APPROXIMATED ANALYTICAL FIRST ORDER SOLUTIONS FOR OPTIMAL LOW-THRUST LIMITED-POWER TRANSFERS BETWEEN ELLIPTICAL ORBITS

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Abstract. *In this work, approximated analytical solutions, which include short periodic terms, are presented for three different problems involving optimal low-thrust limited-power transfers between elliptical orbits in a Newtonian central gravity field. These problems are classified as: transfers between coplanar orbits, transfers between non-coplanar coaxial orbits and transfers between non-coplanar co-parameters orbits. The optimization problem associated to the general space transfer problem is formulated as a Mayer problem of optimal control theory with Cartesian elements – position and velocity vectors – as state variables. After applying the Pontryagin Maximum Principle, classical orbital elements are introduced through a canonical transformation. Short periodic terms are eliminated from the maximum Hamiltonian function through an infinitesimal canonical transformation. The new Hamiltonian function, resulting from the infinitesimal canonical transformation, describes the extremal trajectories associated with the long duration maneuvers for simple transfers (no rendez-vous). This new Hamiltonian function can be simplified for the three special classes of maneuvers described above and closed-form analytical solutions can be obtained through Hamilton-Jacobi theory.*

Keywords*: Optimization of space trajectories, low-thrust limited-power trajectories, transfers between elliptical orbits.*

1. INTRODUCTION

In this work, approximated analytical solutions, which include short periodic terms, are presented for three different problems involving optimal low-thrust limited-power transfers between elliptical orbits in a Newtonian central gravity field. These problems are classified as: transfers between coplanar orbits, transfers between non-coplanar coaxial orbits and transfers between non-coplanar co-parameters orbits. This analysis has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions in the last twenty years. Two important space missions have made use of low-thrust propulsion systems: NASA-JPL Deep Space One and ESA-SMART1. Low-thrust electric propulsion systems are characterized by high specific impulse and low-thrust capability and have their greatest benefits for high-energy planetary missions (Marec, 1979; Racca, 2003). Several researchers have obtained numerical and analytical solutions for a number of specific initial orbits and specific thrust profiles (Edelbaum, 1964, 1965; Marec and Vinh, 1977; Haissig et al, 1992; Kiforenko et al, 2003).

The optimization problem associated to the general space transfer problem is formulated as a Mayer problem of optimal control theory with Cartesian elements – position and velocity vectors – as state variables. It is assumed that the thrust direction is free and the thrust magnitude is unbounded, that is, there exist no constraints on control variables (Marec, 1979, 1984). After applying the Pontryagin Maximum Principle and determining the maximum Hamiltonian function, classical orbital elements are introduced through a canonical transformation – Mathieu transformation – defined by the general solution of the canonical system described by the integrable kernel of the maximum Hamiltonian function. Hori method (Hori, 1966) – a perturbation technique based on Lie series – is applied in solving the canonical system of differential equations that governs the optimal trajectories. Short periodic terms are then eliminated from the maximum Hamiltonian function through an infinitesimal canonical transformation described by a generating function obtained at first order in the thrust magnitude. The new Hamiltonian function, resulting from the infinitesimal canonical transformation, describes the extremal trajectories associated with the long duration maneuvers for simple transfers (no rendez-vous). This new Hamiltonian function can be simplified for the three special classes of maneuvers described above and closed-form analytical solutions can be obtained through Hamilton-Jacobi theory. The separation of variables technique (Lanczos, 1971) is applied to solve the Hamilton-Jacobi equation associated to the average canonical system. First order analytical solutions are then obtained in each case by using the generating function built through Hori method.

2. OPTIMAL SPACE TRAJECTORIES

A low-thrust limited-power propulsion system, or LP system, is characterized by low-thrust acceleration level and high specific impulse (Marec, 1979, 1984). The ratio between the maximum thrust acceleration and the gravity acceleration on the ground, $\gamma_{\max}/g_0$, is between 10^{-4} and 10^{-2}. For such system, the fuel consumption is described by the variable J defined as

$$J = \frac{1}{2}\int_{t_0}^{t} \gamma^2 dt \,, \tag{1}$$

where γ is the magnitude of the thrust acceleration vector $\boldsymbol{\gamma}$, used as control variable. The consumption variable J is a monotonic decreasing function of the mass m of space vehicle,

$$J = P_{\max}\left(\frac{1}{m} - \frac{1}{m_0}\right),$$

where $P_{\max}$ is the maximum power and m_0 is the initial mass. The minimization of the final value of the fuel consumption J_f is equivalent to the maximization of m_f.

The general optimization problem concerned with low-thrust limited-power transfers (no rendezvous) will be formulated as a Mayer problem of optimal control by using Cartesian elements as state variables. Consider the motion of a space vehicle M powered by a limited-power engine in a Newtonian central gravity field. At time t, the state of the vehicle is defined by the position vector $\boldsymbol{r}(t)$, the velocity vector $\boldsymbol{v}(t)$ and the consumption variable J. The control $\boldsymbol{\gamma}$ is unconstrained, that is, the thrust direction is free and the thrust magnitude is unbounded.

The optimization problem is formulated as follows: it is proposed to transfer the space vehicle M from the initial state $(\boldsymbol{r}_0, \boldsymbol{v}_0, 0)$ at the initial time $t_0 = 0$ to the final state $(\boldsymbol{r}_f, \boldsymbol{v}_f, J_f)$ at the specified final time t_f, such that the final consumption variable J_f is a minimum. The state equations are

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \qquad \frac{d\boldsymbol{v}}{dt} = -\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{\gamma} \qquad \frac{dJ}{dt} = \frac{1}{2}\gamma^2 \,, \tag{2}$$

where μ is the gravitational parameter.

