ANALYTICAL SOLUTION OF THE POINT-KINETICS EQUATIONS DURING THE START-UP OF A NUCLEAR REACTOR WITH MULTIPLE WITHDRAW STAGES OF THE CONTROL RODS

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Abstract. *The reactivity is one of the most important properties in a nuclear reactor, because it is directly related to the control of the reactor. The process of start-up a nuclear reactor requires insertion of reactivity in the system that occurs with the withdraw of the control rods following a discontinuous procedure which has to be monitored. In this paper the point kinetics equations are solved considering a fixed source of neutrons and with linear insertion of reactivity from the monitored withdraw of the control rods. An analysis of the transient regime between two rod liftings is done and a discussion on the feasibility of extending the method for a large number of consecutive withdraw procedures is also presented. The results showed small deviations in relation to the reference values*

Keywords*: Point kinetic, Reactivity, Nuclear power*

1. INTRODUCTION

Reactivity is one of the most important properties of a nuclear reactor, as it directly relates to reactor control. The process to start-up a nuclear reactor demands the insertion of reactivity in the system, which takes place with the withdraw of the control rods in a non-continuous and monitored manner. In practice, the control rods are raised at set time intervals so to linearly insert reactivity in the reactor core, generating transients that allow criticality to be reached in a slow and safe manner.

Kinetic equations for reactors are the non-static case of the diffusion equation, and can be obtained from the simplifying hypotheses in the transport equation (Stacey, 2001). The use of kinetic equations allows one to obtain point kinetics equations where the spatial variations in the neutron flux are not considered. The point kinetics model is chosen for its being a simple model, capable of offering satisfactory responses from a physical standpoint, that is, point kinetics equations allow the obtaining of reasonably precise responses on global reactor behaviour in time, with the exception of some specific situations such as the ejection of the control bar. Point kinetics equations, in accounting for the existence of a group of delayed neutrons, are written by:

$$\begin{aligned}\frac{dn(t)}{dt} &= \frac{\rho(t)-\beta}{l}n(t)+\lambda C(t)+q(t)\\ \frac{dC(t)}{dt} &= \frac{\beta}{l}n(t)-\lambda C(t),\end{aligned} \tag{1}$$

where $n(t)$ is the number of neutrons inside the reactor, $C(t)$ is the concentration of delayed precursor neutrons, $q(t)$ is the external neutron source, l is the mean generation time between the birth of the neutron and its subsequent absorption inducing fission. Constants β and λ are the fractions of the delayed neutrons in the group and decay constant for group precursors.

This paper presents an analytical approximation for the concentration of precursor neutrons $C(t)$, from the solution of the point kinetics equations, equation (1), considering a fixed source $q(t)=q_0$ and an insertion of reactivity that is variable in time and is represented by:

$$\rho(t)=\begin{cases}\rho_0+rt,(0\le t<t_0)\\ \rho_0+rt_0,(t\ge t_0)\end{cases}, \tag{2}$$

where t_0 is the time to raise the control rods, ρ_0 is the initial system reactivity and r is the linear insertion rate for reactivity.

2. MATHEMATICAL FORMULATION

2.1. Analytical Solution for neutron density $n(t)$

In implementing a calculation method that considers more than one transient from the solution of the point kinetics equations written by equation (1), with the reactivity written by equation (2), it is necessary to predict neutron density beyond the concentration of delayed precursor neutrons. This Section presents two approximations found in the literature that will be used to predict neutron density considering a double reactivity insertion ramp into the system.

2.1.1 Solution proposed by Palma et al. (2009)

After eliminating the dependency for the concentration of precursors (Zhang et al., 2008), the differential equation that rules neutron density as the control rod is raised is thus written:

$$l\frac{d^2n(t)}{dt^2}+\left[\beta+\lambda l-\rho(t)\right]\frac{dn(t)}{dt}-\left[\frac{d\rho(t)}{dt}+\lambda\rho(t)\right]n(t)=\lambda ql. \tag{3}$$

subjected to the initial conditions,

$$\begin{cases} n(0)=n_0 \\ \dfrac{dn(0)}{dt}=0 \end{cases}, \tag{4}$$

The term $l\dfrac{d^2n(t)}{dt^2}$ is small in relation to all of the others in equation (3) and can be disregarded $\left(l\approx 10^{-3}\right)$. After disregarding this term and considering that $\rho-\beta\gg\lambda l$, the system with differential equations that rule the approximate neutron density behaviour for the variation in reactivity as presented in equation (2), is the following:

