ANELASTIC AND OBERBECK-BOUSSINESQ APPROXIMATIONS WITH THERMAL VISCOUS DISSIPATION IN A ROTATING FRAME

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Abstract. *This paper concerns the justification of two important mathematical models to the natural convection problem in a rotating frame (dry atmosphere). The employed procedure has as main ingredients: i) the basic laws: the mass, momentum and energy balance equations, and the entropy inequality in the form of Clausius-Duhem inequality; ii) constitutive theory: use of mechanical incompressibility constraint, whose changes in the volume of a viscous fluid are only due to the thermal effects; the constitutive equations for a elastic fluid with heat conduction and viscosity restricted by the Clausius-Duhem inequality. The approximated equations are obtained after the dimensionless method, assuming the respective dimensionless numbers are of order one; with the use of a power series expansion, whose the parameter associated was imposed as a small compressibility measure. These equations are equivalent to an ordained anelastic model for a dry atmosphere, in which the density variations are only caused by the thermal effects (buoyant term); including the thermal viscous dissipation and the pressure stress work in the respective approximated energy equation. The Oberbeck-Boussinesq model was obtained, assuming just a constant density reference state.*

Keywords: *Natural convection; atmosphere; modern continuum mechanics*

1. INTRODUCTION

The natural convection phenomenon may occurs in a viscous or inviscid fluid flow with heat conduction subject to the gravitational field, where density variations are observed. This fluid is heated from below, and it will be ascendent; when this (thermal) ascension associated to the gravitational field is major than the heat conduction and viscosity effects, the convection it happens. Particularly, the action of this phenomenon in rotating frames is frequently observed in the Earth's atmosphere and oceans, whose the heating is promoted by the solar radiation.

We remark there isn't an exact solution for this type of problem, thus it is possible use mathematical models that produce simplifications directly of the governing equations: the Oberbeck-Boussinesq (Boussinesq 1903, Spiegel and Veronis 1960) and anelastic models (Batchelor 1953, Ogura and Phillips 1962). In these models, constraints are imposed to the density field: a height (constant) function in the anelastic (Oberbeck-Boussinesq model), except when it is associated to the gravitational field - whose the density variations are assumed only due to the thermal effects. The others properties of the fluid are imposed constants; the mechanical viscous dissipation is usually neglected in the approximated energy balance equation. Consequently, we can simplify the natural convection problem in a deep (shallow) fluid layer using the Anelastic (Oberbeck-Boussinesq model), where the treatment of the acoustic oscillations is not possible in those models.

However, we note that some important effects are commonly ignored in the class problem of natural convection, e.g. the thermal viscous dissipation in the respective energy balance equation (see Gebhart 1962, Gray and Giorgini 1976). In atmospheric convection applications, this effect is frequently neglected too, but the inclusion of this term was proposed by Bister and Emanuel (1998) and Zhang and Altshuler (1999) (Businger and Businger, 2001) to modeling a tropical thunderstorm (a isolated thunderstorm), where in both cases this inclusion leaded to intensification of these systems. On the other hand, the thermodynamical approximated equations referent to this class were investigated, through the mutual inclusion of the thermal viscous dissipation and the pressure stress work in the approximated: energy balance equation, e.g. Kagei *et al.* 2000, Pons and Lé Quéré 2004, 2005a,b, Costa 2005; entropy inequality, e.g. Pons and Lé Quéré 2005a,b, Costa 2005.

Here, we justify the anelastic and Oberbeck-Boussinesq approximations to the natural convection problem in a rotating frame for a dry atmosphere, considering the thermal viscous dissipation and the pressure stress work in the approximated energy balance equation. This procedure is based on basic laws, that includes the Clausius-Duhem inequality; a constitutive theory, utilizing the constraint where the volume changes are only due to the thermal effects, with the constitutive response are restricted by the Clausius-Duhem inequality.

This work is organized as: in the section 2 we briefly describe the anelastic and Oberbeck-Boussinesq approximations, discussing their respective simplifications; in the section 3 we formulate the corresponding governing equations. In the section 4 we obtain the respective approximated equations; and finally in the section 5 we show the conclusions.

2. Convective models

2.1 Oberbeck-Boussinesq approximation

A mathematical model commonly used to simplify the natural convection problem is the Oberbeck-Boussinesq (OB) approximation, whose the pioneering contributions are due to Oberbeck (1879, 1888) and Boussinesq (1903). Nevertheless, to obtain the corresponding simplified equations we have many difficulties, then several authors investigated and justified this model (Spiegel and Veronis 1960, Chandrasekhar 1961, Mihaljan 1962, Gray and Giorgini 1976, Rajagopal *et al.* 1996, Kagei *et al.* 2000, Passerini and Thäter 2005, Ramos and Vargas 2005, Pons and Lé Quéré 2005a,b).

Boussinesq (1903) investigated the natural convection problem in a compressible atmospheric layer with Fourier's heat conduction. He noted a reduction on the density field due to the heating from below, generating a thermal expansion associated to the gravitational field, and consequently, these variations are assumed only by thermal effects, in the buoyancy term, in the momentum balance equation. This assumption was justified with scale argues, with a maximum temperature difference in order to 10K. In this case, the fluid properties imposed constant, and the density variations (in the mass and energy balance equations) as a constant value; and the thermal viscous dissipation in the respective energy balance equations ignored too. Then, the final approximated equations were recovered just substituting these assumptions in the respective governing equations.

