normal Modes (LNM) ($\mathbf{K}\mathbf{\Phi}_k = \mathbf{M}\mathbf{\Phi}_k\mathbf{\Omega}_k^2$ with the normalization conditions $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}$ and $\mathbf{\Omega}^2 = \text{diag}(\omega_i^2)$). Introducing the modal-displacement vector $\mathbf{Q}(t)$ with

$$\mathbf{U}(t) = \mathbf{\Phi}\mathbf{Q}(t) = \sum_{i=1}^{n} \mathbf{\Phi}_i Q_i(t), \tag{10}$$

the equation of motion (9) can be equivalently replaced by

$$\ddot{\mathbf{Q}}(t) + \mathbf{\Theta}\dot{\mathbf{Q}}(t) + \mathbf{\Omega}^2\mathbf{Q}(t) = \mathbf{\Phi}^T\mathbf{F}(t) \tag{11}$$

with $\mathbf{\Theta} = \mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi}$.

3.1 SKLD and modal analysis

The objective of this section is to discuss the relationships between the SKLD of the steady-state response of (9) and modal characteristics, normal modes and frequencies.

If the damping is proportional (i.e., $\mathbf{\Phi}^T\mathbf{C}\mathbf{\Phi} = \text{diag}(2\tau_i\omega_i)$ is diagonal) and the matrix $\mathbf{\Phi}^T\mathbf{S}_F\mathbf{\Phi}$ is also diagonal (i.e., if the modal-excitation terms $\mathbf{\Phi}_i^T\mathbf{F}(t)$ in Eq. (11) are uncorrelated) then (see [Bellizzi and Sampaio, 2009]) the covariance matrices $\mathbf{R_Q}$ and $\mathbf{R_{\dot Q}}$ of the stationary responses $\{\mathbf{Q}(t),\ t \in \mathbb{R}\}$ and $\{\dot{\mathbf{Q}}(t),\ t \in \mathbb{R}\}$ are diagonal. Hence the SKLM associated to the process $\{\mathbf{Q}(t), t \in \mathbb{R}\}$ are equal to the vector of the canonical basis of $\mathbb{R}^n$ and the SKLV are given by the diagonal terms of the matrix $\mathbf{R_{\dot Q}^{-1}}\mathbf{R_Q}$ which is equal to $\mathbf{\Sigma} = (\mathbf{\Omega}^2)^{-1}$. Now using the linear relation (10), we can easily deduce that the SKLVs of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ coincide with the SKLV of $\{\mathbf{Q}(t),\ t \in \mathbb{R}\}$ and that the SKLMs of $\{\mathbf{U}(t),\ t \in \mathbb{R}\}$ are given by $\mathbf{\Gamma} = \mathbf{\Phi}^{-T}$. This relationship is determined up to a multiplicative constant.

It is interesting to note that, as indicated in [Chelidze and Zhou, 2006], no assumption on the mass matrix $\mathbf{M}$ is needed to relate the LNMs to the SKLMs whereas the KLMs coincide with the LNMs only when the mass matrix is proportional to the identity matrix (i.e. $\mathbf{M} = c\mathbf{I}$, where c is any positive real number). Moreover, if $\mathbf{M} = c\mathbf{I}$, then the SKLMs coincide with the KLMs, hence, they both, of course, coincide with the LNMs.

To summarize, the SKLD has the nice property to give the resonance frequencies by the inverse of the SKLVs (characteristic which is not easily to obtain from the KLD) and the normal modal vectors inverting the transpose of the SKLM matrix.

3.2 Influence of the mass inhomogeneity on the SKLM

An interesting property of the SKLM is its sensitivity to the mass-inhomogeneity. Combining the following two equations, $\mathbf{\Gamma} = \mathbf{\Phi}^{-T}$ and $\mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}$, the SKLM matrix reads as $\mathbf{\Gamma} = \mathbf{M}\mathbf{\Phi}$ and in case of mass-inhomogeneity, that is to say when the mass matrix is diagonal, $\mathbf{M} = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$, then $\mathbf{\Gamma}_k = \boldsymbol{D}.\mathbf{\Phi}_k$ where $\boldsymbol{D} = (d_1, \cdots, d_n)^T$ and "." denotes the element-by-element product.

Each SKLM differs from a LNM by a scaling vector factor given by the mass-inhomogeneity.