$$\left[\frac{\rho_0-\beta}{r}+t\right]\frac{dn_1(t)}{dt}-\left[\frac{r+\lambda\rho_0}{r}+\lambda t\right]n_1(t)=\frac{\lambda ql}{r},\left(0\le t\le t_0\right) \tag{5}$$

$$\frac{dn_2(t)}{dt}+\left[\frac{r+\lambda\left(\rho_0+rt_0\right)}{\rho_0+rt_0-\beta}\right]n_2(t)=\frac{\lambda ql}{\beta-\left(\rho_0+rt_0\right)},\left(t\ge t_0\right), \tag{6}$$

subjected to the conditions expressed by equation (5) and the following condition of continuity

$$n_1(t_0)=n_2(t_0). \tag{7}$$

When solving the set of differential equations formed by equations (5) and (6) one can write the following expressions for neutron density (Palma et al., 2009):

$$n(t)=\begin{cases} A_1\dfrac{e^{-\lambda t}}{\left(t-k_2\right)^{k_1+\lambda k_2}}\left[\Gamma\left(k_1+\lambda k_2,\lambda\left(k_2-t\right)\right)+A_2\right], 0\le t<t_0 \\ -\dfrac{\lambda ql}{r+\lambda\left(\rho_0+rt_0\right)}+A_3e^{-\xi\left(t-t_0\right)}, t\ge t_0 \end{cases}. \tag{8}$$

where constants A_1, A_2 and A_3 are defined by:

$$A_1 = \frac{k_3 e^{\lambda k_2}}{\left(-\lambda\right)^{k_1+\lambda k_2}} \tag{9}$$

$$A_2 = -\Gamma\left(k_1+\lambda k_2, \lambda k_2\right) + \frac{n_0\left(\lambda k_2\right)^{k_1+\lambda k_2}}{k_3 e^{\lambda k_2}} \tag{10}$$

$$A_3 = \frac{A_1 e^{-\lambda t_0}}{\left(t_0-k_2\right)^{k_1+\lambda k_2}}\left[\Gamma\left(k_1+\lambda k_2, \lambda k_2-\lambda t_0\right)+A_2\right] + \frac{\lambda q l}{r+\lambda\left(\rho_0+r t_0\right)}, \tag{11}$$

and too:

$$k_1 = \frac{\lambda\rho_0 + r}{r} \tag{12}$$

$$k_2 = \frac{\beta-\rho_0}{r} \tag{13}$$

$$k_3 = \frac{\lambda q l}{r}. \tag{14}$$

2.1.2 Solution proposed by Zhang et al. (2008)

After disregarding the term $l\frac{d^2 n(t)}{dt^2}$ in equation (3) and adopting the prompt jump and constant source approximations, the following expression for neutron density in the event of linear reactivity insertion was proposed by Zhang et al. (2008):

$$n(t) = \begin{cases} \dfrac{\beta n_0 + q l}{\beta + |\rho_0| - r t}, 0 \le t < t_0 \\ \dfrac{q l}{|\rho_0| - r t_0}\left[1 - e^{-\frac{\lambda\left(|\rho_0|-r t_0\right)}{\beta+|\rho_0|-r t_0}(t-t_0)}\right] + \dfrac{\beta n_0 + q l}{\beta+|\rho_0|-r t_0} e^{-\frac{\lambda\left(|\rho_0|-r t_0\right)}{\beta+|\rho_0|-r t_0}(t-t_0)}, t \ge t_0 \end{cases} \tag{15}$$

2.2. Analytical solution for the concentration of precursor neutrons $C(t)$

After the elimination of the dependency of neutron density $n(t)$, the differential equation that rules the density of precursor neutrons during the withdraw of the control rods is thus written:

$$l\frac{d^2 C(t)}{dt^2} + \left[\beta - \rho(t) + l\lambda\right]\frac{dC(t)}{dt} - \lambda\rho(t)C(t) = \beta q \ . \tag{16}$$