Spiegel and Veronis (1960) reaches the OB system for a viscous compressible atmospheric (ideal gas) flow, in a thin layer by scale analysis method. Mihaljan (1962) utilized a power series expansion in terms of two small parameters, following the Buckingham-Pi theorem. Their final equations are obtained in the zero order of this expansion, but the author discusses the influence of the velocity divergence and the heat conduction terms on elevated order expansion, in the context of a thermodynamic analysis.

Gray and Giorgini (1976) imposed the properties of a viscous fluid (gas or liquid) was a function of the temperature and pressure; and these respective dimensionless parameters as a small parameter - posteriorly neglected - arriving a simplified equations system, that includes the thermal viscous dissipation and the pressure stress work in their energy balance equation, called Boussinesq extended equations. With two additional hypotheses, they recovered the usual OB system, for which those terms are ignored. Recently, Ramos and Vargas (2005) extended the procedure of Gray and Giorgini (1976) to promote a justification for a rotating frame. In this case, the corresponding extended system includes the centrifugal force on the buoyancy force.

Only in 1996, Rajagopal *et al.* (1996) produced a justification of the OB system, in the context of the continuum mechanics: they introduce the mechanical incompressibility constraint, imposing that a viscous fluid only changes his volume due to thermal effects; and using a power series expansion, where the small parameter associated was choose as a ratio of characteristic measure. Kagei *et al.* (2000) extended the procedure of Rajagopal *et al.* (1996), but using a power series expansion in terms of a small thermal compressibility, and obtaining the corresponding OB system that admittes the inclusion of the thermal viscous dissipation and the pressure stress work in the respective energy balance equation. In the same fashion, Passerini and Thäter (2005) reached a similar system of Kagei *et al.* (2000) for a non-Newtonian fluid.

Briefly, the approximated system can be reached, after the following considerations, directly of the governing equations:

1. The fluid properties are assumed as constant values;
2. The density is a exception, where its variations are due to thermal effects - only in the buoyancy term in the momentum balance equation;
3. The viscous dissipation may be ignored in the energy balance equation.

A direct consequence is that the acoustic oscillations cannot be treated by this system.

Then, adopting a compressible fluid flow with heat conduction and viscosity (see section 3.1), considering a Fourier conduction, and assuming the before affirmations, we obtain the OB system called as the usual OB approximation, given by

$$
\begin{aligned}
div\mathbf{v} &= 0, \\
\dot{\mathbf{v}} &= -\frac{1}{\rho_0} gradp + [1 - \alpha_0(\theta - \theta_0)]\mathbf{g} + \frac{\mu_0}{\rho_0} div\mathbf{D}, \\
\dot{\theta} &= \kappa_0 div\,(grad\theta)\,,
\end{aligned} \tag{1}
$$

where $\dot{(\ )} := \frac{\partial(\ )}{\partial t} + grad(\ ).\mathbf{v}$ denotes the corresponding material derivative; $\mathbf{v}$ the velocity; p the pressure; θ the absolute temperature; $\mathbf{g}$ the gravitational acceleration vector; $\mathbf{D} := \frac{1}{2}(grad\mathbf{v} + grad\mathbf{v}^T)$ the symmetric velocity gradient; ρ_0 and θ_0 the respective reference values (constants) of density and temperature; and α_0, μ_0 and κ_0 the constant coefficient of thermal expansion, dynamical viscosity and thermal diffusivity.

2.2 Anelastic approximation

In the manifestation of natural convection phenomenon in extended vertical (deep) layers, the use of OB system is unsatisfactory, because this assumes that the density variations are only in the buoyancy term by thermal effects. In these cases, a alternative model can be used: the anelastic approximation, considering the density field decomposed in a average state (a height function), added to deviations due to the thermal contributions associated to the buoyancy force (see Batchelor 1953, Ogura and Phillips 1962, Dutton and Fichtl 1969, Durran 1989, Bannon 1995, 2002).

Originally, Batchelor (1953) proposed a simplified set for a atmospheric flow with variations in the density field, where the gravitational force is acting (after, this set was referred as anelastic equations); recovering the OB system with an additional hypotheses. Then, Ogura and Phillips (1962) promoted a derivation of Batchelor's anelastic (OB) systems, in which in the place of the density field, they used the potential temperature - a function of temperature and pressure, used only for a ideal gas - but we note that this variable is very useful in atmospheric sciences. They suggest the use of anelastic (OB) model to a deep (shallow) fluid layer.

After these pioneering works, many investigators discussed the anelastic model offering many formulations, where the scale analysis method is commonly employed. Nevertheless, in general, the mass balance anelastic equation has a similar form, where the exception is the pseudo-incompressible equation proposed by Durran (1989). Therefore, some authors investigate different forms to the momentum and energy balance equations in the case of a shallow or deep fluid layer, e.g. Dutton and Fichtl (1969). Finally, in the occurrence of this phenomenon in a multicomponent or/and multiphases fluid flow, we must incorporate these effects in the respective buoyancy term (see Dutton and Fichtl 1969, Bannon 1996, Bannon 2002).