4. MODEL-REDUCTION PROCEDURE

Consider the equation of motion of a general n degrees of freedom dynamical system in the form

$$\mathbf{M}\ddot{\mathbf{U}}(t) + \mathbf{C}\dot{\mathbf{U}}(t) + \mathbf{K}\mathbf{U}(t) + \mathbf{G}(\mathbf{U}(t), \dot{\mathbf{U}}(t)) = \mathbf{F}(t) \tag{12}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are symmetric square matrices with dimensions $n \times n$, $\mathbf{G}$ is a non-linear n-vector function of the generalized vector $\mathbf{U}(t)$ and its derivatives and $\{\mathbf{F}(t), t \in \mathbb{R}\}$ is a n-vector random process.

The model-reduction technique considered here is classical and preserves the second-order structure of the original system. It may be viewed as a projection of the n-dimensional displacement field $\mathbf{U}(t)$ onto a m-dimensional subspace $\mathcal{E}_m$ with $m < n$.

Let $(\mathbf{E}_1, \cdots, \mathbf{E}_m)$ be an orthogonal basis of $\mathcal{E}_m$, i.e. $\mathcal{E}_m = \mathrm{span}(\mathbf{E}_1, \cdots, \mathbf{E}_m)$. The projection matrix, $\mathbf{P}_S$ onto the subspace $\mathcal{E}_m$ is defined by $\mathbf{P}_S = \mathbf{E}^T \in \mathbf{R}^{m \times n}$ with $\mathbf{E} = [\mathbf{E}_1 \cdots \mathbf{E}_m]$. In the full space the same projection is defined by $\mathbf{P} = \mathbf{P}_S^T \mathbf{P}_S$.

An approximation of $\mathbf{U}(t)$ is sought in the subspace $\mathcal{E}_m$ as

$$\mathbf{U}(t) \approx \mathbf{U}^m(t) = \mathbf{E}\mathbf{Q}(t). \tag{13}$$

Substituting Eq. (13) into Eq. (12) and imposing that the residue $\mathcal{R}(\mathbf{U}(t), \mathbf{Q}(t))$

$$\mathcal{R}(\mathbf{U}(t), \mathbf{Q}(t)) = \mathbf{M}\mathbf{E}\ddot{\mathbf{Q}}(t) + \mathbf{C}\mathbf{E}\dot{\mathbf{Q}}(t) + \mathbf{K}\mathbf{E}\mathbf{Q}(t) + \mathbf{G}(\mathbf{E}\mathbf{Q}(t), \mathbf{E}\dot{\mathbf{Q}}(t)) - \mathbf{F}(t) \tag{14}$$

is orthogonal to $\mathcal{E}_m$, the following reduced-order system can be deduced:

$$\mathbf{E}^T \mathbf{M}\mathbf{E}\,\ddot{\mathbf{Q}}(t) + \mathbf{E}^T \mathbf{C}\mathbf{E}\,\dot{\mathbf{Q}}(t) + \mathbf{E}^T \mathbf{K}\,\mathbf{E}\mathbf{Q}(t) + \mathbf{E}^T \mathbf{G}(\mathbf{E}\mathbf{Q}(t), \mathbf{E}\dot{\mathbf{Q}}(t)) = \mathbf{E}^T \mathbf{F}(t). \tag{15}$$

Applying this reduction procedure, it is necessary to keep in mind that generally the approximation $\mathbf{U}^m(t)$ does not coincide with the orthogonal projection $\mathbf{P}\mathbf{U}(t)$ of $\mathbf{U}(t)$ in $\mathcal{E}_m$ even for a linear system ($\mathbf{G} = 0$) and using modal subspaces. The procedure (13)(15) is also used when the set of vectors $(\mathbf{E}_1, \cdots, \mathbf{E}_m)$ is non orthogonal this is the case applying the modal troncature based on the linear normal modes of the underlying linear system when the mass matrix differs from the identity matrix.

5. EXAMPLES AND METHODS

We consider a finite chain of n mass points with the first one linked by a linear spring to a fixed point, the others consecutively linked one to the other, with the last one linked only to the previous mass. All the stiffness coefficients of the strings are equal and their common value is 1. The mass values are denotes m_i ($m_i > 0$). The system can also include isolated nonlinearities between consecutive masses of the form $\lambda_i(U_i(t) - U_{i-1}(t))^3$ for $i = 2, \cdots, n$. The associated equations of motion is of the form (12) with

$$\mathbf{M} = \begin{pmatrix} m_1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & m_2 & 0 & & 0 & 0 \\ 0 & 0 & m_3 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & m_{n-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & m_n \end{pmatrix}, \; \mathbf{K} = \begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \tag{16}$$

and $\mathbf{G}$ which only depends on $\mathbf{U}(t)$ is easily deduced from the form of the nonlinearity. The damping matrix is chosen to be $\mathbf{C} = 2\tau_1\omega_1\mathbf{M}$ with $\tau_1 > 0$, which assures that the damping is proportional and fixes the damping ratio of the first linear mode. Note that the linear version of this system has been discussed in [Farooq and Feeny, 2008].

