NUMERICAL SIMULATION OF FREE SURFACE DISTURBANCES INDUCED BY VORTEX FLOWS

Roger Matsumoto Moreira, roger@vm.uff.br
Pâmella da Silva Estolano, pestolano22@hotmail.com
LabCFD / School of Engineering / Fluminense Federal University
R. Passos da Pátria, 156, bl.D, sl.563A, Niterói, R.J., Brazil.

Abstract. *At an initially undisturbed free surface, considerable disturbances due to vortex interactions have been reported by several authors, based on linearised approaches, numerical simulations and experiments for various Froude numbers. This work aims to study numerically the generation of water waves induced by vortices located underneath the free surface. To solve the unsteady nonlinear two-dimensional free surface potential flow problem, a boundary-integral method is used. Different positions and strengths of vortex dipoles are set up in order to analyse their contribution for free surface disturbances and mixing. The fluid flow is assumed to be in deep water and irrotational, except at the location of the discrete vortices. Two different features are observed when a pair of counterrotating point vortices approaches the free surface. For a weak circulation, no wave breaking occurs and a small depression on the free surface is formed. For larger circulations, a central hump is formed followed by free surface breaking.*

Keywords: *Free surface flows, water waves, boundary integral method.*

1. INTRODUCTION

A wide variety of forms can be exhibited by the surface of a body of water depending on the conditions that are applied to it. Water waves may change significantly their properties when meeting underlying flows induced by internal waves or currents, or when approaching the coast. Waves can also be generated when water flows past obstacles or when ships move over the surface. At an initially undisturbed free surface, considerable disturbances due to vortex interactions have been reported by several authors, based on linearised approaches and numerical simulations for various Froude numbers. Experiments were also carried out showing this evidence (Willmarth *et al.* 1989).

In the theoretical models here reviewed the fluid flow is assumed to be in deep water and irrotational, except at the location of the discrete vortices. To characterise the fluid motion two different Froude numbers are defined based on different length scales,

$$Fr = \frac{k}{\sqrt{gd^3}}, \qquad Fr_s = \frac{k}{\sqrt{gl^3}}, \tag{1}$$

where k is the strength of the vortices, g is the acceleration due to gravity, d is the initial depth of the point vortices below the free surface and l is the initial separation between two vortices; Fr_s is called the spacing Froude number. For clarity, in the present work a vortex pair (or eddy pair) is assumed to be two vortices with the same sign, while a vortex dipole (or eddy couple) has opposite signs.

Telste (1989) observed two different free surface features when a pair of counter-rotating point vortices approaches a free surface. For a weak circulation ($Fr = 0.04$, $Fr_s = 0.5$) no wave breaking occurs and a small depression on the free surface is formed. If the reference frame is considered moving with one of the point vortices then this solution approaches the linear steady state profile predicted by Novikov (1981). For larger circulations ($Fr = 0.2$ and 0.6, $Fr_s = 2.2$ and 7.1) a central hump is formed followed by free surface breaking. To solve the unsteady nonlinear two-dimensional free surface potential flow problem Telste used a boundary-integral method. Marcus & Berger (1989) solved the same unsteady nonlinear problem via a finite-difference method. In their work strong vortices ($Fr = 0.5$, $Fr_s = 2.8$) also displace a mound of fluid before the free surface breaks. For weaker vortices ($Fr = 0.06$, $Fr_s = 0.3$) disturbances are gentler, with a shallower scar. Nevertheless their numerical method suffered from instabilities which excluded longer simulations.

The effect of a single vortex near a free surface was investigated by Tyvand (1991), who observed that all supercritical vortices ($Fr \geq 6.3$) lead to surface breaking while subcritical vortices ($Fr < 6.3$) tend to accumulate a surface mound until surface breaking eventually occurs. Tong (1991) introduced a point vortex into the boundary-integral method developed by Dold & Peregrine (1986), showing that small waves could be generated by a single weak vortex ($Fr < 0.2$), with the steepness of the waves depending on the direction of the vortex circulation, while a stronger vortex ($Fr \geq 0.2$) leads to free surface breaking.

