USE OF 3D-TRANSIENT ANALYTICAL SOLUTION BASED ON GREEN'S FUNCTION TO REDUCE COMPUTATIONAL TIME IN INVERSE HEAT CONDUCTION PROBLEMS

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Abstract. *Inverse problems can be found in many areas of science and engineering and can be applied in different ways. Two examples can be cited: thermal properties estimation or heat flux function estimation in some engineering thermal process. The great advantage of the inverse technique is the ability of obtaining a physical problem solution that cannot be solved directly. Different techniques for the solution of inverse heat conduction problem (IHCP) can be found in literature. However, any inverse or optimization technique has a basic and common characteristic: both of them need to solve the direct solution several times. This characteristic is the cause of the large time consumed. In heat conduction problem, the time consumed is, usually, due to the use of numerical solutions of multidimensional models with refined mesh. In this case, if analytical solutions are available the computational time can be reduced drastically. This study presents the development and application of a 3D-transient analytical solution based on Green's function. The inverse problem is due to the thermal properties estimation of conductors. The method is based on experimental determination of thermal conductivity and diffusivity using partially heated surface method without heat flux transducer. First developed to use numerical solution, this technique can, using analytical solution, estimate thermal properties faster and with more accuracy.*

Keywords: *heat transfer, analytical solution, Green's Function, inverse heat conduction*

1. INTRODUCTION

Analytical solutions are an important tool for solution of engineering problems (Beck *et al.*, 2008), since they can be used: to validate approximate solutions; to facilitate the analysis and understanding of physical problems; in construction of physical problems; in construction of new numerical algorithms such as the transient method of surface element (Beck *et al.*, 1992) or in direct application in real problems reducing the computational cost and allowing exact solutions of the model studied be obtained.

Examples of analytical solutions developed for verification or validation of approximate solutions obtained by numerical methods, such as algorithms based on finite differences, finite volumes or finite elements can be found in papers of Beck *et al.* (2006), Beck *et al.* (2004) or Macmasters *et al.* (2002) and others.

To mention, also, various thermal models involving heat conduction, such as, multidimensional problems in rectangular solids Beck and McMasters (2004), bidimensional transient problems with periodic heating in multi-layer cylindrical bars Milosevic and Raynoud (2004) or heated solids in motion (Haji-Sheikh and Beck, 2002), (Haji-Sheikh *et al.*, 2003), (Beck and McMasters, 2004) and (Haji-Sheikh *et al.*, 2009).

The complexity of a thermal model, from the view-point of analytical solution, normally is in multidimensional transient problems subjected to non-homogeneities such as prescribed heat flux or transient temperature or transient heat generation. Since practical engineering problems normally involve all these aspects the customary procedure is the use of simplifying hypotheses or the application of numerical models.

The principal objective of this study is the use direct of analytical solutions to develop techniques of inverse problems applied to thermal problems, more specifically in experimental techniques to obtain thermal properties.

Various researches have dedicated special attention of obtain analytical solutions for the thermal problem in orthogonal machining processes, always using previous knowledge machining processes as knowledge of the heat generated at the tool-chip interface or simplified models.

A thermal model adequate for this type of problem, should consider the multidimensional transient characteristics, environment (convection) and the heat generated at the tool-workpiece interface. As mentioned, the construction of an analytical model with the above characteristics is one of the reasons of this study.

Normally the objective of development of analytical solutions is to gain knowledge about the phenomenon, to validate numerical models or application direct in engineering problems such as, for example, thermal modeling of an orthogonal cutting temperature (Longbottom and Lanham, 2005). However, in this study, the principal objective is the use of analytical solutions in optimization problems. The inverse algorithms usually have to calculate the direct problem several times. In this case, the use of analytical solutions not only increases the precision but also reduces greatly the computational time. A typical application, shown here, is in the inclusion of analytical solutions in the DPT optimization code for estimations of thermal properties of rectangular solid samples using the partial heating method developed by Borges *et al.* (2006).