Two excitation conditions will be considered:

• uncorrelated excitation: the system is excited by a standard vector-valued white-noise process with matrix intensity

$$\mathbf{S}_F = S_0\mathbf{M}, \tag{17}$$

with $S_0 > 0$. This choice ensures that, for all mass values m_i, $\mathbf{\Phi}^T \mathbf{S}_F \mathbf{\Phi} = S_0\mathbf{I}$ is always diagonal.

• correlated excitation: the system is excited by a white-noise scalar process applied to the mass numbered i_{excit}, i.e.

$$\mathbf{F}(t) = (0 \cdots 010 \cdots 0)^T f(t) = \mathbf{P}f(t), \tag{18}$$

with $\{f(t), t \in \mathbb{R}\}$ being a white-noise process with intensity $S_0 > 0$. The intensity matrix of $\{\mathbf{F}(t), t \in \mathbb{R}\}$ is given by $\mathbf{S}_F = S_0\mathbf{P}\mathbf{P}^T$ and hence $\mathbf{\Phi}^T \mathbf{S}_F \mathbf{\Phi} = S_0(\Phi_{1i}\Phi_{1j})$ is not a diagonal matrix.

Based on the choice of $\mathbf{E}$, different reduced models can be developed. Three families of vectors will be considered in the next section:

- LNM: $\mathbf{E}_{LNM} = [\Phi_1 \cdots \Phi_m]$ where Φ_i denotes the linear normal modes of the underlying linear system;

- KLM: $\mathbf{E}_{KLM} = [\Psi_1 \cdots \Psi_m]$ where Ψ_i denotes the KL modes obtained from the steady-state response;

- SKLM: $\mathbf{E}_{SKLM} = [\Gamma_1 \cdots \Gamma_m]$ where Γ_i denotes the SKL modes obtained from the steady-state response.

The reduced-order models have to reproduce the steady-state response of the original system. The Power Spectral Density Matrix (PSDM) function of the steady-state response will be considered to compare the efficiency of the reduced-order models. The PSDM function $\mathbf{S}_U(f)$ is defined as the Fourier transform of the stationary covariance function of the steady-state response.

6. COMPARISON OF THE REDUCED-ORDER MODELS

In all the numerical simulation, $n = 10$, and the damping parameter is $\tau_1 = 0.01$.

6.1 Linear case ($\lambda_i = 0$ for $i = 2, \cdots, n$)

The covariance matrices $\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$ have been obtained solving the associated Lyapounov equations, an advantage of our intrinsic definitions, [Bellizzi and Sampaio, 2006].

6.1.1 Uncorrelated excitation ((17) with $S_0 = 1$)

Case 1: Homogeneous mass ($m_i = 1$ for $i = 1, \cdots, n$)

For this configuration, the three vector families: SKLM, KLM, and LNM coincide giving the same reduced-order model. The Frobenius norm of the PSDM response of the full system is shown in Fig. 1 and compared with the PSDM response of the reduced-order models obtained with $m = 1$ and $m = 5$. The reduced-order model corresponds to the
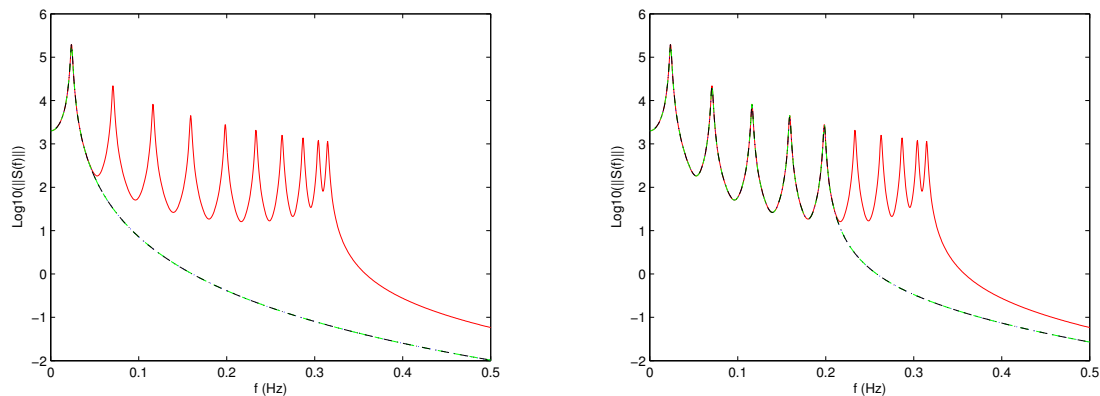

Figure 1. Frobenius norm of the PSDM response: full system (red continuous line), reduced model with $m = 1$ (left), and reduced model with $m = 5$ (right). (LNM: blue dotted line, KLM: green dashed line, SKLM: black dotted-dashed line).

standard modal truncation used intensively in the industry for structural dynamics problem. The justification is that higher modes generally have much less influence in the response of the system.