The term $l\frac{d^2 C(t)}{dt^2}$ is small in relation to all of the others in equation (16) and can be disregarded $\left(l \approx 10^{-3}\right)$. After disregarding this term and considering that $\rho - \beta \gg \lambda l$, the system of differential equations that rule the approximate concentration of precursors neutrons behaviour for the variation of reactivity as presented in Eq. (2) is the following:

$$\left[\frac{\rho_0-\beta}{r}+t\right]\frac{dC_1(t)}{dt}+\left[\frac{\lambda\rho_0}{r}+\lambda t\right]C_1(t)=\frac{\beta q}{r},\left(0\le t<t_0\right) \tag{17}$$

$$\frac{dC_2(t)}{dt}+C_2(t)\left[\frac{\lambda(\rho_0+rt_0)}{(\rho_0+rt_0-\beta)}\right]=\frac{\beta q}{(\rho_0+rt_0)-\beta},(t\ge t_0) \tag{18}$$

subjected to the initial condition

$$C\left(0\right)=C_0, \tag{19}$$

and of continuity:

$$C_1\left(t_0\right)=C_2\left(t_0\right). \tag{20}$$

The system formed by Eq.s (17) and (18) consists of ordinary, non-homogeneous differential equations and can be written thus:

$$\frac{dC\left(t\right)}{dt}+f\left(t\right)C\left(t\right)=g\left(t\right), \tag{21}$$

and may be solved using the integrating factor method (Arfken, 2001), providing solutions represented by:

$$C\left(t\right)=e^{-\int f(t)dt}\left[\int e^{\int f(t)dt}g\left(t\right)dt+\Omega\right], \tag{22}$$

where, for each reactivity insertion regime the integration constants Ω can be determined from the conditions expressed by the initial and continuity conditions.

2.2.1 Solution during reactivity ramp $0\le t\le t_0$

During the insertion of reactivity, which occurs in an interval $\left[t,t_0\right]$, functions $f\left(t\right)$ and $g\left(t\right)$ are written by:

$$f\left(t\right)=\frac{k_4+\lambda t}{k_2+t} \tag{23}$$

$$g\left(t\right)=\frac{k_5}{k_2+t}, \tag{24}$$

where constants k_4 and k_5 are defined thus:

$$k_4=\frac{\lambda\rho_0}{r} \tag{25}$$

$$k_5=-\frac{\mathrm{q}\beta}{r}. \tag{26}$$

In replacing Eq.s (10) and (11) in Eq. (9) one can write:

$$C_1(t) = e^{\int\left(\frac{k_4+\lambda t}{k_2+t}\right)dt}\left[\int e^{-\int\left(\frac{k_4+\lambda t}{k_2+t}\right)dt}\frac{k_5}{k_2+t}dt + \Omega_1\right] \tag{27}$$

Denoting $\omega = k_4 + \lambda k_2 - 1$, integrating Eq. (27) and imposing the initial condition as expressed by Eq. (19) one obtains the following expression for the precursor concentration during the reactivity ramp in the system:

$$C_1(t) = A_4\frac{e^{-\lambda t}}{(t-k_5)^{k_4+\lambda k_5}}\left[\Gamma(\omega+1, \lambda k_5 - \lambda t) + A_5\right] \tag{28}$$

where constants A_4 and A_5 are defined by:

$$A_4 = \frac{e^{\lambda k_5}k_6}{(-\lambda)^{(\omega+1)}} \tag{29}$$

$$A_5 = -\Gamma(\omega+1, \lambda k_5) + \frac{C_0(\lambda k_5)^{\omega+1}}{k_6 e^{\lambda k_5}} \tag{30}$$