The majority of experimental techniques that determining thermal properties uses analytical solutions. For example, the classical techniques to measure properties such as the flash method (Parker *et al.*, 1961) and its variations of estimation of thermal diffusivity (Degiovanni, 1988), (Sramkova and Log, 1995), (Thermitus and Laurent, 1997), (Albers *et al.*, 2001), (Maillet *et al.*, 2000) or the hot wire method proposed by Blackwell (1954), and applied by various authors to estimate thermal conductivity of solids (Grazzini *et al.*, 1996), (Gross and Le-Thanh-Son, 2004), (Abu-Hamdeh *et al.*, 2001), (Xie *et al.*, 2006), (Coquard *et al.*, 2006). There are still other studies that use analytical solutions for estimations properties such as Guimarães *et al.* (1995), Nicolau *et al.* (2002), and Lima *et al.* (2003). However, all these techniques are applied to measure thermal properties of nonconductors using one-dimensional thermal models. Usually, the hypothesis of one-dimensional model is guaranteed by a high geometrical ratio (area/thickness). The thermal gradient in the directions of the heat flux is thus obtained. This procedure is difficult to apply in conductor materials which due to the low sensitivity of the thermal properties in relation to the temperature variation in the heat flux direction will require high-power equipment or very thick samples (Borges *et al.*, 2006). This is one of the reasons for the trend to search of more complex thermal models which approximates to real experimental conditions and thus allowing the development of more simple and less costly experimental project with greater flexibility in relation to the sample geometry. Multidimensional models with numerical solutions were, then, introduced and incorporated in experimental techniques such as in the papers of de Dowding *et al.* (1996), Aviles-Ramos *et al.* (2001), Murphy *et al.* (2005) and Borges *et al.* (2006).

The technique developed by Borges *et al.* (2006) is quite robust and competitive in terms of range of application (conductors and non-conductors) and geometry (disks, rectangular and irregular samples) principally due to the adaptability and capacity of solution of thermal models with are more complex, such as a three-dimensional transient model with transient boundary conditions (heat flux varying with time and in space). The possibility of incorporating exact solutions in this method (code) represents a great contribution in reduction of computational time, in numerical stability and in more precise estimations. This is one of the main objectives of this work.

2. THEORETICAL FUNDAMENTALS

2.1 Description of Experimental Technique to Obtain Thermal Conductivity and Thermal Diffusivity

The procedure is divided in five steps: development of an experimental apparatus which permits a heat flux at one part of the sample while the remaining surfaces are kept isolated; ii) obtain a thermal model of the sample; iii) obtain a parameter proportional to the heat flux at the sample $q^+(t)$, using the sequential method with specified function Beck *et al.* (1995); iv) obtain the thermal diffusivity; v) comparison between the total heat rate supplied by the heater element and the parameter, $q^+(t)$, proportional to the heat flux at the sample during the on-off heating cycle and consequently obtain the thermal conductivity.

Thermal model

The proposed thermal model to be reproduced experimentally is given by a sample initially at uniform temperature, T_0. The sample is then submitted to a heat flux $[W/m^2]$ while all other surfaces are kept isolated. Figure 1 shows the thermal model.

Experimental set-up

The boundary conditions of the theoretical model must be guaranteed experimentally. This means that the isolated condition at the surfaces must be obtained. An efficient way to obtain isolation experimentally from the viewpoint of convection heat loss is an environment in vacuum. Figure 2 shows an experimental apparatus which is basically a vacuum furnace subjecting the heat and the sample to an environment free from convection and a computer controlled data acquisition system.

A cast iron sample with thickness of $65mm$ and lateral dimensions of $80,5mm \times 80mm$ initially in thermal equilibrium at T_0 is then submitted to a unidirectional and uniform heat flux. A total heat rate, P, is supplied by a 318Ω electrical resistance heater, covered with silicone rubber, with lateral dimensions of $50 \times 50mm$ and thickness $0,3mm$. The magnitude of P can be obtained just by multiplying the voltage difference versus current value.

The temperatures are measured using two surface thermocouples (type K). The signals of temperatures are acquired by a data acquisition system HP Series 75000 with voltmeter E1326B controlled by a personal computer.