According to the Pontryagin Maximum Principle (Pontryagin et al, 1962), the optimal thrust acceleration $\boldsymbol{\gamma}^*$ must be selected from the admissible controls such that the Hamiltonian function H reaches its maximum. The Hamiltonian function is formed using Eq. (2),

$$H = \boldsymbol{p}_r \bullet \boldsymbol{v} + \boldsymbol{p}_v \bullet \left(-\frac{\mu}{r^3}\boldsymbol{r} + \boldsymbol{\gamma}\right) + \frac{1}{2} p_J \gamma^2 \,, \tag{3}$$

where $\boldsymbol{p}_r$, $\boldsymbol{p}_v$ and p_J are the adjoint variables and dot denotes the dot product. Since the optimization problem is unconstrained, $\boldsymbol{\gamma}^*$ is given by

$$\boldsymbol{\gamma}^* = -\frac{\boldsymbol{p}_v}{p_J} \,. \tag{4}$$

The optimal thrust acceleration $\boldsymbol{\gamma}^*$ is modulated (Marec, 1979) and the optimal trajectories are governed by the maximum Hamiltonian function H^*, obtained from Eqns (3) and (4),

$$H^* = \boldsymbol{p}_r \bullet \boldsymbol{v} - \boldsymbol{p}_v \bullet \frac{\mu}{r^3}\boldsymbol{r} - \frac{p_v^{\;2}}{2 p_J} \,. \tag{5}$$

The consumption variable J is ignorable and p_J is a first integral. From the transversality conditions, $p_J(t_f) = -1$; thus, $p_J(t) = -1$. Equation (5) reduces to

$$H = \boldsymbol{p}_r \bullet \boldsymbol{v} - \boldsymbol{p}_v \bullet \frac{\mu}{r^3}\boldsymbol{r} + \frac{p_v^{\;2}}{2} \,. \tag{6}$$

Using Eqns (6) and (7), the maximum Hamiltonian function can be written in the form $H^* = H_0 + H_{\gamma^*}$, where $H_0 = \boldsymbol{p}_r \bullet \boldsymbol{v} - \boldsymbol{p}_v \bullet \frac{\mu}{r^3}\boldsymbol{r}$ denotes the undisturbed Hamiltonian function and $H_{\gamma^*} = \frac{{p_v}^2}{2}$ denotes the disturbing function concerning the optimal thrust acceleration.

3. TRANSFORMATION FROM CARTESIAN ELEMENTS TO A SET OF ORBITAL ELEMENTS

Consider the canonical system of differential equations governed by the undisturbed Hamiltonian function H_0,

$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v} \qquad \frac{d\boldsymbol{v}}{dt} = -\frac{\mu}{r^3}\boldsymbol{r} \qquad \frac{d\boldsymbol{p}_r}{dt} = \frac{\mu}{r^3}\left(\boldsymbol{p}_v - 3(\boldsymbol{p}_v \bullet \boldsymbol{e}_r)\boldsymbol{e}_r\right) \qquad \frac{d\boldsymbol{p}_v}{dt} = -\boldsymbol{p}_r\,, \tag{7}$$

where $\boldsymbol{e}_r$ is the unit vector pointing radially outward of the moving frame of reference (Fig. 1). The general solution of the state equations is well-known in Astrodynamics (Battin, 1987) and the general solution of the adjoint equations is obtained through properties of generalized canonical systems (da Silva Fernandes, 1994). Thus,

$$\boldsymbol{r} = \frac{a(1-e^2)}{1+e\cos f}\boldsymbol{e}_r\ , \tag{8}$$

$$\boldsymbol{v} = \sqrt{\frac{\mu}{a(1-e^2)}}\left[(e\sin f)\boldsymbol{e}_r + (1+e\cos f)\boldsymbol{e}_s\right], \tag{9}$$

$$\begin{aligned}\boldsymbol{p}_r = {} & \frac{a}{r^2}\left\{2ap_a + \left((1-e^2)\cos E\right)p_e + \left(\frac{r}{a}\right)\frac{\sin f}{e}\left(p_\omega - \frac{(1-e^3\cos E)}{\sqrt{1-e^2}}p_M\right)\right\}\boldsymbol{e}_r + \left\{\frac{\sin f}{a}p_e - \frac{(e+\cos f)}{ae(1-e^2)}p_\omega \right. \\ & \left. + \frac{\sqrt{1-e^2}\cos f}{ae}p_M\right\}\boldsymbol{e}_s + \frac{1}{a\sqrt{1-e^2}}\left\{\left(\frac{a}{r}\right)\sin E\left[p_I\cos\omega + \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\sin\omega\right]\right. \\ & \left. + \sqrt{1-e^2}\left(\frac{a}{r}\right)\cos E\left[p_I\sin\omega - \left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\cos\omega\right]\right\}\boldsymbol{e}_w\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\boldsymbol{p}_v = {} & \frac{1}{na\sqrt{1-e^2}}\left\{\left\{2ae\sin f\ p_a + \left((1-e^2)\sin f\right)p_e - \frac{(1-e^2)\cos f}{e}p_\omega + \frac{(1-e^2)^{3/2}}{e}\left(\cos f - \frac{2e}{1+e\cos f}\right)p_M\right\}\boldsymbol{e}_r\right. \\ & + \left\{2a(1-e^2)\left(\frac{a}{r}\right)p_a + (1-e^2)(\cos f + \cos E)p_e + \frac{(1-e^2)\sin f}{e}\left(1+\frac{1}{1+e\cos f}\right)\left(p_\omega - \sqrt{1-e^2}\,p_M\right)\right\}\boldsymbol{e}_s \\ & \left. + \left\{\left(\frac{r}{a}\right)\cos(\omega+f)p_I + \left(\frac{r}{a}\right)\sin(\omega+f)\left(\frac{p_\Omega}{\sin I} - p_\omega\cot I\right)\right\}\boldsymbol{e}_w\right\}.\end{aligned} \tag{11}$$