Case 2: Inhomogeneous mass ($m_i = 1$ except $m_4 = 3$)

For this configuration, the vector family SKLM differs from the LNM but also the vector family KLM differ from the LNM (see Fig. 2). As mentioned in Section 3.2 each SKLM differs from the corresponding LNM by a scaling vector factor ($= \text{diag}(\mathbf{M})$ (see the dashed-dotted curves Fig 2). Only the first three KLMs are similar to the first three LNMs. It is interesting to note the modes approximated from the SKLMs coincide with the LNMs as predicted by the theoretical results, Section 3.1.

The Frobenius norm of the PSDM response of the full system is shown in Fig. 1 and compared with the PSDM response of the reduced-order models obtained with $m = 1$ and $m = 5$. The behavior of the LNM reduced model and the KLM reduced model are similar and reproduce correctly the full system around the resonance frequencies taken into
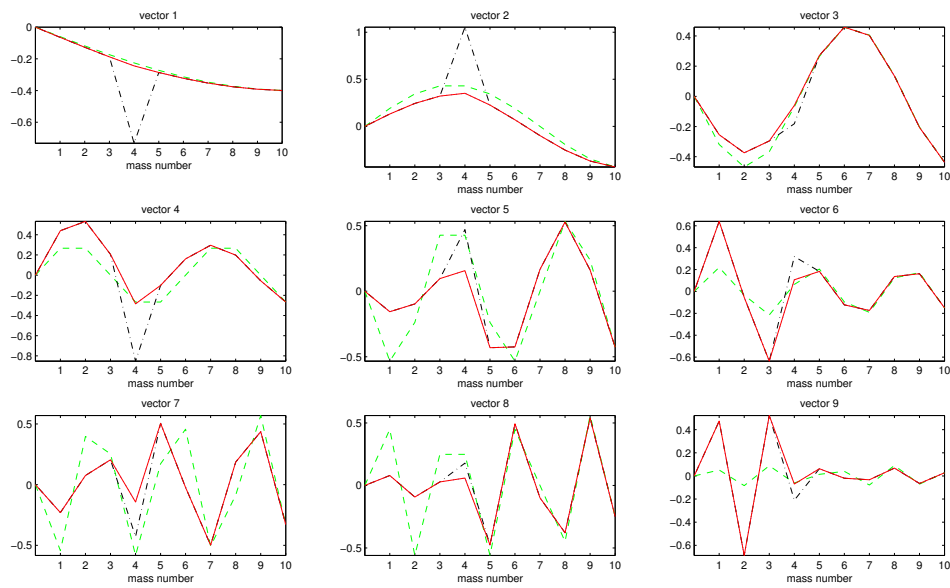
Figure 2. Linear case with inhomogeneous mass and uncorrelated excitation: the LNMs (blue dotted line), the KLMs (green dashed line), the SKLMs (black dotted-dashed line) and the modes approximated from the SKLMs (red solid line).
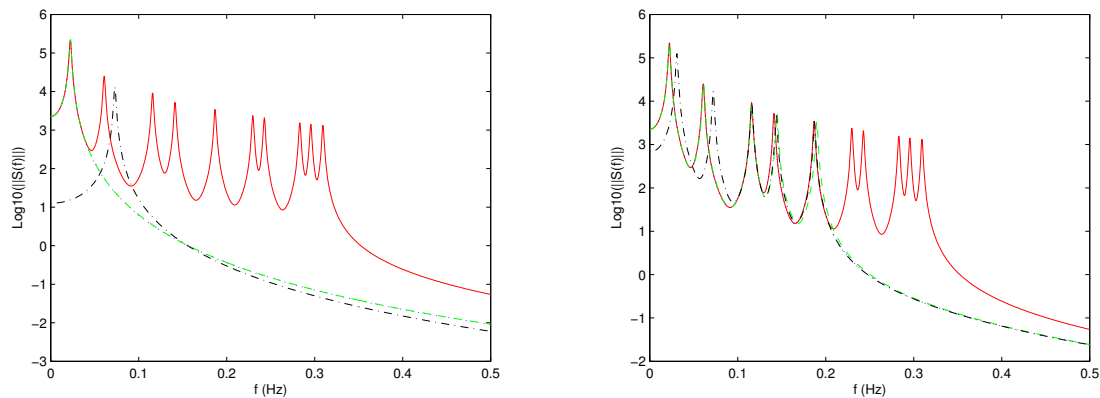