2.2.2 Solution after reactivity ramp $t \geq t_0$

Following the insertion of reactivity during a time interval t_0 the reactivity holds constant and equal to $\rho_0 + rt_0$. Thus, the differential equation to be solved becomes simpler:

$$\frac{dC_2(t)}{dt} + \psi_1 C_2(t) = \psi_2, (t \geq t_0) \tag{31}$$

where constants ψ_1 and ψ_2 are defined thus:

$$\psi_1 = \frac{\lambda(\rho_0 + rt_0)}{(\rho_0 + rt_0 - \beta)} \tag{32}$$

$$\psi_2 = \frac{\beta q}{(\rho_0 + rt_0) - \beta}. \tag{33}$$

Solving Eq. (31) and imposing the condition of continuity as expressed by Eq. (20) one can write the following expression for the concentration of precursor neutrons following the reactivity ramp:

$$\mathrm{C}_2(t) = -\frac{q_0\beta}{\lambda(\rho_0 + rt_0)} + \mathrm{A}_6 e^{-\xi(t-t_0)} \tag{34}$$

where constant A_6 is defined thus:

$$A_6 = \frac{A_4 e^{-\lambda t_0}}{(t_0 - k_5)^{(k_4+\lambda k_5)}}\left[\Gamma(\omega+1, \lambda k_5 - \lambda t_0) + A_5\right] + \frac{q_0\beta}{\lambda(\rho_0 + rt_0)} \tag{35}$$

Therefore, without adopting the prompt jump approximation in instant t_0, the solution proposed in this paper for the concentration of precursor neutrons is:

$$C(t)=\begin{cases} A_4\dfrac{e^{-\lambda t}}{(t-k_5)^{k_4+\lambda k_5}}\left[\Gamma(\omega+1,\lambda k_5-\lambda t)+A_5\right] & ,0\le t<t_0 \\ -\dfrac{q_0\beta}{\lambda(\rho_0+rt_0)}+\mathrm{A}_6 e^{-\xi(t-t_0)} & ,t\ge t_0 \end{cases}. \tag{36}$$

The results obtained for the concentration of precursor neutrons in both reactivity insertion regimes, calculated from Eq. (36) are shown in the results section.

It is possible to approximate the incomplete gamma functions that exist in the functional form of neutron density $n(t)$, as written by Eq. (36), with no significant accuracy loss. For that, the following relation will be used (Gradshteyn & Ryzhik, 2007):

$$\Gamma(a,x)=\Gamma(a)-x^a\sum_{n=0}^{\infty}\frac{(-x)^n}{(a+n)n!}. \tag{37}$$

Numerical studies indicate that with only two terms in the existing expansion in Eq. (37) the required accuracy is achieved. In using the gamma function approximation proposed by Nemmes (Nemmes, 2007), one obtains the following expression for the incomplete gamma function that will be used in this paper:

$$\Gamma(a,x)\approx\sqrt{\frac{2\pi}{a}}\left[\frac{1}{e}\left(a+\frac{10a}{120a^2-1}\right)\right]^a+x^a\left[\frac{x}{a+1}-\frac{x}{2(a+2)}-\frac{1}{a}\right], \tag{38}$$

where $e\approx 2.71828$ is the basis for the Neperian logarithms.

The results obtained for concentration of precursor neutron in both reactivity insertion regimes, calculated from Eq. (36) and with the incomplete gamma function calculated from Eq. (38) are shown in the results section.

3. RESULTS

As a reference in the validation of the analytical approximation for the concentration of precursor neutrons obtained in this paper, the method of finite differences will be used for the numerical solution of the point kinetics equations, Eq. (1). The following expressions were used to implement the implicit temporal integration method (Hashimoto et al., 2000):

$$\begin{aligned} \frac{n^{i+1}-n^i}{\Delta t}&=\frac{\rho^{i+1}-\beta}{l}n^{i+1}+\lambda C^{i+1}+q \\ \frac{C^{i+1}-C^i}{\Delta t}&=\frac{\beta}{l}n^{i+1}-\lambda C^{i+1}, \end{aligned} \tag{39}$$

where $\Delta t=t_{i+1}-t_i$, $n^i=n(t_i)$, $C^i=C(t_i)$, $\rho^i=\rho(t_i)$ and t_i is the time from the i[th] iteration. All validations used the mesh point $\Delta t=10^{-6}s$. The nuclear parameters used in this paper for the validation of the approximation presented, Eq. (36), are using fuel material ^{235}U and assuming that $\beta=0.0065$, $l=0.0001s$, $\lambda=0.07741s^{-1}$ and $q=10^8\,neutrons/cm^3.s$. As demonstrated in the above reference (Hashimoto et al., 2000) this numerical method presents unconditional stability.