where $\boldsymbol{e}_s$ and $\boldsymbol{e}_w$ are unit vectors along circumferential and normal directions of the moving frame of reference, respectively (Fig. 1); a is the semi-major axis, e is the eccentricity, I is the inclination of orbital plane, Ω is the longitude of the ascending node, ω is the argument of pericenter, f is the true anomaly, E is the eccentric anomaly, M is the mean anomaly, $n = \sqrt{\mu/a^3}$ is the mean motion, and (r/a), $(r/a)\sin f$, ... etc are functions of the elliptic motion which can be expressed explicitly in terms of the eccentricity and the mean anomaly through Lagrange series (Battin, 1987). The anomalies are related through the equations:

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\,, \tag{12}$$

$$M = E - e\sin E\ . \tag{13}$$

The unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_s$ and $\boldsymbol{e}_w$ of the moving frame of reference are written in the fixed frame of reference as

$$
\begin{aligned}
\boldsymbol{e}_r &= \left(\cos\Omega\cos(\omega+f)-\sin\Omega\sin(\omega+f)\cos I\right)\boldsymbol{i} \\
&\quad+\left(\sin\Omega\cos(\omega+f)+\cos\Omega\sin(\omega+f)\cos I\right)\boldsymbol{j}+\sin(\omega+f)\sin I\,\boldsymbol{k}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{e}_s &= -\left(\cos\Omega\sin(\omega+f)+\sin\Omega\cos(\omega+f)\cos I\right)\boldsymbol{i} \\
&\quad+\left(-\sin\Omega\sin(\omega+f)+\cos\Omega\cos(\omega+f)\cos I\right)\boldsymbol{j}+\cos(\omega+f)\sin I\,\boldsymbol{k}
\end{aligned}
$$

$$
\boldsymbol{e}_w = \sin\Omega\sin I\,\boldsymbol{i}-\cos\Omega\sin I\,\boldsymbol{j}+\cos I\,\boldsymbol{k}\,.
$$

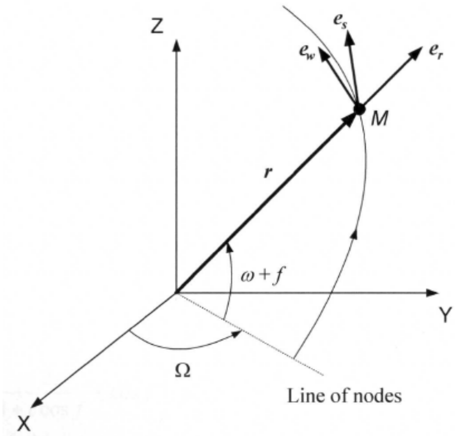

Figure 1 – Frames of reference.

Equations (8) – (11) define a Mathieu transformation between the Cartesian elements $(\boldsymbol{r}, \boldsymbol{v}, \boldsymbol{p_r}, \boldsymbol{p_v})$ and the orbital ones $(a, e, I, \Omega, \omega, M, p_a, p_e, p_I, p_\Omega, p_\omega, p_M)$. The Hamiltonian function is invariant with respect to this canonical transformation, thus

$$
H_0 = np_M\,, \tag{14}
$$

$$
\begin{aligned}
H_\gamma^* = \frac{1}{2n^2a^2(1-e^2)}\Bigg\{ & \frac{1}{2}(1-\cos 2f)\left[2aep_a+(1-e^2)p_e\right]^2+2(1-e^2)\sin 2f\left[-ap_ap_\omega-\frac{(1-e^2)}{2e}p_ep_\omega\right] \\
& +4(1-e^2)^{3/2}\sin f\left(\frac{-2e}{1+e\cos f}+\cos f\right)\left[ap_ap_M+\frac{(1-e^2)}{2e}p_ep_M\right]+\frac{(1-e^2)^2}{2e^2}(1+\cos 2f)p_\omega^{\,2} \\
& -\frac{2(1-e^2)^{5/2}}{e^2}\left(\frac{-2e}{1+e\cos f}+\cos f\right)\cos f\,p_\omega p_M+\frac{(1-e^2)^3}{e^2}\left(\frac{-2e}{1+e\cos f}+\cos f\right)^2 p_M^{\,2}+4a^2(1-e^2)^2\left(\frac{a}{r}\right)^2 p_a^{\,2} \\
& +4a(1-e^2)^2\left(\frac{a}{r}\right)(\cos E+\cos f)p_ap_e+(1-e^2)^2(\cos E+\cos f)^2 p_e^{\,2} \\
& +\frac{4a(1-e^2)^2}{e}\left(\frac{a}{r}\right)\sin f\left(1+\frac{1}{1+e\cos f}\right)\left[p_ap_\omega-(1-e^2)^{1/2}p_ap_M\right] \\
& +\frac{2(1-e^2)^2}{e}(\cos E+\cos f)\left(1+\frac{1}{1+e\cos f}\right)\sin f\left[p_ep_\omega-\sqrt{1-e^2}\,p_ep_M\right] \\
& +\left[\frac{(1-e^2)}{e}\left(1+\frac{1}{1+e\cos f}\right)\sin f\left[p_\omega-\sqrt{1-e^2}\,p_M\right]\right]^2+\frac{1}{2}\left(\frac{r}{a}\right)^2\left[p_I^{\,2}+\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\right] \\
& +\frac{1}{2}\left(\frac{r}{a}\right)^2\cos 2(\omega+f)\left[p_I^{\,2}-\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)^2\right]+\left(\frac{r}{a}\right)^2\sin 2(\omega+f)\,p_I\left(\frac{p_\Omega}{\sin I}-p_\omega\cot I\right)\Bigg\}.
\end{aligned} \tag{15}
$$

The new Hamiltonian function describes the optimal low-thrust limited-power trajectories in a Newtonian central gravity field. Note that new Hamiltonian function becomes singular for circular and/or equatorial orbits.