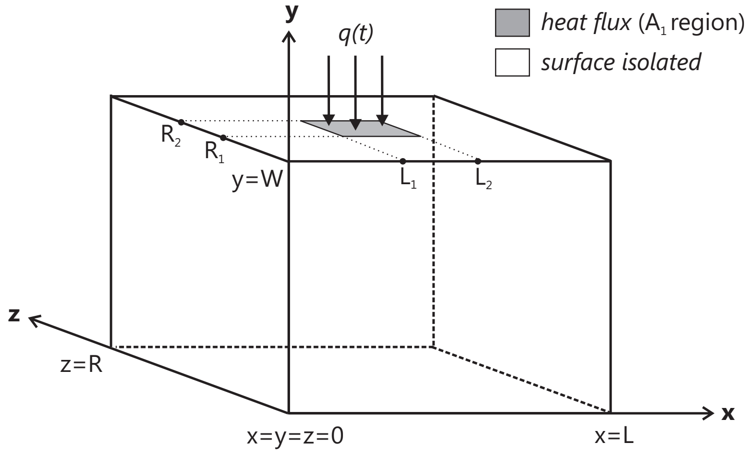
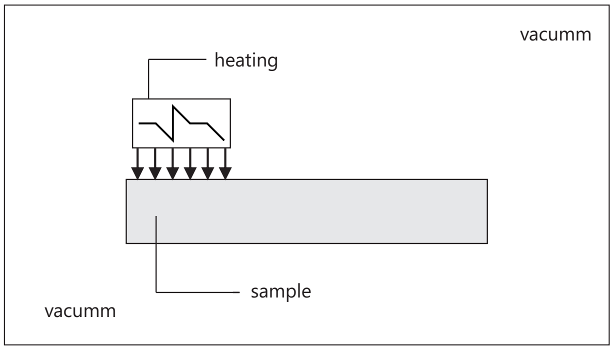


Figure 1. Equivalent three-dimensional transient thermal model

(a)



(b)

Figure 2. (a) Experimental apparatus. (b) Schematic of the heater and the sample

Inverse problem: obtaining the dimensionless heat flux $q^{+}(t)$

Various techniques of inverse problems can be used to estimate the heat flux applied. The main difficulty is that the thermal properties of the sample are also unknown. In which case, the technique of Borges *et al.* (2006) proposes the obtaining of a dimensionless heat flux $q^{+}(t)$, proportional to the real heat flux applied $q(t)$. Later the real heat flux $q(t)$ is then identified. This study uses the function specification sequential method described by Beck *et al.* (1995) to estimate $q^{+}(t)$.

Function Specification Sequential Method Beck *et al.* (1995)

The function specification sequential method is based on mean squared error, S, between measured temperatures, Y, and calculated temperatures, T, such that

$$S = \sum_{i=1}^{r} \left(T(t_{M+i-1}) - Y(t_{M+i-1})\right)^2 \tag{1}$$

where $i = 1, 2, ..., r$ is the number of future time steps, $j = 1, 2, ..., J$ is the number of thermocouple, M is the computed instant of time.

This method is called function specification because it assumes a functional form of the surface heat flux variation with time and space. The function can be a sequence of constants segments, straightline segments or it can be one of many others forms such as parabolic, cubic or exponential Beck *et al.* (1995). In this work, the heat flux is considered as constants segments over future time steps, then

$$q^+_{M+1} = ... = q^+_{M+i-1} \tag{2}$$

The function specification method can be resumed in following computational algorithm:
Step 1: Calculate the sensitivity coefficients to whole domain time and save them;
Step 2: Assume a functional form for $q^+(t)$ for times t_M, t_{M+1},...,t_{M+r-1};
Step 3: Minimize the sum of squares function S;
Step 4: Obtain the estimated heat flux;
Step 5: Retain and save the first heat flux component q^+_M for all positions;
Step 6: Increase M by one and repeat the procedure since **Step 2**.
The sequential method will be applied in solution of the proposed inverse directly without alteration.
The principle is applied to estimate q^+_M , which is proportional to the heat flux $q(t)$ (Fig.1), and is defined by

$$q^+(t) = \frac{q(t)}{k}\frac{k_{ref}}{q_{ref}} \tag{3}$$

where k_{ref} and q_{ref} are reference values of conductivity and heat flux density respectively. They can assume, whatever value, for example unitary.

The solution of the dimensionless direct problem is presented in the next section.

Obtaining the thermal diffusivity, α

To obtain the thermal diffusivity, a dynamic system equivalent to the thermal model shown in Fig.1 was chosen. The system is characterized by input whose signal $X(t)$ is the dimensionless heat flux $X(t) = q^+(t)$, and an output represented by the temperature difference between two distinct positions of the sample, $Y(t) = T_1(t) - T_2(t)$.

The frequency response, $H(f)$, of the system is defined by

$$H(f) = \frac{T_1(t) - T_2(t)}{\varphi(f)} \tag{4}$$

And its phase factor, φ, can be calculated as

$$\varphi = \arctan\left(\frac{\Im H(f)}{\Re H(f)}\right) \tag{5}$$

where $\Im H(f)$ and $\Re H(f)$ are the imaginary and real parts of $H(f)$, respectively.