Dutton and Fichtl (1969) proposed the use of approximated equations to a viscous compressible fluid flow for the shallow/deep atmospheric convection problem. In the shallow convection, the pressure variations appears only in the pressure gradient force, otherwise, it is necessary to consider these variations also in the energy balance equation, and in the state equation. Therefore, the mass flux (velocity field) is solenoidal in the mass balance for the deep (shallow) convection. Lipps and Hemler (1982) justified the use of anelastic system to the troposphere, where they assumed the thermodynamic variables decomposed in a average state added to a deviation, utilizing a scale analysis method. Durran (1989) proposed a alternative form to the anelastic system - via scale analysis - called pseudo-incompressible system, considering in his mass balance equation, the deviations in the density field due to the thermal effects and the heating rate.

Lipps (1990), Nance and Durran (1994) and Bannon (1995) compared different forms of the anelastic equations. Particularly, Bannon (1995) was the first to investigate the performance of many types of anelastic equations, and their ability on describe the natural convection in a atmospheric flow with large horizontal extensions, where is recognized that the natural atmospheric convection is relevant in the small until large atmospheric scales.

Bannon (1996) presented a hybrid formulation between the Dutton and Fichtl (1969) and Lipps and Hemler (1982), and he developed a anelastic system to the moist air, according to Bannon (1995). Finally, Bannon (2002) promoted a study about natural atmospheric convection in a cloud environment, in which he considered the all hydrometeors (water vapor, and liquid and ice particles), preserving the Bannon (1995) requirements.

We note there are many forms of anelastic equations, but in general, the approximated mass balance equation has the same form. Then, the anelastic system, denoted as the usual system, can be obtained, following the next assumptions:

1. The fluid properties are considered as constants;
2. The density field, however is admitted as a average function in the horizontal direction, i.e. it is variable just in the vertical spacial coordinate, excepting when this is related to the gravitational field - in the momentum balance equation;
3. The density variations on the gravitational term are only due to thermal effects.

Then, if we adopt a compressible fluid flow with (Fourier) heat conduction and viscosity, and assuming the before affirmations, we have:

$$
\begin{aligned}
div\left[\rho(\mathbf{z})\mathbf{v}\right] &= 0,\\
\dot{\mathbf{v}} &= -\frac{1}{\rho(\mathbf{z})}gradp + \left[1-\alpha_0(\theta-\theta(\mathbf{z}))\right]\mathbf{g} + \frac{\mu_0}{\rho(\mathbf{z})}\left[div(2\mathbf{D}) - \frac{1}{3}grad(div\mathbf{v})\right],\\
C_0\dot{\theta} &= K_0 div\left(grad\theta\right) + \left[\theta\alpha_0^2 p\dot{\theta} + \dot{p}\alpha_0\theta + 2\mu_0\left(\|\mathbf{D}\|^2 - \frac{1}{3}|tr\mathbf{D}|^2\right)\right],
\end{aligned}
\tag{2}
$$

where $\rho(\mathbf{z})$ represents a density height function, C is the specific heat, and the zero indices represent constant values related to reference state adopted. Additionally, we note that the last term in the right side represents the effects of the pressure stress work and the thermal viscous dissipation, respectively, where we its maintained and will be discuss below.

2.3 Dissipative effects

The thermal viscous dissipation turn it appreciable in a natural convection problem, in situations where the kinetic energy is appreciable when it is compared to the total transferred energy; this occurs for elevated body forces or in a extended convective regions (Gebhart, 1962).

In atmospheric applications related to natural convection, this effect is frequently neglected, specially in large atmospheric systems. In this sense, Bister and Emanuel (1998) and Zhang and Whang (1999) investigated the relevance of this effect, includes it in the treatment of hurricanes; and Businger and Businger (2001) for isolated thunderstorms. In these works, were noted that there was a intensification of these systems, depending on the structure, and the stage of development of the respective system.

Therefore, the thermodynamic approximated equations used to the natural convection problem were investigated, for instance, Kagei *et al.* (2000), Pons and Lé Quéré (2004, 2005a,b) and Costa (2005), discussing the simultaneous inclusion of thermal viscous dissipation and the pressure stress work in these equations. Kagei *et al.* (2000) proposed the inclusion of these terms in the approximated energy balance equation for the natural convection problem, in the context of modern continuum mechanics. Pons and Lé Quéré (2004, 2005a,b) include these in their approximated energy balance equation and in the entropy inequality. Finally, Costa (2005) also incorporated this simultaneous terms in their energy balance equation and entropy inequality. In the main, they are in concordance taht it is necessary to include these two terms together, showing that if only the thermal viscous dissipation (a positive quantity) is added, the heat will be multiplied.

3. Formulation

3.1 Basic laws

The basic laws related to a compressible fluid flow with heat conduction and viscosity are expressed here, through the mass, momentum, energy balance equations, and the entropy inequality, in the Clausius-Duhem version, in their local representations, respectively, given by:

$$\begin{aligned} \dot{\rho} + \rho div\mathbf{v} &= 0, \\ \rho\dot{\mathbf{v}} &= div\mathbf{T} + \rho\mathbf{b}, \\ \rho\dot{e} &= \mathbf{T}.\mathbf{L} - div\mathbf{q} + \rho r, \\ \rho\dot{\eta} - div\left(\frac{\mathbf{q}}{\theta}\right) + \frac{\rho r}{\theta} &\geq 0, \end{aligned} \tag{3}$$

where ρ, e and η represent the specific mass, internal energy and entropy, $\mathbf{T}$ the Cauchy (symmetric) stress tensor ($\mathbf{T} := \mathbf{T}^T$), $\mathbf{b}$ the specific body force, $\mathbf{q}$ (r) the heat flux (supply), $\mathbf{L}$ $(grad\theta)$ the velocity (temperature) gradient.