4. AVERAGED MAXIMUM HAMILTONIAN FOR OPTIMAL TRANSFERS

In order to eliminate the short periodic terms from the maximum Hamiltonian function H^*, Hori method (Hori, 1966) is applied. It is assumed that H_0 is of zero order and H_γ^* is of the first order in a small parameter defined by the magnitude of the thrust acceleration.

Consider an infinitesimal canonical transformation,

$$(a, e, I, \Omega, \omega, M, p_a, p_e, p_I, p_\Omega, p_\omega, p_M) \to (a', e', I', \Omega', \omega', M', p'_a, p'_e, p'_I, p'_\Omega, p'_\omega, p'_M).$$

The new variables are designated by the prime. According to the algorithm of Hori method, at order 0, one finds

$$F_0 = n' p'_M .$$

F_0 denotes the new undisturbed Hamiltonian. Now, consider the canonical system described by F_0:

$$\frac{da'}{dt} = 0 \qquad \frac{de'}{dt} = 0 \qquad \frac{dI'}{dt} = 0 \qquad \frac{d\Omega'}{dt} = 0 \qquad \frac{d\omega'}{dt} = 0 \qquad \frac{dM'}{dt} = n',$$

and,

$$\frac{dp'_a}{dt} = \frac{3}{2}\frac{n'}{a'} p'_M \qquad \frac{dp'_e}{dt} = 0 \qquad \frac{dp'_I}{dt} = 0 \qquad \frac{dp'_\Omega}{dt} = 0 \qquad \frac{dp'_\omega}{dt} = 0 \qquad \frac{dp'_M}{dt} = 0,$$

general solution of which is given by

$$a' = a'_0 \qquad e' = e'_0 \qquad I' = I'_0 \qquad \Omega' = \Omega'_0 \qquad \omega' = \omega'_0 \qquad M' = M'_0 + n'(t - t_0)$$

and,

$$p'_a = p'_{a_0} + \frac{3}{2}\frac{n'(t - t_0)}{a'} p'_M \qquad p'_e = p'_{e_0} \qquad p'_I = p'_{I_0} \qquad p'_\Omega = p'_{\Omega_0} \qquad p'_\omega = p'_{\omega_0} \qquad p'_M = p'_{M_0} .$$

The subscript 0 denotes the constants of integration.

This general solution is introduced into the equation of order 1 of the algorithm of Hori method and the mean value of H_{γ^*} must be calculated from the resulting equation. S_1 is obtained through integration of the remaining part. F_1 and S_1 are then given by the following equations:

$$\begin{aligned} F_1 = \frac{a'}{2\mu}\Bigg\{ & 4a'^2 p'^2_a + \frac{5}{2}\left(1 - e'^2\right)p'^2_e + \frac{\left(5 - 4e'^2\right)}{2e'^2} p'^2_\omega + \frac{p'^2_I}{2\left(1 - e'^2\right)}\left[\left(1 + \frac{3}{2}e'^2\right) + \frac{5}{2}e'^2 \cos 2\omega'\right] \\ & + \frac{5e'^2 \sin 2\omega'}{2\left(1 - e'^2\right)} p'_I \left(\frac{p'_\Omega}{\sin I'} - \cot I' \, p'_\omega\right) + \frac{1}{2\left(1 - e'^2\right)}\left(\frac{p'_\Omega}{\sin I'} - \cot I' \, p'_\omega\right)^2 \left[\left(1 + \frac{3}{2}e'^2\right) - \frac{5}{2}e'^2 \cos 2\omega'\right]\Bigg\}, \end{aligned} \tag{16}$$

$$\begin{aligned} S_1 = \frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}}\Bigg\{ & 8e' \sin E' \, a'^2 p'^2_a + 8\left(1 - e'^2\right)\sin E' \, a' p'_a p'_e - \frac{8\sqrt{1 - e'^2}}{e'} \cos E' p'_a p'_\omega \\ & + \left(1 - e'^2\right)\left[-\frac{5}{4}e' \sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e' \sin 3E'\right]p'^2_e + \frac{\sqrt{1 - e'^2}}{e'}\left[\frac{5}{2}e' \cos E' - \frac{1}{2}\left(3 - e'^2\right)\cos 2E' + \frac{1}{6}e' \cos 3E'\right]p'_e p'_\omega \\ & + \left(1 - e'^2\right)^{-1}\left[p'^2_I + \left(\frac{p'_\Omega}{\sin I'} - p'_\omega \cot I'\right)^2\right]\left[\left(-e' + \frac{3}{8}e'^3\right)\sin E' + \frac{3}{8}e'^2 \sin 2E' - \frac{1}{24}e'^3 \sin 3E'\right] + \left(1 - e'^2\right)^{-1}\Big[p'^2_I \cos 2\omega' \\ & + 2p'_I\left(\frac{p'_\Omega}{\sin I'} - p'_\omega \cot I'\right)\sin 2\omega' - \left(\frac{p'_\Omega}{\sin I'} - p'_\omega \cot I'\right)^2 \cos 2\omega'\Big] \end{aligned}$$