Guimarães *et al.* (1995) observed that the phase factor, φ, is a function exclusively of the thermal diffusivity. This fact is the base of the procedure for obtaining the thermal diffusivity thorough minimization of an objective based on the difference between experimental and calculated values of φ. This objective function can be written as

$$S_\varphi = \sum_{i=1}^{Nf} \left(\varphi_e(i) - \varphi(i)\right)^2 \tag{6}$$

where φ_e and φ are the experimental and calculated values of the phase factor of $H(f)$, respectively.

The value of α are the values which minimizes Eq.(6). A method indicated to this minimization is the golden section method with polynomial approximation (Vanderplaats, 1984), since Eq.(6) is an unimodal function.

Obtaining heat flux, $q(t)$ and thermal conductivity, k

Once the thermal diffusivity and the dimensionless heat flux are determined, it remains to obtain the real heat flux, $q(t)$ and the thermal conductivity, k, of the sample.

The principle consists of applying a heat flux supplied by a resistance glued to the surface of the sample. For all heat flux generated be totally absorbed by the sample the environment has to be a vacuum. The total heat supplied to the sample can be obtained using the current and the voltage of the electric resistance.

It was concluded that the heat flux will be totally absorbed by the surface only after a time t_f. If $P = V \cdot I$ represents the power per unit area of the heater dissipated buy the resistance and $q(t)$ represents the heat flux effectively supplied to the sample. Then applying the energy conservation principle after a specific time t_f, we can write

$$\int_0^{t_f} q(t)dt = \int_0^{t_f} V(t)I(t)dt \tag{7}$$

where $V(t)$ and $I(t)$ represent the voltage and the current supplied. But Eq.(3) allows the thermal conductivity k, be obtained as

$$k = \int_0^{t_f} V(t)I(t) \left[\frac{q_{ref}}{k_{ref}} \int_0^{t_f} q^+(t)dt\right] dt \tag{8}$$

3. DIRECT PROBLEM

The procedure to obtain the thermal properties involves two optimization problems (inverse problem) which obtain $q^+(t)$ and α through the minimization of Eq.(1) and Eq.(6) respectively. Both estimations involve the evaluation of temperatures calculated using the thermal model, $T_1(t)$ and $T_2(t)$, which represent the solution of the direct problem.

From the viewpoint of application of the experimental technique proposed by Borges *et al.* (2006), the choice of the method of solution of the direct problem is in open. In other words, whichever numerical method such as finite volume or finite elements or if possible analytical solutions can be used. As mentioned, this study proposes the incorporation of analytical solutions in the experimental technique of determination of thermal conductivity and thermal diffusivity using the method of a partially heated surface without heat flux transducer proposed by Borges *et al.* (2006) reducing the computational cost and increasing the precision of the numerical calculations.

In following, the thermal model represented by Fig.1 and the obtaining of its analytical solution using Green's function (Fernandes, 2009) is obtained.

3.1 Exact solution of the thermal model using Green's function

The thermal problem proposed to be reproduced experimentally is given by a sample initially at a uniform temperature T_0. The sample is then subjected to a heat flux $[W/m^2]$ while all other surfaces are maintained isolated. Figure 1 shows the thermal model.

$$\frac{\partial^2\vartheta}{\partial u} + \frac{\partial^2\vartheta}{\partial v} + \frac{\partial^2\vartheta}{\partial w} = \frac{\partial\vartheta}{\partial\mu} \tag{9a}$$

In region A and $\mu^+ > 0$, subject to the boundary conditions:

$$-\left.\frac{\partial\vartheta}{\partial v}\right|_{v=W} = q^+(t) \qquad \text{in } A_1 \text{ region: } \frac{L_1}{L} \le u \le \frac{L_2}{L} \text{ and } \frac{R_1}{R} \le w \le \frac{R_2}{R} \tag{9b}$$

$$\left.\frac{\partial\vartheta}{\partial v}\right|_{v=W} = 0 \qquad \text{in } A - A_1 \tag{9c}$$

$$\left.\frac{\partial\vartheta}{\partial u}\right|_{u=0} = \left.\frac{\partial\vartheta}{\partial u}\right|_{u=L} = \left.\frac{\partial\vartheta}{\partial v}\right|_{v=0} = \left.\frac{\partial\vartheta}{\partial w}\right|_{w=0} = \left.\frac{\partial\vartheta}{\partial w}\right|_{w=W} = 0 \tag{9d}$$

and the initial condition,

$$\vartheta(u, v, w, 0) = 0 \tag{9e}$$

where A is defined by $(0 \le u \le 1, 0 \le w \le 1)$ and A_1 is region where the heat flux is applied.