Here, we are adopting a rotating frame, then it is convenient to assume the body force including the rotate forces, i.e.,

$$\mathbf{b} = \mathbf{g} - 2\omega \times \mathbf{v} - \omega \times (\omega \times \mathbf{r}) \tag{4}$$

where $\mathbf{g}$ is the gravitational acceleration vector, ω the (uniform) angular velocity, and $\mathbf{r}$ is the position vector (with the origin in the rotate axes - Earth's axes), and the two terms are the specific Coriolis and centrifugal forces, respectively.

3.2 Constitutive theory

Here, the constitutive theory is based on the following constitutive hypotheses:

3.2.1 Mechanical incompressibility constraint

Our first constitutive hypotheses is the use of the mechanical incompressibility constraint, imposing that only isochoric motions are possible in isothermal process, but in non-isochoric process, the motions are not necessarily isochoric, i.e., the changes in the fluid volume are only due to the thermal contributions (see Rajagopal *et al.* 1996, Kagei *et al.* 2000), express as:

$$div\mathbf{v} = \alpha(\theta)\dot{\theta}, \tag{5}$$

where α is the thermal expansion coefficient.

3.2.2 Clausius-Duhem inequality

The Clausius-Duhem inequality added to energy balance equation offers restrictions to the material feature, in the thermodynamic context. Then, it is admitted that the respective constitutive response must satisfy these principle too (see Coleman and Noll, 1963). With the use of the eq. $(3)_3$, the inequality $(3)_4$ turn it

$$-\rho\left(\dot{\varphi} + \eta\dot{\theta}\right) + \mathbf{T}.\mathbf{L} - \frac{\mathbf{q}.grad\theta}{\theta} \geq 0 \tag{6}$$

where $(\varphi := e - \theta\eta)$ is the specific Helmholtz's free energy.

The inequality (6) can be rewrite as:

$$-\rho\dot{\varphi} - \rho\eta\dot{\theta} - p div\mathbf{v} + \mathbf{T}_d.\mathbf{D}_d - \frac{\mathbf{q}.grad\theta}{\theta} \geq 0, \tag{7}$$

where was utilized the tensor decomposition $\mathbf{T}(\mathbf{D})$ in their normal components $p := -\frac{1}{3}tr\mathbf{T}\left(\frac{1}{3}tr\mathbf{D}\right)$, and their respective deviations portions $\mathbf{T}_d := \mathbf{T} - \frac{1}{3}tr\mathbf{T}\left(\mathbf{D}_d := \mathbf{D} - \frac{1}{3}tr\mathbf{D}\right)$, with $(tr\mathbf{T}_d = tr\mathbf{D}_d := 0)$ and $(\mathbf{I}\cdot\mathbf{D} = tr\mathbf{D} = div\mathbf{v})$.

Finally, manipulating $(3)_1$ and (7), we obtain:

$$-\rho\dot{\varphi} - \pi\dot{\theta} + \mathbf{T}_d.\mathbf{D}_d + \frac{\mathbf{q}.grad\theta}{\theta} \geq 0, \tag{8}$$

where $\pi := \rho\left(\eta + \frac{\alpha p}{\rho}\right)$.

Then, the inequality (8) suggests that will be provided constitutive equations to φ, π, $\mathbf{q}$ and $\mathbf{T}_d$.

3.2.3 Constitutive responses

Combining the inequality (8), it is suggested that we have to formulate constitutive equations to for φ, π, $\mathbf{q}$ and $\mathbf{T}_d$, then adopting the class of elastic fluids with heat conduction and viscosity (see Coleman and Noll 1963, Šilhavý 1997), we consider the corresponding constitutive equations are given in a specific point and time when

$$\sigma = (\theta, grad\theta, \mathbf{D}_d) \tag{9}$$

is know in this point and time, i.e., $\hat{\varphi}(\sigma), \hat{\pi}(\sigma), \hat{\mathbf{T}}_d(\sigma), \hat{\mathbf{q}}(\sigma)$, admitting that these functions are smooth. We note also, that (9) defines a incompressible fluid with heat conduction and viscosity, and in this case $\mathbf{D}_d = \mathbf{D}$ and $div\mathbf{v} = tr\mathbf{D} := 0$.

3.2.4 Thermodynamic Compatibility

Thus, considering the eq. (9) in the inequality (8), we obtain the following functional inequality:

$$-\rho\dot{\hat{\varphi}}(\sigma) - \hat{\pi}(\sigma)\dot{\theta} + \hat{\mathbf{T}}_d(\sigma).\mathbf{D}_d - \frac{\hat{\mathbf{q}}(\sigma).grad\theta}{\theta} \geq 0, \tag{10}$$

Then, following the Coleman-Noll procedure (see Coleman and Noll, 1963) we can postulate that all the constitutive, where in a some point , θ, $\dot{\theta}$, $grad\theta$, $\mathbf{D}_d$, $\dot{grad\theta}$, $\dot{\mathbf{D}}_d$ may be specified arbitrarily, conform the below declarations:

i) the response function $\hat{\varphi}$ ia a only temperature function, i.e., $\hat{\varphi} = \varphi(\theta)$;

ii) the response functions $\hat{\pi}$ and $\hat{\varphi}$ are related to $\hat{\pi} = -\rho\frac{\partial\hat{\varphi}}{\partial\theta}$;

iii) the response functions $\hat{\mathbf{T}}_d$ e $\hat{\mathbf{q}}$ must satisfy the reduced inequality: $\hat{\mathbf{T}}_d.\mathbf{D}_d - \frac{\hat{\mathbf{q}}.grad\theta}{\theta} \geq 0$, with $\hat{\mathbf{T}}_d(\theta, 0, 0) = 0; \hat{\mathbf{q}}(\theta, 0, 0) = 0$.