$$\times\left[\left(-\frac{5}{4}e'+\frac{5}{8}e'^3\right)\sin E'+\left(\frac{1}{4}+\frac{1}{8}e'^2\right)\sin 2E'+\left(-\frac{1}{12}e'+\frac{1}{24}e'^3\right)\sin 3E'\right]$$
$$+\left(1-e'^2\right)^{-1/2}\left[-p_I'^2\sin 2\omega'+2p_I'\left(\frac{p_\Omega'}{\sin I'}-p_\omega'\cot I'\right)\cos 2\omega'+\left(\frac{p_\Omega'}{\sin I'}-p_\omega'\cot I'\right)^2\sin 2\omega'\right]\left[\frac{5}{4}e'\cos E'\right. \quad (17)$$
$$\left.-\left(\frac{1}{4}+\frac{1}{4}e'^2\right)\cos 2E'+\frac{1}{12}e'\cos 3E'\right]+\frac{p_\omega'^2}{e'^2}\left[\left(\frac{5}{4}e'-e'^3\right)\sin E'+\left(-\frac{3}{4}+\frac{1}{2}e'^2\right)\sin 2E'+\frac{1}{12}e'\sin 3E'\right]\Bigg\}.$$

Terms factored by p_M' have been omitted in equations above, since only transfers (no rendez-vous) are considered.

5. SPECIAL CLASSES OF MANEUVERS

In this section, complete first order solutions for three special classes of maneuvers – transfers between coplanar orbits, transfers between non-coplanar coaxial orbits and transfers between non-coplanar co-parameters orbits – are presented. These maneuvers correspond to integrable canonical systems described by $F' = F_0 + F_1$, whose solutions are obtained by applying Hamilton-Jacobi theory.

5.1 Transfers between coplanar orbits

For transfers between coplanar orbits F_1 and S_1 simplify and are given by:

$$F_1 = \frac{a'}{2\mu}\left\{4a'^2 p_a'^2 + \frac{5}{2}\left(1-e'^2\right)p_e'^2 + \frac{\left(5-4e'^2\right)}{2e'^2}p_\omega'^2\right\}, \quad (18)$$

$$S_1 = \frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}}\left\{8e'\sin E'\, a'^2 p_a'^2 + 8\left(1-e'^2\right)\sin E'\, a' p_a' p_e' - 8\frac{\left(1-e'^2\right)^{1/2}}{e'}\cos E'\, a' p_a' p_\omega'\right.$$
$$+\left(1-e'^2\right)\left[-\frac{5}{4}e'\sin E'+\frac{3}{4}\sin 2E'-\frac{1}{12}e'\sin 3E'\right]p_e'^2$$
$$+\frac{2\left(1-e'^2\right)^{1/2}}{e'}\left[\frac{5}{4}e'\cos E'+\frac{1}{4}\left(e'^2-3\right)\cos 2E'+\frac{1}{12}e'\cos 3E'\right]p_e' p_\omega' \quad (19)$$
$$\left.+\frac{1}{e'^2}\left[\left(\frac{5}{4}-e'^2\right)e'\sin E'-\frac{1}{2}\left(\frac{3}{2}-e'^2\right)\sin 2E'+\frac{1}{12}e'\sin 3E'\right]p_\omega'^2\right\}.$$

The general solution of the canonical system described by the new average Hamiltonian function is obtained through two canonical transformations as described in da Silva Fernandes and Carvalho (2008). First, consider the Mathieu transformation, $(a', e', \omega', p_a', p_e', p_\omega') \to (a'', \phi, \omega'', p_a'', p_\phi, p_\omega'')$, defined by the following equations:

$$a' = a'' \qquad p_a' = p_a'' \qquad e' = \sin\phi \qquad p_e' = \frac{p_\phi}{\cos\phi} \qquad \omega' = \omega'' \qquad p_\omega' = p_\omega''. \quad (20)$$

The Hamiltonian function F' is invariant with respect to this transformation. Thus,

$$F'' = \frac{a''}{2\mu}\left\{4a''^2 p_a''^2 + \frac{5}{2}p_\phi''^2 + \left(\frac{5}{2}\csc^2\phi - 2\right)p_\omega''^2\right\}. \quad (21)$$

Now, consider the canonical transformation, $\left(a'', \phi, \omega'', p_a'', p_\phi, p_\omega''\right) \xrightarrow{W} \left(C_1, C_2, \mathrm{E}, p_{C_1}, p_{C_2}, p_{\mathrm{E}}\right)$, defined by a generating function *W* such that the constants C_1, C_2 and E become the new generalized coordinates. These constants are defined by

$$p_\omega'' = C_1 \qquad p_\phi^2 + p_\omega''^2\csc^2\phi = C_2^2 \qquad \frac{a''}{2\mu}\left\{4a''^2 p_a''^2 + \frac{5}{2}p_\phi''^2 + \left(\frac{5}{2}\csc^2\phi - 2\right)p_\omega''^2\right\} = \mathrm{E}.$$

Constant E should not be confused with the eccentric anomaly E. By applying the separation of variables technique for solving the Hamilton-Jacobi equation (Lanczos, 1971), one gets:

$$W(a'',\phi,\omega'',C_1,C_2,\mathsf{E}) = W_1(a'',C_1,C_2,\mathsf{E}) + W_2(\phi,C_1,C_2,\mathsf{E}) + W_3(\omega'',C_1,C_2,\mathsf{E}),$$

with $W_1 = -\sqrt{\frac{5C^2}{2}}\left\{\sqrt{\frac{4\mu\mathsf{E}}{5C^2a''}-1} - \tan^{-1}\sqrt{\frac{4\mu\mathsf{E}}{5C^2a''}-1}\right\}$, $W_2 = \int\sqrt{C_2^2 - C_1^2\csc^2\phi}\,d\phi$, $W_3 = C_1\omega''$ and $5C_2^2 - 4C_1^2 = 5C^2$.