Figure 3. Frobenius norm of the PSDM response: full system (red continuous line), reduced model with $m = 1$ (left), and reduced model with $m = 5$ (right).(LNM: blue dotted line, KLM: green dashed line, SKLM: black dotted-dashed line).

account. One rather surprising finding obtained is the behavior of the SKLM reduced model. For $m = 1$, the resonance peak does not coincide with a resonance frequency of the full system. Increasing m, the different between the resonance peaks and the resonance frequencies of the full system persist to disappear completely for $m = n$, since now we have two bases of the same space.

6.1.2 Correlated excitation ((18) with $S_0 = 1$)

Case 1: Homogeneous mass ($m_i = 1$ for $i = 1, \cdots, n$)

For this configuration, the results are similar to the results obtained in the case of uncorrelated excitation. The three vectors families: SKLMs, KLM and LNM coincide (see Fig. 4) and give only one reduced models (see Fig. 5).

Case 2: Inhomogeneous mass ($m_i = 1$ except $m_4 = 3$)

For this configuration, the vector families SKLM, LNM, and KLM are distinct as seen from Fig. 6. As mentioned in Section 3.2, each SKLM differs from the corresponding LNM by a scaling vector factor ($= \text{diag}(\mathbf{M})$) (see the dashed-dotted curves Fig 2). New here is that each KLM differs from the corresponding LNM. It is interesting to note the modes
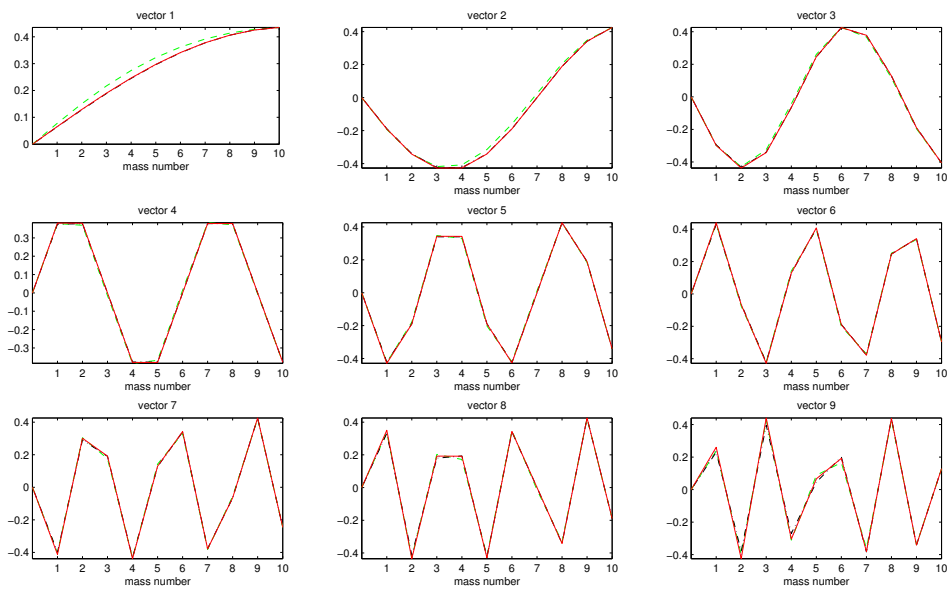
Figure 4. Linear case with homogeneous mass and correlated excitation: the LNMs (blue dotted line), the KLMs (green dashed line), the SKLMs (black dotted-dashed line) and the modes approximated from the SKLMs (red solid line).
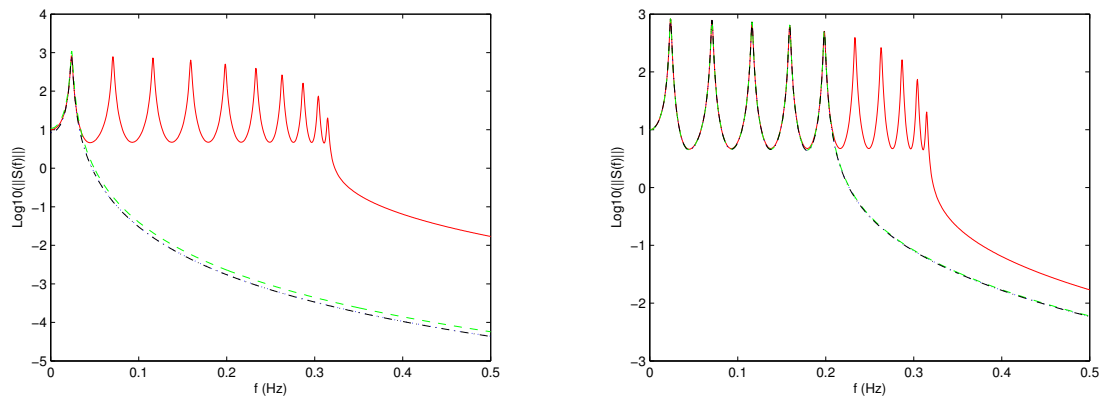