In this moment, we impose that the response functions for $\mathbf{q}$ and $\mathbf{T}_d$ are linear in $grad\theta$ and $\mathbf{D}_d$, respectively, like the incompressible Navier-Stokes fluid, i.e.,

$$\mathbf{q} = K(\theta)grad\theta,$$
$$\mathbf{T}_d = 2\mu(\theta)\mathbf{D}_d, \tag{11}$$

where K and μ are the respective thermal conductivity and dynamical viscosity coefficients, with non-negative values for to guarantee the second law requirements.

3.3 Governing equations

So, the governing equations are obtained, substituting the constitutive responses into the respective basic laws. First, the Cauchy stress tensor may be rewrite as

$$\mathbf{T} = -p\mathbf{I} + 2\mu(\theta)\mathbf{D} - \frac{2}{3}\mu(tr\mathbf{D})\mathbf{I}, \tag{12}$$

after the use of the definitions of p and $\mathbf{D}_d$, and the constitutive equation $(11)_2$. Then, substituting the equations (4), the Coleman-Noll consequences, eq. $(11)_1$ and eq. (12) into the equations (3), we obtain the following governing equations:

$$\dot{\rho} + \rho div\mathbf{v} = 0,$$

$$div\mathbf{v} = \alpha\dot{\theta},$$
$$\rho\dot{\mathbf{v}} = -gradp + 2\mu\left[div\mathbf{D} - \frac{1}{3}grad(div\mathbf{v})\right] + \rho\left[\mathbf{g} - 2\omega\times\mathbf{v} - \omega\times(\omega\times\mathbf{r})\right],$$
$$\left[C(\theta) - \alpha^2\theta p\right]\dot{\theta} - \alpha\theta\dot{p} = K\triangle\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}|tr\mathbf{D}|^2\right], \tag{13}$$

where $\triangle := div(grad)$ represents the Laplacian operator, $C(\theta) := -\rho\theta\partial_{\theta\theta}\varphi$ the specific heat coefficient, and as the others coefficients α, β, μ and K it is a constant value, assuming $r := 0$ for simplicity.

4. Approximated equations

4.1 Convective problem

At this moment, we consider a layer with depth L (infinitive horizontally) of a fluid with heat conduction and viscosity, heated frow below, where is acting a gravitational field, and the rotate forces - Coriolis and centrifugal. In this situation, there is a constant temperature difference (Θ) between the bottom (θ_b) and top layer (θ_t) $(\Theta := \theta_b - \theta_t)$, $(\Theta > 0)$, and the convection phenomenon will be manifested if this difference be greater than the heat conduction and viscosity effects, simultaneously.

4.2 Dimensionless method

Let U a representative value to the velocity, L to the length and t_r to the time; $g_r := |\mathbf{g}|$ to the gravitational acceleration vector and Ω to the angular velocity characteristic; θ_r, p_r, ρ_r and C_r the respective reference values to temperature, pressure, density and specific heat.

The following dimensionless quantities are introduced now:

$$\overline{\mathbf{x}} = \frac{\mathbf{x}}{L}, \quad \overline{\mathbf{r}} = \frac{\mathbf{r}}{L}, \quad \overline{t} = \frac{t}{t_r}, \quad \overline{\mathbf{v}} = \frac{\mathbf{v}}{U}, \quad \overline{\omega} = \frac{\omega}{\Omega},$$
$$\overline{\theta} = \frac{\theta - \theta_r}{\Theta}, \quad \overline{\rho} = \frac{\rho}{\rho_r}, \quad \overline{p} = \frac{p}{p_r}, \quad \overline{\mathbf{g}} = \frac{\mathbf{g}}{|\mathbf{g}|} \quad \overline{C} = \frac{C}{C_r}. \tag{14}$$

- *Characteristic scales*

  The choice of the respective characteristic scales is a very important step, and it depends on the particularities of the kind of problem. Then, the natural convection problem will be treated here, admitting the following reference scales.

  First, the reference scale to the velocity was choose as the buoyancy term has the same order to the inertial term, in the momentum balance equation, according to Chandrasekhar (1961), i.e.,$U := (\alpha g_r L\Theta)^{\frac{1}{2}}$. The characteristic temperature is choose as the average temperature in the layer, in agreement with Gray and Giorgini (1976), Rajagopal *et al.* (1996) given by $\theta_r = \frac{\theta_b+\theta_t}{2} := \theta_m$. The reference scale to pressure adopts the hydrostatic case, where there is a balance between the pressure and gravitational forces, represented by $p_r := \rho_r g_r L$; the reference density is assumed as know in a specific height, express as $\rho_r := \rho(z_0)$. We note that any considerations was imposed, a priori, to the choice of time and angular velocity characteristic.