After some calculations (details can be found in da Silva Fernandes and Carvalho, 2008), one finds the solution of the canonical system governed by the Hamiltonian F'' for a given set of initial conditions:

$$a''(t) = \frac{a_0''}{1+\frac{4a_0''}{\mu}\left(\frac{1}{2}\mathsf{E}t^2 - a_0''p_{a_0}''t\right)} \qquad a''\sin^2 k_0 = a_0''\sin^2\left(\sqrt{2}\psi + k_0\right) \qquad \psi = \frac{1}{5}(\tau-\tau_0)\sqrt{1+4\cos^2 k_1}$$

$$\cos\phi = \cos k_1\cos\tau \qquad \omega'' = k_2 + \tan^{-1}(\tan\tau\csc k_1) - \frac{4}{5}\tau\sin k_1$$

$$p_a''^2 = \left(\frac{a_0''}{a''}\right)^3 p_{a_0}''^2 + \frac{1}{8}p_{\omega_0}''^2\left(5\csc^2 k_1 - 4\right)\left(\frac{a_0''}{a''^3} - \frac{1}{a''^2}\right) \qquad p_\phi^2 = p_{\omega_0}''^2\left(\csc^2 k_1 - \csc^2\phi\right) \qquad p_\omega'' = p_{\omega_0}'', \tag{22}$$

with the auxiliary constants k_0, k_1 and k_2 defined as functions of the initial value of the adjoint variables by

$$\csc^2 k_0 = \frac{8\left(a_0''p_{a_0}''\right)^2 + p_{\omega_0}''^2\left(5\csc^2 k_1 - 4\right)}{p_{\omega_0}''^2\left(5\csc^2 k_1 - 4\right)}, \csc^2 k_1 = \frac{p_{\phi_0}^2 + p_{\omega_0}''^2\csc^2\phi_0}{p_{\omega_0}''^2}, k_2 = \omega_0'' + \frac{4}{5}\tau_0\sin k_1 - \tan^{-1}\left(\tan\tau_0\csc k_1\right).$$

The constants C, C_1, C_2 and E can also be written as functions of the initial value of the adjoint variables:

$$C^2 = \frac{1}{5}p_{\omega_0}''^2\left(5\csc^2 k_1 - 4\right),\quad C_1 = p_{\omega_0}'',\quad C_2^2 = p_{\phi_0}^2 + p_{\omega_0}''^2\csc^2\phi_0,\quad 4\mu\mathsf{E} = a_0''\left(8\left(a_0''p_{a_0}''\right)^2 + p_{\omega_0}''^2\left(5\csc^2 k_1 - 4\right)\right).$$

The initial conditions are $a''(0) = a_0''$, $e''(0) = \sin\phi_0$ and $\omega''(0) = \omega_0''$, and, τ_0 is obtained from $\cos\phi_0 = \cos k_1\cos\tau_0$.

Following Hori method (Hori, 1966) and applying the initial conditions, one finds:

$$a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}}\left[8e'\sin E'a'^2p_a' + 4\left(1-e'^2\right)\sin E'a'p_e' - 4\frac{\left(1-e'^2\right)^{1/2}}{e'}\cos E'a'p_\omega'\right]_{E_0'}^{E'}, \tag{23}$$

$$\begin{aligned} e(t) = e'(t) + \sqrt{\frac{a'^5}{\mu^3}}&\left[4\left(1-e'^2\right)\sin E'a'p_a' + \left(1-e'^2\right)\left[-\frac{5}{4}e'\sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_e'\right.\\ &\left.+\frac{\left(1-e'^2\right)^{1/2}}{e'}\left[\frac{5}{4}e'\cos E' + \frac{1}{4}\left(e'^2-3\right)\cos 2E' + \frac{1}{12}e'\cos 3E'\right]p_\omega'\right]_{E_0'}^{E'}, \end{aligned} \tag{24}$$

$$\begin{aligned} \omega(t) = \omega'(t) + \sqrt{\frac{a'^5}{\mu^3}}&\left[-4\frac{\left(1-e'^2\right)^{1/2}}{e'}\cos E'a'p_a' + \frac{\left(1-e'^2\right)^{1/2}}{e'}\left[\frac{5}{4}e'\cos E' + \frac{1}{4}\left(e'^2-3\right)\cos 2E' + \frac{1}{12}e'\cos 3E'\right]p_e'\right.\\ &\left.+\frac{1}{e'^2}\left[\left(\frac{5}{4}-e'^2\right)e'\sin E' - \frac{1}{2}\left(\frac{3}{2}-e'^2\right)\sin 2E' + \frac{1}{12}e'\sin 3E'\right]p_\omega'\right]_{E_0'}^{E'}, \end{aligned} \tag{25}$$

with a', e', …, p_ω' given through Eqs (20) and (22). These equations become singular for circular orbits. The eccentric anomaly E' is computed from Kepler´s equation with the mean anomaly M' given by

$$M'(t) = M'(t_0) + \int_{t_0}^{t} \left[\sqrt{\frac{\mu}{a'^3}} - \left(\frac{5+2e'^2}{2} \right) \sqrt{\frac{a'^5}{\mu^3}} \frac{\sqrt{1-e'^2}}{e'^2} p'_\omega \right] dt .$$