Figure 5. Frobenius norm of the PSDM responses of the full system (red continuous line), the reduced models with $m = 1$ (left), and the reduced models with $m = 5$ (right). (LNM: blue dotted line, KLM: green dashed line, SKLM: black dotted-dashed line).

approximated from the SKLMs coincide with the LNMs as predicted by the theoretical results, Section 3.1.

The Frobenius norm of the PSDM response of the full system is shown in Fig. 7 and compared with the PSDM response of the reduced-order models obtained with $m = 1$ and $m = 5$. The behavior of the LNM reduced model reproduces correctly the full system behavior around the resonance frequencies taken into account. One rather surprising finding obtained is the behaviors of the KLM reduced model. For $m = 1$, the resonance peak does not coincide with a resonance frequency of the full system. Increasing m, the different between the resonance peaks and the resonance frequencies of the full system persist and disappear completely for $m = n$ (not show here). The same comment can be made for the behaviors of the SKLM reduced model as already noted in case of homogeneous mass.

6.2 Nonlinear case

We consider now a configuration with an homogenous mass, a local nonlinearity between the masses m_5 and m_6 with $\lambda_5 = 10$ and an uncorrelated excitation. Due to the nonlinearity term, Eq. (12) was solved numerically using the Newmark method with null initial conditions and a discretized white-noise trajectory. Eq. (12) was integrated over a long time $T = 6550$ sec with the sampling frequency $f_e = 10$ Hz. From the sampling trajectories, the covariance matrices
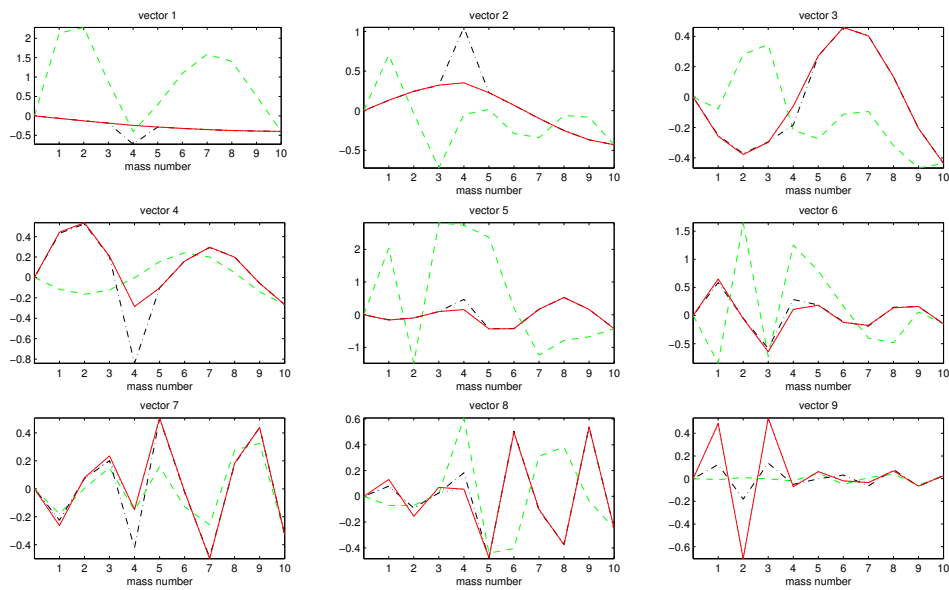
Figure 6. Linear case with inhomogeneous mass and correlated excitation: the LNMs (blue dotted line), the KLMs (green dashed line), the SKLMs (black dotted-dashed line) and the modes approximated from the SKLMs (red solid line).
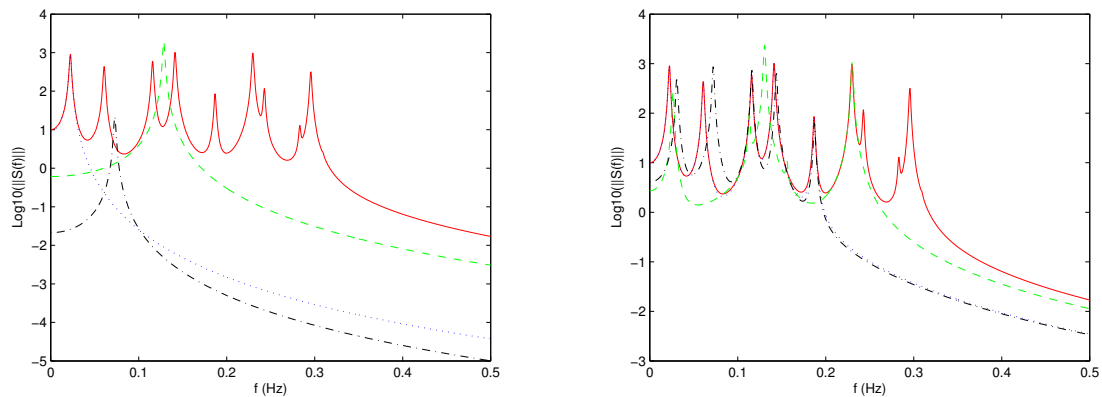