In Equation (9) dimensionless group are defined as:

$$u = \frac{x}{L}; \quad v = \frac{y}{W}; \quad w = \frac{z}{R} \tag{10a}$$

$$\mu = \frac{\alpha_{ref}t}{W^2}; \quad \vartheta(u, v, w, \mu) = \frac{T(x, y, z, t) - T_0}{\frac{q_{ref}L}{k_{ref}}} \quad q^+ = \frac{q(t)}{k}\frac{k_{ref}}{q_{ref}} \tag{10b}$$

where α_{ref}, represent a reference value of thermal diffusivity.

Equation (9) represent a direct problem of heat conduction. If the dimensionless heat flux $q^+(t)$ is specified. On the other hand, the inverse problem is established when $q^+(t)$ is unknown. The various solution techniques of inverse problems (or optimization problems) have as procedure the evaluation repetitive of direct problems, always using with estimated or calculated values of heat flux density in an iterative process. Thus the proposed solution of the direct problem Eqs.(9) will be presented considering a known transient heat flux. In order to simplify the results, the direct problem solution is presented in its dimensional form considering definitions in Eqs.(10).

$$
\begin{aligned}
& T(x,y,z,t) = T_0 \\
& + \frac{\alpha}{k}\frac{1}{LWR}[L_2 - L_1][R_2 - R_1]\int_0^t q(\tau)d\tau \\
& + \frac{\alpha}{k}\frac{2}{WR}[R_2 - R_1]\sum_{m=1}^{\infty} e^{-\frac{m^2}{L^2}\pi^2\alpha t}\cos\left(\frac{m\pi x}{L}\right)\left[\sin\left(\frac{m\pi L_2}{L}\right) - \sin\left(\frac{m\pi L_1}{L}\right)\right]\frac{1}{m\pi}\int_0^t\left[q(\tau)e^{\frac{m^2}{L^2}\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{2}{LWR}[L_2 - L_1][R_2 - R_1]\sum_{n=1}^{\infty} e^{-\frac{n^2}{W^2}\pi^2\alpha t}\cos\left(\frac{n\pi y}{W}\right)\cos(n\pi)\int_0^t\left[q(\tau)e^{\frac{n^2}{W^2}\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{2}{LW}[L_2 - L_1]\sum_{p=1}^{\infty} e^{-\frac{p^2}{R^2}\pi^2\alpha t}\cos\left(\frac{p\pi z}{R}\right)\left[\sin\left(\frac{p\pi R_2}{R}\right) - \sin\left(\frac{p\pi R_1}{R}\right)\right]\frac{1}{p\pi}\int_0^t\left[q(\tau)e^{\frac{p^2}{R^2}\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{4}{WR}[R_2 - R_1]\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e^{-\left(\frac{m^2}{L^2}+\frac{n^2}{W^2}\right)\pi^2\alpha t}\cos\left(\frac{m\pi x}{L}\right)\left[\sin\left(\frac{m\pi L_2}{L}\right) - \sin\left(\frac{m\pi L_1}{L}\right)\right]\frac{1}{m\pi}\cos\left(\frac{n\pi y}{W}\right)\cos(n\pi) \\
& \times\int_0^t\left[q(\tau)e^{\left(\frac{m^2}{L^2}+\frac{n^2}{W^2}\right)\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{4}{W}\sum_{m=1}^{\infty}\sum_{p=1}^{\infty} e^{-\left(\frac{m^2}{L^2}+\frac{p^2}{R^2}\right)\pi^2\alpha t}\cos\left(\frac{m\pi x}{L}\right)\left[\sin\left(\frac{m\pi L_2}{L}\right) - \sin\left(\frac{m\pi L_1}{L}\right)\right]\frac{1}{m\pi} \\
& \times\cos\left(\frac{p\pi z}{R}\right)\left[\sin\left(\frac{p\pi R_2}{R}\right) - \sin\left(\frac{p\pi R_1}{R}\right)\right]\frac{1}{p\pi} \\
& \times\int_0^t\left[q(\tau)e^{\left(\frac{m^2}{L^2}+\frac{p^2}{R^2}\right)\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{4}{LW}[L_2 - L_1]\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\left(\frac{n^2}{W^2}+\frac{p^2}{R^2}\right)\pi^2\alpha t}\cos\left(\frac{n\pi y}{W}\right)\cos(n\pi)\cos\left(\frac{p\pi z}{R}\right)\left[\sin\left(\frac{p\pi R_2}{R}\right) - \sin\left(\frac{p\pi R_1}{R}\right)\right]\frac{1}{p\pi} \\
& \times\int_0^t\left[q(\tau)e^{\left(\frac{n^2}{W^2}+\frac{p^2}{R^2}\right)\pi^2\alpha\tau}\right]d\tau \\
& + \frac{\alpha}{k}\frac{8}{W}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} e^{-\left(\frac{m^2}{L^2}+\frac{n^2}{W^2}+\frac{p^2}{R^2}\right)\pi^2\alpha t}\cos\left(\frac{m\pi x}{L}\right)\left[\sin\left(\frac{m\pi L_2}{L}\right) - \sin\left(\frac{m\pi L_1}{L}\right)\right]\frac{1}{m\pi} \\
& \times\cos\left(\frac{n\pi y}{W}\right)\cos(n\pi)\cos\left(\frac{p\pi z}{R}\right)\left[\sin\left(\frac{p\pi R_2}{R}\right) - \sin\left(\frac{p\pi R_1}{R}\right)\right]\frac{1}{p\pi} \\
& \times\int_0^t\left[q(\tau)e^{\left(\frac{m^2}{L^2}+\frac{n^2}{W^2}+\frac{p^2}{R^2}\right)\pi^2\alpha\tau}\right]d\tau
\end{aligned}
\tag{11}
$$