5.2 Transfers between non-coplanar coaxial orbits

For transfers between non-coplanar coaxial orbits F_1 and S_1 simplify and are given by:

$$F_1 = \frac{a'}{2\mu} \left\{ 4a'^2 p'^2_a + \frac{5}{2}(1-e'^2) p'^2_e + \frac{(1+4e'^2)}{2(1-e'^2)} p'^2_I \right\}, \tag{26}$$

$$S_1 = \frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}} \left\{ 8e' \sin E' a'^2 p'^2_a + 8(1-e'^2) \sin E' a' p'_a p'_e + (1-e'^2)\left[-\frac{5}{4} e' \sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12} e' \sin 3E' \right] p'^2_e \right.$$
$$\left. + (1-e'^2)^{-1} p'^2_I \left[\left(-\frac{9}{4}e' + e'^3 \right) \sin E' + \left(\frac{1}{4} + \frac{1}{2}e'^2 \right) \sin 2E' - \frac{1}{12} e'^3 \sin 3E' \right] \right\}. \tag{27}$$

Consider the Mathieu transformation defined by Eq. (20) with p'_I replacing p'_ω. The Hamiltonian function F' is invariant with respect to this transformation and it is given by

$$F'' = \frac{a''}{2\mu} \left\{ 4a''^2 p''^2_a + \frac{5}{2} p''^2_\phi + \frac{1}{2}(1+5\tan^2\phi) p''^2_I \right\} . \tag{28}$$

Now, consider the canonical transformation, $(a'', \phi, I'', p''_a, p_\phi, p''_I) \xrightarrow{W} (C_1, C_2, \mathrm{E}, p_{C_1}, p_{C_2}, p_{\mathrm{E}})$, defined by a generating function W such that the constants C_1, C_2 and E become the new generalized coordinates. These constants are defined by

$$p^2_\phi + p''^2_I \tan^2\phi = C_1^2 \qquad p''_I = C_2 \qquad \frac{a''}{2\mu}\left\{ 4a''^2 p''^2_a + \frac{5}{2} p''^2_\phi + \frac{1}{2}(1+5\tan^2\phi) p''^2_I \right\} = \mathrm{E} .$$

By applying the separation of variables technique for solving the Hamilton-Jacobi equation (Lanczos, 1971), one gets:

$$W(a'', \phi, I'', C_1, C_2, \mathrm{E}) = W_1(a'', C_1, C_2, \mathrm{E}) + W_2(\phi, C_1, C_2, \mathrm{E}) + W_3(I'', C_1, C_2, \mathrm{E}) ,$$

with W_1 given as defined in Section 5.1, $W_2 = \int \sqrt{C_1^2 - C_2^2 \tan^2\phi}\, d\phi$, $W_3 = C_2 I''$ and $5C_1^2 + C_2^2 = 5C^2$.

After some calculations, one finds the solution of the canonical system governed by the Hamiltonian F'' for a given set of initial conditions:

$$a''(t) = \frac{a''_0}{1 + \frac{4a''_0}{\mu}\left(\frac{1}{2}\mathrm{E}t^2 - a''_0 p''_{a_0} t \right)} \qquad a'' \sin^2 k_0 = a''_0 \sin^2\left(\sqrt{2}\psi + k_0\right) \qquad \psi = \frac{C}{\sqrt{5(C_1^2 + C_2^2)}}(\tau - \tau_0)$$

$$\sin\phi = \sin k_1 \sin\tau \qquad I'' = k_2 + \tan^{-1}(\tan\tau / \sec k_1) - \frac{4}{5}\tau\cos k_1$$

$$p''^2_a = \left(\frac{a''_0}{a''} \right)^3 p''^2_{a_0} + \frac{1}{8} p''^2_{I_0}\left(5\sec^2 k_1 - 4\right)\left(\frac{a''_0}{a''^3} - \frac{1}{a''^2} \right) \qquad p^2_\phi = p''^2_{I_0}\left(\sec^2 k_1 - \sec^2\phi\right) \qquad p''_I = p''_{I_0} , \tag{29}$$

with the auxiliary constants k_0, k_1 and k_2 defined as functions of the initial value of the adjoint variables by

$$\csc^2 k_0 = \frac{8\left(a''_0 p''_{a_0}\right)^2 + p''^2_{I_0}\left(5\sec^2 k_1 - 4\right)}{p''^2_{I_0}\left(5\sec^2 k_1 - 4\right)} , \sec^2 k_1 = \frac{p^2_{\phi_0} + p''^2_{I_0}\sec^2\phi_0}{p''^2_{I_0}} , k_2 = I''_0 - \tan^{-1}(\tan\tau_0 / \sec k_1) + \frac{4}{5}\tau_0 \cos k_1 .$$

The constants C, C_1, C_2 and E can also be written as functions of the initial value of the adjoint variables:

$$C^2 = \frac{1}{5} p_{I_0}''^2 \left(5\sec^2 k_1 - 4\right), \quad C_2 = p_{I_0}'', \quad C_1^2 = p_{\phi_0}^2 + p_{I_0}''^2 \tan^2 \phi_0, \quad 4\mu \mathsf{E} = a_0''\left(8\left(a_0'' p_{a_0}''\right)^2 + p_{I_0}''^2 \left(5\sec^2 k_1 - 4\right)\right).$$

The initial conditions are $a''(0) = a_0''$, $e''(0) = \sin\phi_0$ and $I''(0) = I_0''$, and, τ_0 is obtained from $\sin\phi_0 = \sin k_1 \sin\tau_0$.