Figure 7. Frobenius norm of the PSDM responses of the full system (red continuous line), the reduced models with $m = 1$ (left), and the reduced models with $m = 5$ (right). (LNM: blue dotted line, KLM: green dashed line, SKLM: black dotted-dashed line).

$\mathbf{R_U}$ and $\mathbf{R_{\dot{U}}}$ were obtained using the classical time-average estimates (valid under the ergodic assumption). The same numerical procedure (with the null initial conditions and the same excitation trajectory) was used to solve the reduced-order models (14). The PSDM were estimated using Welch's averaged modified periodogram method with a window length of 4096 points ($\Delta f = 0.0024$ Hz) and without overlap.

The KLMs and the SKLMs are shown in Fig. 8 and compared to the LNMs of the underlying linear systems. The KLM and the SKLM are very close except for the mode vectors 8 and 9. In fact, the eighth SKM are close to the nineth KLM and the nineth SKM are close to the eighth KLM. The KLM and SKLM vectors 10 are only influenced by the local nonlinearity.

The Frobenius norm of the PSDM response of the full system is shown in Fig. 9 and compared with the PSDM response of the reduced-order models obtained with $m = 1$ and $m = 5$. The KLM and SKLM reduced-order models give similar results and they approximate better the full system than the reduced-order model obtained with the LNM of the underlying linear system for the low-freqency range. Moreover, the KLM and SKLM reduced-order models do not reproduce the nonlinear effect (between 0.5 and 2 Hz) whereas a nonlinear effect is present in case of reduced-order model obtained with the LNM but out of the range 0.5 to 2 Hz.
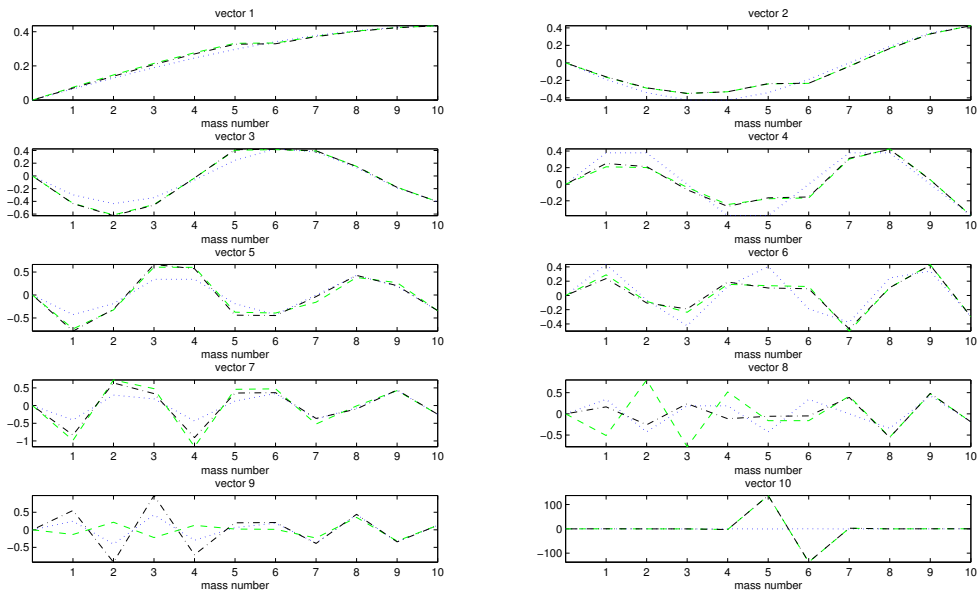
Figure 8. Nonlinear case with homogeneous mass and uncorrelated excitation: the LNMs (blue dotted line) of the underlying linear system, the KLMs (green dashed line), the SKLMs (black dotted-dashed line).