Figure 1 considers the regime of linear reactivity insertion such that:

$$\rho(t)=\begin{cases} -0.006+0.0001t, & 0\le t\le t_0 \\ -0.0055, & t>t_0 \end{cases} \tag{40}$$

where $t_0 = 5s$.

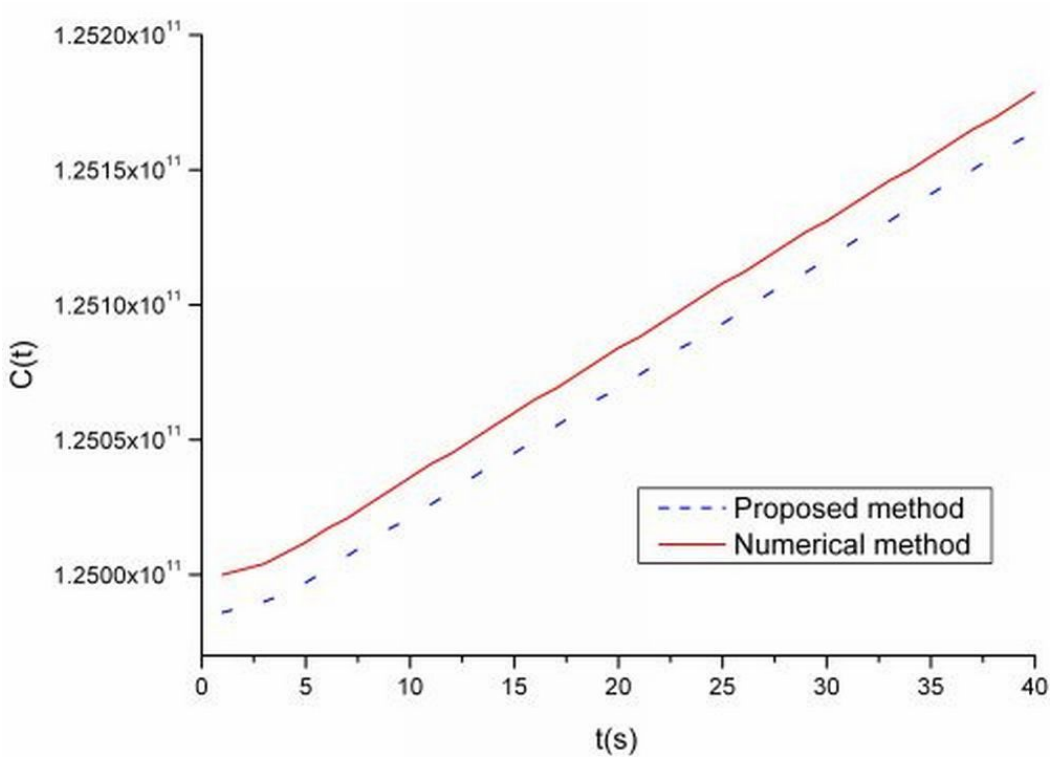

Figure 1. Comparison between the calculation methods for $n(t)$ with ρ(t) calculated from Eq. (39).

During the reactivity ramp one can see the concordance between the analytical methods and the numerical one for the relatively big time interval for the control rods withdrawal. Maximum percentage deviation obtained in the entire simulation period was of 0.05 %.

From the accurate expressions for neutron density $n(t)$, Eq. (8), and from the concentration of precursors $C(t)$, Eq. (35), one obtains a full set of point kinetics solutions.

Another important application for the solutions presented in this paper is the possibility of extending the prediction of the transients for an additional ramp, which takes place in practice. For that, it is necessary to add, apart from the condition of continuity expressed by Eq. (4), the condition that

$$n_2(t_1) = n_3(t_1), \tag{40}$$

where t_1 is the instant when the second ramp starts, that is, the interface between two consecutive transients, and $n_3(t_1)$ is a solution with the same functional form of Eq. (8) in the interval $[0, t_0]$ with constants A_1 and A_2 being updated at each process.

Figure 2 shows the comparison of the methods proposed by Palma et al. (2009) and Zhang et al. (2008) for two reactivity ramps separated by a constant reactivity plateau, with it written thus:

$$\rho(t) = \begin{cases} -0.006 + 0.0001t & 0 \le t \le t_0 \\ -0.0055 & t_0 \ge t \ge t_1 \\ \text{-0.0055+0.0001t} & t \ge t_1 \end{cases}, \tag{41}$$

where $t_0 = 5s$ and $t_1 = 50s$.