Following Hori method (Hori, 1966) and applying the initial conditions, one finds:

$$a(t) = a'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[8e'\sin E' a'^2 p_a' + 4\left(1 - e'^2\right)\sin E' a' p_e'\right]_{E_0'}^{E'}, \tag{30}$$

$$e(t) = e'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[4\left(1 - e'^2\right)\sin E' a' p_a' + \left(1 - e'^2\right)\left[-\frac{5}{4}e'\sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_e'\right]_{E_0'}^{E'}, \tag{31}$$

$$I(t) = I'(t) + \sqrt{\frac{a'^5}{\mu^3}} \left[\left(1 - e'^2\right)^{-1}\left[\left(-\frac{9}{4}e' + e'^3\right)\sin E' + \left(\frac{1}{4} + \frac{1}{2}e'^2\right)\sin 2E' - \frac{1}{12}e'^3 \sin 3E'\right]p_I'\right]_{E_0'}^{E'} \tag{32}$$

with a', e', …, p_I' given through Eqs (20) (with p_I' replacing p_ω') and (29). The eccentric anomaly E' is computed from Kepler´s equation with the mean anomaly $M'(t) = M'(t_0) + \int_{t_0}^{t} n' dt$.

5.3 Transfers between non-coplanar co-parameters orbits

For transfers between non-coplanar co-parameters orbits F_1 and S_1 simplify and are given by:

$$F_1 = \frac{a'}{2\mu}\left\{4a'^2 p_a'^2 + \frac{5}{2}\left(1 - e'^2\right)p_e'^2 + \frac{1}{2}p_I'^2\right\}, \tag{33}$$

$$\begin{aligned} S_1 = \frac{1}{2}\sqrt{\frac{a'^5}{\mu^3}} \Big\{ & 8e'\sin E' a'^2 p_a'^2 + 8\left(1 - e'^2\right)\sin E' a' p_a' p_e' + \left(1 - e'^2\right)\left[-\frac{5}{4}e'\sin E' + \frac{3}{4}\sin 2E' - \frac{1}{12}e'\sin 3E'\right]p_e'^2 \\ & + p_I'^2\left[\frac{1}{4}e'\sin E' - \frac{1}{4}\sin 2E' + \frac{1}{12}e'\sin 3E'\right]\Big\}. \end{aligned} \tag{34}$$

Consider the Mathieu transformation defined by Eq. (20) with p_I' replacing p_ω'. The Hamiltonian function F' is invariant with respect to this transformation and it is given by

$$F'' = \frac{a''}{2\mu}\left\{4a''^2 p_a''^2 + \frac{5}{2}p_\phi''^2 + \frac{1}{2}p_I''^2\right\}. \tag{35}$$

Note that Eq. (28) reduces to Eq. (35), taking $\tan\phi = 0$. Thus, Eq. (29) simplify and the solution of the canonical system governed by the Hamiltonian F'' for a given set of initial conditions is given by:

$$a''(t) = \frac{a_0''}{1 + \frac{4a_0''}{\mu}\left(\frac{1}{2}\mathsf{E}t^2 - a_0'' p_{a_0}'' t\right)} \qquad a''\sin^2 k_0 = a_0''\sin^2\left(\sqrt{2}\psi + k_0\right) \qquad \psi = \frac{C}{\sqrt{5}C_1}\left(\phi - \phi_0\right)$$

$$I'' = I_0'' + \frac{C_2}{\sqrt{5}C}\psi \qquad p_a''^2 = \left(\frac{a_0''}{a''}\right)^3 p_{a_0}''^2 + \frac{1}{8}\left(5p_{\phi_0}''^2 + p_{I_0}''^2\right)\left(\frac{a_0''}{a''^3} - \frac{1}{a''^2}\right) \qquad p_\phi = p_{\phi_0} \qquad p_I'' = p_{I_0}'', \tag{36}$$

with the auxiliary constant k_0 defined by $\csc^2 k_0 = \frac{8\left(a_0'' p_{a_0}''\right)^2 + 5p_{\phi_0}''^2 + p_{I_0}''^2}{5p_{\phi_0}''^2 + p_{I_0}''^2}$. The constants C, C_1, C_2 and E can also be written as functions of the initial value of the adjoint variables:

$$C^2 = p_{\phi_0}^2 + \frac{1}{5} p_{I_0}''^2, \qquad C_2 = p_{I_0}'', \qquad C_1 = p_{\phi_0}, \qquad 4\mu \mathrm{E} = a_0''\left(8\left(a_0'' p_{a_0}''\right)^2 + 5 p_{\phi_0}''^2 + p_{I_0}''^2\right).$$

The initial conditions are $a''(0) = a_0''$, $e''(0) = \sin\phi_0$ and $I''(0) = I_0''$.

Following Hori method (Hori, 1966) and applying the initial conditions, one finds that $a(t)$ and $e(t)$ are given by Eqs (30) and (31), respectively, and

$$I(t) = I'(t) + \sqrt{\frac{a'^5}{\mu^3}}\left[\left[\frac{1}{4} e' \sin E' - \frac{1}{4}\sin 2E' + \frac{1}{12} e' \sin 3E'\right] p_I'\right]_{E_0'}^{E'}. \qquad (37)$$

with a', e', …, p_I' given through Eqs (20) (with p_I' replacing p_ω') and (36). The eccentric anomaly E' is computed as described in Section 5.2.

6. CONCLUSION

Approximated analytical solutions, which include short periodic terms, have been obtained for three different problems involving optimal low-thrust limited-power transfers between elliptical orbits in a Newtonian central gravity field using an approach based on canonical transformations. The two-point boundary value problem of going from an initial orbit to a given final orbit can be solved through a Newton-Raphson algorithm using these solutions, as described in da Silva Fernandes e Carvalho (2008). Finally, it should be noted that similar results are obtained for maneuvers between non-coplanar orbits involving changes in the longitude of the ascending node.

7. ACKNOWLEDGEMENTS

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