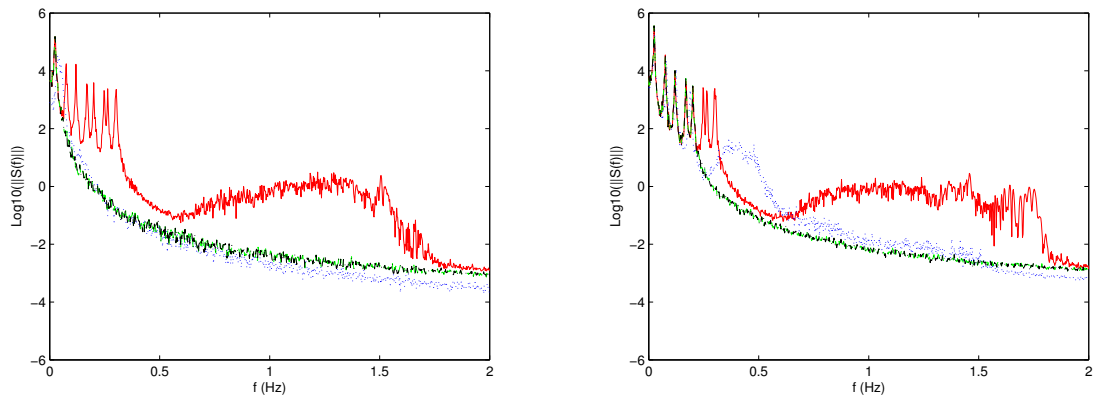

Figure 9. Frobenius norm of the PSDM responses of the full system (red continuous line), the reduced models with $m = 1$ (left), and the reduced models with $m = 5$ (right). (LNM: blue dotted line, KLM: green dashed line, SKLM: black dotted-dashed line).

The nonlinear effect can be reproduced with a reduced-order model based on the following familly vectors

$$\mathbf{E}_{modKLM} = \left[\Psi_1 \cdots \Psi_{m-1} \Psi_n \right] \tag{19}$$

where the mth KLM has been replaced by the last KLM. The KLM Ψ_n (see Fig. 8) contain information about the local nonlinearity, hence the reduced-order model constructed with it reproduces the nonlinear effect (see Fig. 10).

CONCLUSIONS

It was shown that the SKLD is nicely related to LNM independent of the mass matrix and that it gives also the natural frequencies of a conservative linear system. This property indicates that the SKLD is an important tool for modal analysis.

However the reduced-order models constructed with the SKLD are not good. Except in the case of nonhomogeneous mass, it is better to use KLM that besides having the property of best basis when m is fixed, they work very well also for nonlinear systems. In this case, the reduced-order models constructed with KLM give better results that the reduced-order models constructed with the LNM of the underlying linear system. Moreover, the behaviour of the reduced-order model can be improve including in the set of vectors a KLM with small energy but containing nonlinear information.
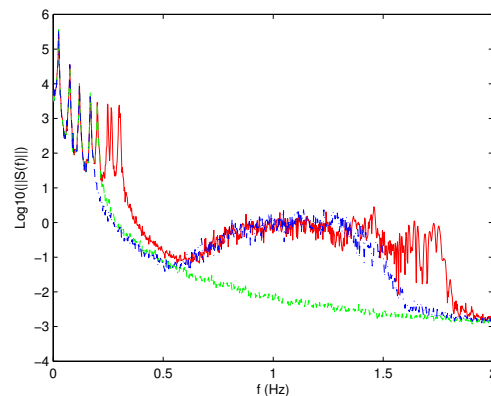
Figure 10. Frobenius norm of the PSDM responses of the full system (red continuous line), the reduced models with $m = 5$ (right). (LNM: blue dotted line, KLM: green dashed line, modified KLM: black dotted-dashed line).

Acknowledgments

7. REFERENCES

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