Barnes *et al.* (1996) used Tong's algorithm as an approach to model the region of strong vorticity generated by a plunging breaker. In their model vorticity is represented by two-dimensional discrete vortices interacting with a nonlinear free surface initially at rest. Each point vortex moves under the influence of the free surface, the other vortices and its images, while the free surface moves due to the vortices and gravity. Figure 1 shows the free surface displacement due to a single vortex (continuous lines) and an equivalent system of 10 vortices distributed initially around the same point (dashed lines). The two cases have $Fr = 0.5$ and the results are remarkably similar, showing that the point approximation can model patches of vorticity quite well under certain conditions.

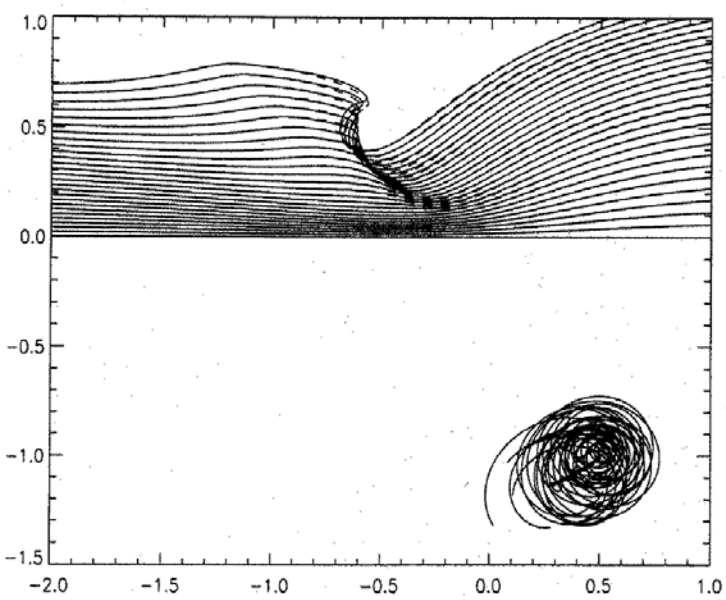

Figure 1. Stacked free surface displacement due to a single vortex (solid lines) and 10 vortices (dashed lines) and the corresponding vortex paths. $Fr = 0.5$. (Barnes *et al.* 1996)

It is not the purpose of this work to discuss the formation of cusps at the free surface, or to carry out a detailed investigation on the mechanisms of surface breaking due to a vortices, despite their importance in the understanding of certain free surface flows. Our main interest is to model numerically the free surface transformation induced by vortex interactions and to analyse if our predictions agree with theory and experiments. By extending our model to nonlinear waves, a better understanding of this phenomenon can be gained.

2. BOUNDARY VALUE PROBLEM

The aim of this work is to model the unsteady motion of a body of fluid in deep water that is bounded above by a free surface $\mathcal{F}$ under the influence of gravity, and that interacts with a vortex flow. The first major simplifying assumption is that only motions that are essentially two-dimensional are considered in our model. All the information needed to describe the flow is therefore contained in an appropriately oriented plane through the fluid and its surface. Displacements of the free surface are measured by the distance from a rest state in which the surface is flat and a fluid domain $\mathcal{D}$ is chosen by taking a plane perpendicular to this "still-water level". $\mathcal{D}$ is chosen to be periodic in x. Cartesian coordinates are defined by setting the x-axis in the undisturbed surface with the y-axis vertically upwards so that the fluid occupies the half-plane $y \leq 0$ when at rest.

The fluid flow is assumed to be inviscid and incompressible. The singularities – a pair of fixed counter-rotating vortices – are distributed below the free surface and are defined in terms of their position and strength according to the required underlying flow. It is assumed that the flow is irrotational outside the singular cores and away from the free surface. The irrotational velocity field $\mathbf{u}(x, y, t)$ is then given by the gradient of a full velocity potential $\Phi(x, y, t)$ which satisfies Laplace's equation in the fluid domain $\mathcal{D}$, excluding the singular points $\mathbf{x}_{\mathbf{s}_i}$,

$$\nabla^2\Phi = 0, \qquad \text{in } \mathcal{D} - \bigcup_{i=1}^{n} \mathbf{x}_{\mathbf{s}_i}. \tag{2}$$

From Green's theorem all the interior properties of the fluid can be determined by its properties at the boundaries alone. The entire motion can then be modelled by considering a point discretisation of the surface. The velocity of the fluid at the surface is determined using,

$$\mathbf{u} = \nabla\Phi = \frac{\partial\Phi}{\partial n_1}\mathbf{n_1} + \frac{\partial\Phi}{\partial n_2}\mathbf{n_2}, \tag{3}$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are the tangential and normal unit vectors respectively.