where m, n, p are the number of terms required for the convergence of series.

4. ANALYSIS AND DISCUSSIONS OF RESULTS

4.1 Analytical Results Comparison

Some results obtained during development of the tridimensional thermal model proposed are shown here. Comparison with an established analytical solution of a simpler problem to ensure the precision of the proposed analytical solution is also shown. In following, a comparison with the numerical solution of the proposed problem is made showing the thermal properties and the computational time.

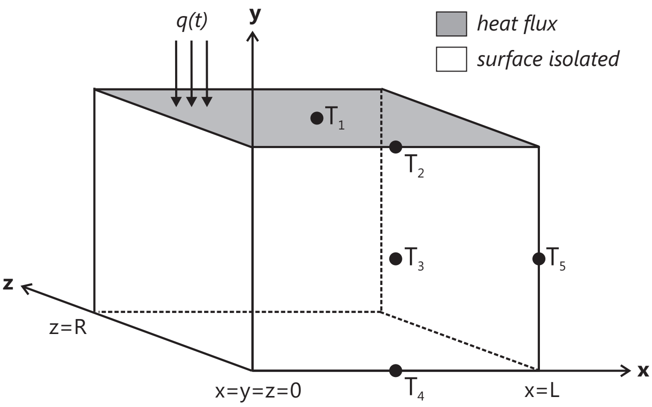


Figure 3. Schematic of a retangular solid with heated superior surface and all other surfaces isolated with locations of positions used for comparison.

In the simpler problem the entire superior surface is heated and the other surfaces are isolated as shown in Fig.3. In this case, consolidated results are available in literature (Walker and Beck, 2008). The cube of sides $1cm$ of AISI304 stainless steel ($k = 14.9[W/mK]$ and $\alpha = 3.95 \times 10^{-6}[m^2/s]$) is subjected to a constant heat flux $q = 10^5[W/m^2]$ for $t > 0$.

Initially the cube is at uniform a temperature de 30^oC. Figure 3 shows the locations of positions used for comparison. Table 1 shows the temperature values obtained by the proposed analytical solution (new) and those obtained by Walker and Beck (2008) the different locations proving the accuracy of the proposed analytical solution.

Table 1. Comparison calculated using analytical solutions

Time[s]	$T_1[^oC]$ This work	$T_1[^oC]$ Ref.*	$T_2[^oC]$ This work	$T_2[^oC]$ Ref.*	$T_3[^oC]$ This work	$T_3[^oC]$ Ref.*	$T_4[^oC]$ This work	$T_4[^oC]$ Ref.*	$T_5[^oC]$ This work	$T_5[^oC]$ Ref.*
0	30,00000	30,00000	30,00000	30,00000	30,00000	30,00000	30,00000	30,00000	30,00000	30,00000
20	105,38577	105,38591	105,38577	105,38591	80,22371	80,22371	71,84004	71,84004	80,22371	80,22371
40	158,41149	158,41163	158,41149	158,41163	133,24385	133,24385	124,85459	124,85459	133,24385	133,24385
60	211,43163	211,43177	211,43163	211,43177	186,26398	186,26398	177,87472	177,87472	186,26398	186,26398
80	264,45177	264,45190	264,45177	264,45190	239,28412	239,28412	230,89485	230,89485	239,28412	239,28412
100	317,47190	317,47204	317,47190	317,47204	292,30425	292,30425	283,91499	283,91499	292,30425	292,30425