Now, inserting eq. $(13)_2$ into $(13)_1$, the final equation can be integrated, turn it in

$$\rho(\mathbf{x}, t) = \rho^0(\mathbf{x})\exp[-\alpha(\theta(\mathbf{x}, t) - \theta^0(\mathbf{x}))], \tag{15}$$

where ρ^0 and θ^0 are reference values, assuming that α is a constant, and a uniform state in the horizontal directions (x and y components of the position vector) is assumed, that is, the density only varies with the vertical direction (height), with $\rho^0(\mathbf{x}) := \rho^0(\mathbf{z})$. Then, the dimensionless density change into:

$$\overline{\rho} = \frac{\rho^0(\mathbf{z})\exp[-\alpha(\theta - \theta^0(\mathbf{z}))]}{\rho(\mathbf{z}_0)} \sim \tilde{\rho}(\mathbf{z})[1 - \alpha\Theta\overline{\theta}], \tag{16}$$

where is admitted $\theta^0(\mathbf{z}) := \theta_m$, $\tilde{\rho}(\mathbf{z}) := \frac{\rho^0(\mathbf{z})}{\rho(\mathbf{z}_0)}$.

The governing equations in their dimensionless form are obtained, after the utilization of reference values in eq. (13) given by:

$$-\alpha\Theta\left[\tilde{\rho}(\mathbf{z})S\frac{\partial\overline{\theta}}{\partial\overline{t}} + \overline{div}(\tilde{\rho}(\mathbf{z})\overline{\theta}\overline{\mathbf{v}})\right] + \overline{div}(\tilde{\rho}(\mathbf{z})\overline{\mathbf{v}}) = 0. \tag{17}$$

$$(\alpha\Theta)\left[\tilde{\rho}(\mathbf{z})(1-\alpha\Theta\overline{\theta})\right]\left[S\frac{\partial\overline{\mathbf{v}}}{\partial\overline{t}}+(\overline{grad}\overline{\mathbf{v}})\overline{\mathbf{v}}\right]=-\overline{grad}\overline{p}+\tilde{\rho}(\mathbf{z})(1-\alpha\Theta\overline{\theta})\overline{\mathbf{g}}$$
$$+\alpha\Theta\frac{2}{Re}\left[\overline{div}\overline{\mathbf{D}}-\frac{1}{3}\overline{grad}(\overline{div}\overline{\mathbf{v}})\right]-\frac{\alpha\Theta\left[\tilde{\rho}(\mathbf{z})(1-\alpha\Theta\overline{\theta})\right]}{Ro}\left[2\overline{\omega}\times\overline{\mathbf{v}}+\frac{1}{Ro}\overline{\omega}\times(\overline{\omega}\times\overline{\mathbf{r}})\right], \tag{18}$$

$$\left[\overline{C}(\overline{\theta})-(\alpha\Theta)Di\left(\overline{\theta}+\frac{\theta_m}{\Theta}\right)\overline{p}\right]\left(S\partial_{\overline{t}}\overline{\theta}+\overline{grad}\overline{\theta}.\overline{\mathbf{v}}\right)-Di\left(\overline{\theta}+\frac{\theta_m}{\Theta}\right)\left(S\partial_{\overline{t}}\overline{p}+\overline{grad}\overline{p}.\overline{\mathbf{v}}\right)$$
$$=\frac{1}{RePr}\overline{\triangle}\overline{\theta}+2\frac{Di}{Re}\left[\|\overline{\mathbf{D}}\|^2-\frac{1}{3}|tr\overline{\mathbf{D}}|^2\right], \tag{19}$$

where are adopted the posterior dimensionless numbers: Reynolds (Re), Rossby (Ro), Dissipation (Di), Prandtl (Pr) and a additional number (S) defines as:

$$Re:=\frac{LU}{\nu},\quad Ro:=\frac{U}{\Omega L},\quad Di:=\frac{\alpha\rho_r|\mathbf{g}|L}{C_r},\quad Pr:=\frac{C_R\nu}{K},\quad S:=\frac{L}{Ut_R}. \tag{20}$$

The above numbers are related to, respectively, the ratio of the inertial and viscous forces; ratio of the inertial and Coriolis forces; the ratio of the product of thermal expansion coefficient, body force acceleration and length, and the specific heat; the ratio of the mechanical and thermal diffusivity, and finally the ratio of the advection and time terms in the material derivative. It is observed that the product $RePr$ represents the ratio of a energy transfer by conduction and convection.

4.3 Power series expansions

The starting point of our justification is the dimensionless equations, whose the respective variables dependents will be expanded in power series in terms of a small parameter, collecting the first orders of expanded equations.