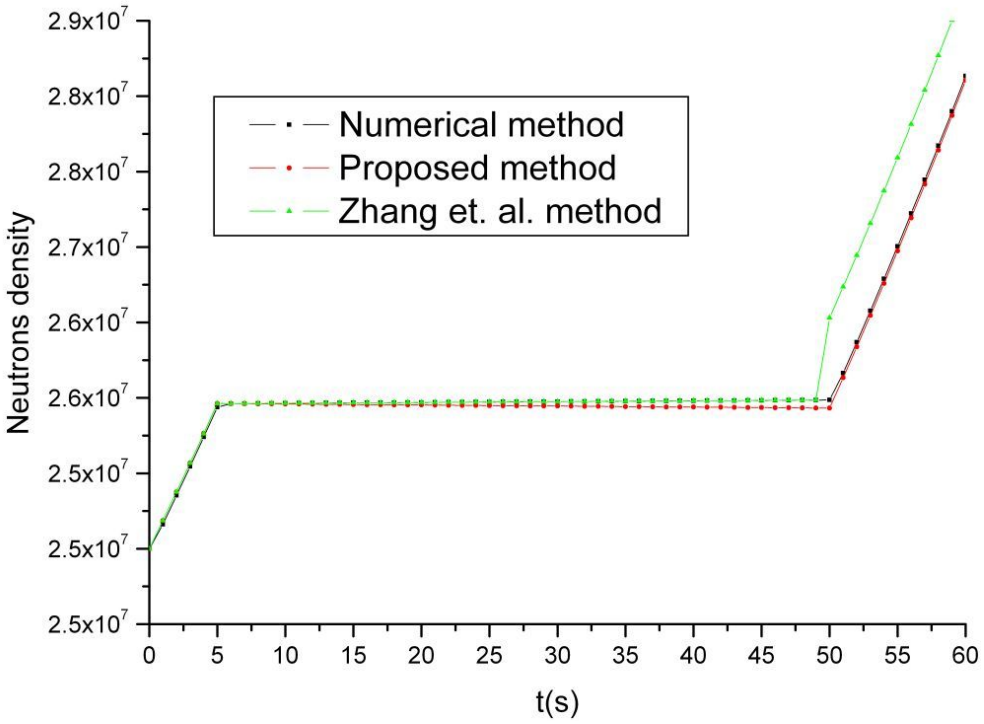

Figure 2. Comparison between the calculation methods for $n(t)$ with ρ(t) calculated from Eq. (41)

It is possible to see from Figure 2 that the method presented in this paper presents accurate results across the entire simulation whilst the method proposed by Zhang et al presents accurate results only during the first transient, that is, during interval $[0, t_1]$. The maximum percentage deviation obtained from the method proposed in this paper was of 1.3% whilst the method proposed by Zhang et al produced deviations to the order of 4.2% for $t > t_1 = 50s$. One of the explanations for this fact is that the prompt jump approximation is not used in the expression presented in this paper, whereas it can be found in the set of approximations proposed by Zhang et al. (2008).

4. CONCLUSIONS

An analytical approximation was developed in this paper, seeking to predict the concentration of precursor neutrons $C(t)$ during the removal of the control rods with the resulting linear reactivity insertion in the system. The formulation proposed consists of the solution of the point kinetics equations for a group of precursors without resorting to the prompt jump approximation. The results obtained have been shown to be compatible with those obtained from the reference method that was the numerical solution of the point kinetics equations.

For the periods of time for the removal of the control rods (t_0) used in the practice, the percentage deviations obtained are found within the acceptable limit for engineering applications, not reaching 0.1 % in relation to the reference values.

Another result obtained was the extension of the expressions proposed by Palma et al. (2009) for an additional ramp that simulates the raising of the control rods separated by stagnation periods, with the reactivity being written by Eq. (41) and considering an external and constant source of neutrons. The results obtained were accurate and were systematically better than those achieved by the expressions proposed by Zhang et al., Eq. (15) when $t > t_1 = 50s$.

5. ACKNOWLEDGEMENTS

The authors acknowledge the support provided by FAPERJ and CNPq in the developing of this research.

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