(*) Walker and Beck (2008)

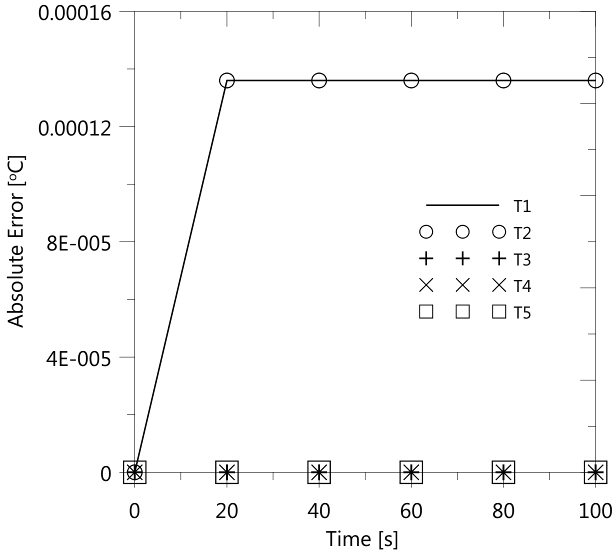


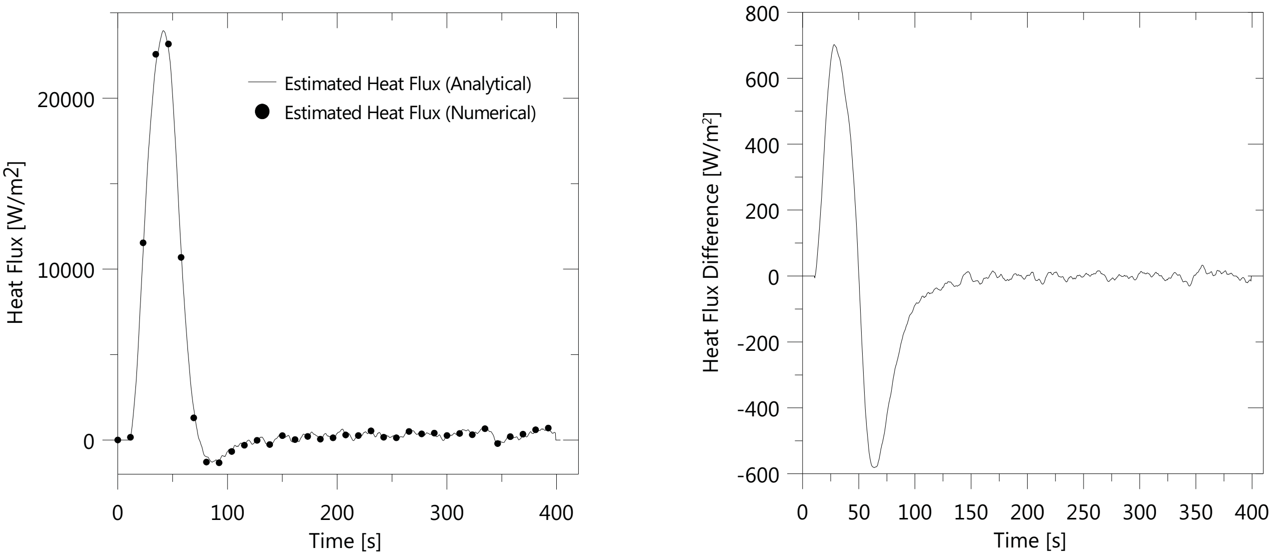
Figure 4. Error between the analytical solution of this work and those obtained by Walker and Beck (2008) for different locations

4.2 DPT Code Using Analytical Solution

In the following will be presented results for the use of analytical solution in the thermal diffusivity and conductivity estimation by using DTP code as describe in section 2. The comparison with the numerical solution is made using the three dimensional transient thermal model shown in Fig.1, In this case, only part of the superior surface is heated. The cast iron block has dimensions of $L = 0,0805m$, $W = 0,008m$ and $Z = 0,06m$ and the heated region, S_1, is defined by $0 < x < 0,005m$ and $0 < z < 0,005m$.