First, the small parameter choose is a thermal compressibility measure, according to Kagei *et al.* (2000), given by:

$$\varepsilon:=\sqrt[3]{\alpha\Theta}\ll 1. \tag{21}$$

We assume also, that all the effects involved are retained, that is, the viscous, inertial, rotating and dissipative effects are preserved, so we adopt the respective dimensionless numbers have a unit order, related to the small parameter ε, i.e.,

$$\mathcal{O}(Re)=\mathcal{O}(\varepsilon^0);\quad \mathcal{O}(Ro)=\mathcal{O}(\varepsilon^0);\quad \mathcal{O}(Di)=\mathcal{O}(\varepsilon^0);\quad \mathcal{O}(S)=\mathcal{O}(\varepsilon^0). \tag{22}$$

Then, using eq. (20) and eq. (22), we obtain the following restrictions:

$$\frac{g_R L^3}{\nu^2}\sim\frac{1}{\varepsilon^3},\quad \omega_R\sim\frac{U}{L},\quad \frac{\rho_R g_R L}{C_R}\sim\frac{1}{\varepsilon^3},\quad t_R\sim\frac{L}{U}. \tag{23}$$

We recognize the restrictions are measurable in the specific fluid flow, additionally the time and angular velocity scale have a inverse relation, and finally the the time derivative has the same order to the advection term in the material derivative.

The quantities $\mathbf{v}$, θ and p are expanded in a power series in respect to the small perturbation parameter as

$$\mathbf{v}=\sum_{n=0}^{\infty}\varepsilon^n\mathbf{v}_n,\quad \theta=\sum_{n=0}^{\infty}\varepsilon^n\theta_n,\quad p=\sum_{n=0}^{\infty}\varepsilon^n p_n. \tag{24}$$

The boundary conditions are annexed in the respective term of zero order, and consequently, the term of high orders are assumed with null boundary conditions. Therefore, all the quantities are imposed sufficient smooth.

Now, substituting these expanded variables, with the use of eq. (21), eq. (22), eq. (24), into the dimensionless equations (17)-(19), we analysis the first orders of ε powers, or ε levels.

Collecting the zero order of expended equations, that is, $O(\varepsilon^0)$, we obtain:

$$\overline{div}(\tilde{\rho}(\mathbf{z})\overline{\mathbf{v}}_0)=0,$$
$$0=-\overline{grad}\overline{p}_0+\tilde{\rho}(\mathbf{z})\overline{\mathbf{g}},$$
$$\tilde{C}(\overline{\theta}_0)\dot{\overline{\theta}}_0-Di\left(\overline{\theta}_0+\frac{\Theta_M}{\Theta}\right)\dot{\overline{p}}_0=\frac{1}{PrRe}\triangle\overline{\theta}_0+2\frac{Di}{Re}\left[\|\overline{\mathbf{D}}(\overline{\mathbf{v}})_0\|^2-\frac{1}{3}|tr\overline{\mathbf{D}}(\overline{\mathbf{v}}_0)|^2\right], \tag{25}$$

where for the specific heat we consider

$$\tilde{C}(\overline{\theta})\sim\tilde{C}(\overline{\theta}_0)+\varepsilon\tilde{C}^{'}(\overline{\theta}_0)\sum_{k=1}^{\infty}\varepsilon^{k-1}\theta_k. \tag{26}$$

For the third order $O(\varepsilon)^3$, the momentum balance equation turn it:

$$\tilde{\rho}(\mathbf{z})\dot{\overline{\mathbf{v}}}_0 = -\overline{grad}\overline{p}_3 - \overline{\theta}_0\overline{\mathbf{g}} + \frac{2}{Re}\left[div\overline{\mathbf{D}}(\overline{\mathbf{v}}_0) - \frac{1}{3}\overline{grad}(\overline{div}\overline{\mathbf{v}}_0)\right] - \frac{\tilde{\rho}(\mathbf{z})}{Ro}\left[2\overline{\omega}\times\overline{\mathbf{v}}_0 + \frac{1}{Ro}(\overline{\omega}\times(\overline{\omega}\times\overline{\mathbf{r}}))\right], \tag{27}$$

We assumed $\mathbf{v} := \mathbf{v}_0$, $\theta := \theta_0$ and $p := p_0 + \varepsilon^3 p_3$, while $p_1 = p_2 = p_4 = p_5 := 0$, $\theta_1 = \theta_2 := 0$, $\mathbf{v}_1 = \mathbf{v}_2 := 0$ satisfy the null boundary conditions to the high orders of the expansions.

4.4 Ordained equations

4.4.1 Anelastic approximation

Finally, written the equations $(25)_{1,3}$ and (27) in their dimensionless forms, we have:

$$\begin{aligned} div[\rho(\mathbf{z})\mathbf{v}] &= 0, \\ C(\theta)\dot{\theta} - \frac{1}{\rho(\mathbf{z})}\alpha\theta\dot{p} &= K\triangle\theta + 2\mu\left[\|\mathbf{D}\|^2 - \frac{1}{3}|tr\mathbf{D}|^2\right], \\ \dot{\mathbf{v}} &= -\frac{1}{\rho(\mathbf{z})}gradp + [1-(\theta-\theta_0(\mathbf{z}))]\mathbf{g} + 2\nu\left[div\mathbf{D} - \frac{1}{3}grad(div\mathbf{v})\right] - [2\omega\times\mathbf{v} + \omega\times(\omega\times\mathbf{r})]. \end{aligned} \tag{28}$$

Then, the equations system (28) corresponds to a anelastic system, that includes the thermal viscous dissipation and the pressure stress work in the approximated energy balance equation, and comprehend the rotate effects in the momentum balance approximated equation.