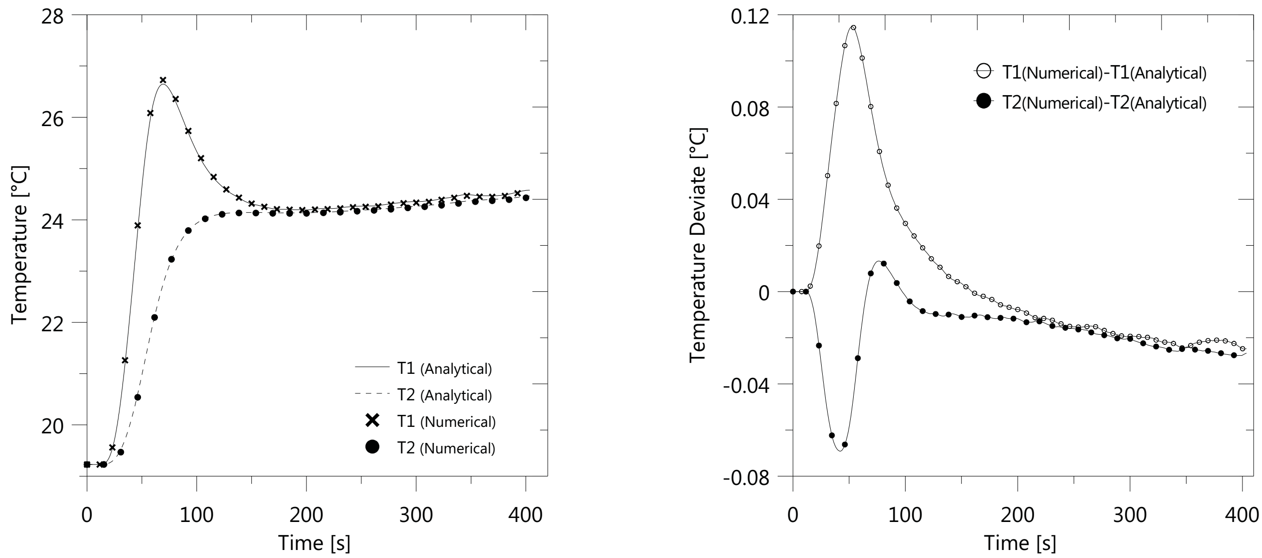
As mentioned two thermocouples are used to measure the temperatures. The thermocouples T_1 and T_2 are located at $(0,0378; 0,008; 0,003)$ and $(0,0345; 0,008; 0,0438)$ respectively. The results obtained numerically and analytically are compared in Figures 5 and 6.

It should be mentioned that with the heat flux estimated in the last step of the algorithm of Beck *et al.* (2006), we have a direct problem whose results are temperatures. In this sense, Figure 5(a) shows the heat fluxes estimated using the analytical and numerical solutions. The good agreement between them is clear. The maximum deviation between the estimated heat fluxes is 3% (Fig.5(b)). Figure 6 shows the temperature evolution at locations T_1 and T_2 using the respective estimated heat flux shown in Fig.5(a).



(a) Estimated heat flux (b) Residues

Figure 5. Heat flux for a cast iron sample



(a) Temperatures (b) Residues

Figure 6. Estimated temperatures (T_1 and T_2) for a cast iron sample

The maximum deviation observed of $0,12^oC$ (0,4%) includes not only numerical dispersion but also the influence of each heat flux on the solution (Fig.6(b)). In this case, the main contribution is in the computational time as shown in Tab.2. The difference between the two is of order of 7500%. The analytical solution took 5 min and the numerical solution took 6,7 hours. This time corresponds to all procedures to obtain the thermal properties, including the optimization by iteration. The deviation in temperatures and in heat fluxes were responsible for deviation in estimated values of thermal conductivity and diffusivity of 1,7% and 2,6% respectively.

Table 2. Comparison between estimated values of thermal diffusivity and conductivity and the computational time using DPT with numerical and analytical solutions

Solution type	Thermal diffusivity $[m^2/s]$	Thermal conductivity $[W/mK]$	Computational Time $[s]$
Analytical	$1.13 \times 10^{-5} \pm 0.023 \times 10^{-5}$	42.59 ± 0.51	305.00s (0.085h)
Numerical	$1.10 \times 10^{-5} \pm 0.017 \times 10^{-5}$	43.32 ± 0.39	23040.00s (6.4h)

5. CONCLUSIONS

The complexity of a thermal model in relation to analytical solutions is in multidimensional transient problems subjected to non-homogeneities such as boundary conditions of prescribed heat flux or boundary conditions such as time-variable temperature. The use of Green's function adapts efficiently to this type of problems. This study shows the great contributions of the use of analytical solutions in inverse problems, once the optimization algorithms usually have to calculate the direct problem several times. The use of analytical solutions not only increases the precision but also reduce "drastically" the computational time.

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