Consequently, the distinct present effects in this class of problem, may be or not considered, it depends of the choice of the relationship of the dimensionless number and the small parameter adopted, i.e., we can neglect or add each effect only modifying the relationship between the small and dimensionless parameters, and following the same procedure. For instance, if we are interesting in the description of natural convection in a inertial frame, we can just assume, $Ro = O(\frac{1}{\varepsilon})$, and preserve the other numbers of unit order; resulting in a anelastic model with thermal viscous dissipation and pressure stress work for a inertial frame. But, if we want to study this problem without considering the dissipative terms, we may admit $Di = O(\varepsilon)$, where the others dimensionless numbers are maintained of unit order, then we recover the usual anelastic system in a rotating frame. Finally, we note that with this procedure we may ignore more than one effect simultaneously, e.g., if we want neglect the rotate and dissipative effects, we assume $Ro = O(\frac{1}{\varepsilon})$ and $Di = O(\varepsilon)$, then we obtain the anelastic equations in a inertial frame without the dissipative and pressure stress work terms in the energy balance equation, given by:

$$\begin{aligned} div[\rho(\mathbf{z})\mathbf{v}] &= 0, \\ C(\theta)\dot{\theta} &= K\triangle\theta, \\ \dot{\mathbf{v}} &= -\frac{1}{\rho(\mathbf{z})}gradp + [1-(\theta-\theta_0(\mathbf{z}))]\mathbf{g} + 2\nu\left[div\mathbf{D} - \frac{1}{3}grad(div\mathbf{v})\right]. \end{aligned} \tag{29}$$

4.4.2 Oberbeck-Boussinesq approximation

The Oberbeck-Boussinesq (OB) model can be reached, only considering the hypotheses that the reference state of density is a special case of our procedure, that is, the term $\rho(\mathbf{z})$ is choose as a constant in eq. (28), so the the resulting model considers yet the thermal viscous dissipation and the pressure stress work, express as:

$$\begin{aligned} div\mathbf{v} &= 0, \\ C(\theta)\dot{\theta} - \frac{1}{\rho_0}\alpha\theta\dot{p} &= K\triangle\theta + 2\mu\|\mathbf{D}\|^2, \\ \dot{\mathbf{v}} &= -\frac{1}{\rho_0}gradp + [1-(\theta-\theta_0)]\mathbf{g} + 2\nu_0 div\mathbf{D} - [2\omega\times\mathbf{v} + (\omega\times(\omega\times\mathbf{r}))]. \end{aligned} \tag{30}$$

Analogously, adopting $\rho(\mathbf{z})$ as a constant, $Ro = O(\frac{1}{\varepsilon})$ and $Di = O(\varepsilon)$, we arrive to the usual OB model, that neglects the thermal viscous dissipation, and the pressure sress work, described in a inertial frame, represents as:

$$\begin{aligned} div\mathbf{v} &= 0, \\ C(\theta)\dot{\theta} &= K\triangle\theta, \\ \dot{\mathbf{v}} &= -\frac{1}{\rho_0}gradp + [1-(\theta-\theta_0)]\mathbf{g} + 2\nu div\mathbf{D}. \end{aligned} \tag{31}$$

5. Conclusions

In this work, we present a unique justification to utilize the anelastic/OB model, applicable to a fluid flow in the presence of natural convection phenomenon in a rotating frame. This justification is based in the governing equation obtained here, through the modern continuum mechanics framework, i.e., we employ the basic laws, including the Clausius-Duhem inequality; and a constitutive theory, for which we introduce the mechanical incompressibility constraint - where the volume changes in a viscous fluid are due to the thermal contributions - and the constitutive response are restricted by the Clausius-Duhem inequality, for the class of elastic fluids with heat conduction and viscosity.

The governing equations submitted to dimensionless method, and the respective dimensionless numbers imposed of order one. A posteriori, we used a power series expansion, with a small parameter choose as a thermal compressible measure. Then, the dependent variables expanded in terms of this small parameter powers, obtaining a ordained system, equivalent to a anelastic system that includes the rotational effects (in the approximated momentum balance equation), and the thermal viscous dissipation and pressure stress work terms in the energy balance approximated equation. Additionally, we obtain the Oberbeck-Boussinesq model as a special case of this system, just assuming a constant characteristic value to density, where is maintained the rotational and dissipative effects before mentioned.

We recognize in this procedure we are interesting in the first expansion orders, where the boundary conditions are related to the respective zero order of dependent variables, and assuming null boundary conditions to the elevated orders. Then, the final corresponding equations are obtained, collecting the zero order ($\mathcal{O}(\varepsilon^0)$) to the mass and energy, and the third order ($\mathcal{O}(\varepsilon^3)$) to the momentum balance approximated equations, with $\mathbf{v} := \mathbf{v}_0$, $\theta := \theta_0$ and $p := p_0 + \varepsilon^3 p_3$, while $p_1 = p_2 = p_4 = p_5 := 0$, $\theta_1 = \theta_2 := 0$, $\mathbf{v}_1 = \mathbf{v}_2 := 0$.

We note that if we impose different relationships between the small parameter and the each dimensionless number involved, we may neglect one or more effects simultaneously, for instance, a anelasic/OB model can be describes in a inertial frame, neglecting the thermal viscous dissipation and the pressure stress work, just choosing mutually the respective Rossby (dissipation) dimensionless number as the inverse (same) order to the small parameter.

Finally, we remark that this procedure can be employed to the treatment of others fluid flow in the presence of natural convection phenomenon in a rotating frame, as the natural convection the oceans in a deep (shallow) layers, with the use of anelastic (Oberbeck-Boussinesq) models, respectively.

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