2.1 Introduction of singularities

The introduction of the singular points in our model is done by decomposing our full velocity potential Φ into a regular part ϕ_w (due to surface waves) and a singular part ϕ_s (due to the singularities),

$$\Phi = \phi_w + \phi_s. \tag{4}$$

Then Laplace's equation can be rewritten as the sum,

$$\nabla^2\phi_w + \nabla^2\phi_s = 0, \qquad \text{in } \mathcal{D} - \bigcup_{i=1}^{n} \mathbf{x}_{\mathbf{s}_i}. \tag{5}$$

and the "total" surface velocity becomes,

$$\mathbf{u} = \nabla\phi_w + \nabla\phi_s. \tag{6}$$

2.2 Boundary conditions

The kinematic boundary condition to be satisfied at the free surface $\mathcal{F}$ is based on a continuum idea that a fluid particle described by a position vector $\mathbf{r} = (x, y, t)$ on the moving free surface remains on it. Therefore,

$$\frac{D\mathbf{r}}{Dt} = \nabla\Phi, \qquad \text{on } \mathcal{F}. \tag{7}$$

The dynamic surface condition is given by Bernoulli's equation,

$$\frac{D\Phi}{Dt} = \frac{1}{2}\,|\nabla\Phi|^2 - gy - \frac{p}{\rho}, \qquad \text{on } \mathcal{F}. \tag{8}$$

Here y is the elevation of the free surface above the undisturbed water level, g is the acceleration due to gravity (acting vertically downwards) and ρ is the fluid density. The pressure p can be chosen to approximate the effects of wind, surface tension or a localised pressure on the surface, though it is not used in the calculations.

As already stated, $\mathcal{D}$ is bounded above by $\mathcal{F}$ and we assume that the water is deep, satisfying the condition $|\nabla\Phi| \to 0$ as $y \to -\infty$. For a periodic domain, the velocity potential $\Phi(x, y, t)$ and the velocity $\mathbf{u}(x, y, t)$ are required to be continuous at the vertical boundaries such that,

$$\nabla\Phi(0, y, t) = \nabla\Phi(2\pi, y, t), \qquad \text{for } -\infty < y \leq 0 \text{ and } t > 0. \tag{9}$$

Here the length units are chosen to make the period equal to 2π.

To complete the model an initial condition for $\mathcal{F}$ is required and is given by,

$$\eta(x) = \eta_0(x), \qquad \Phi(x, \eta) = \Phi_0(x, \eta_0), \qquad \text{for } t = 0. \tag{10}$$

3. LINEAR STEADY FREE SURFACE APPROXIMATION

Under linear theory, it is possible to solve explicitly the problem of a stationary free surface flow due to a generic distribution of singularities. Laplace's equation is valid for the whole fluid domain $\mathcal{D}$, excluding the singular points $\mathbf{x}_{\mathbf{s}_i}$, while at the far field $\Phi \sim 0$ as $|x| \to \infty$, with $-\infty < y \leq 0$. As $\mathbf{x} \to \mathbf{x}_{\mathbf{s}_i}$, the total velocity potential Φ approaches the velocity potential ϕ_s. At the free surface the kinematic and dynamic boundary conditions (7) and (8) take the stationary form,

$$\frac{\partial\Phi}{\partial x}\frac{\partial\eta}{\partial x} = \frac{\partial\Phi}{\partial y}, \tag{11}$$

$$\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)^2 + \left(\frac{\partial\Phi}{\partial y}\right)^2\right] + g\eta = 0, \tag{12}$$

both valid on $y = \eta(x)$.

The most usual and simple way of dealing with the nonlinear boundary condition is to say that in some sense the waves are small and that the nonlinear terms, being of the order of the square of a small quantity, are negligible. A linearised steady free surface solution can then be approximated by defining the total velocity potential $\Phi(x, y)$ and the free surface profile $\eta(x)$ in terms of an algebraic expansion of the form,

$$\Phi(x,y) = \epsilon\Phi_1 + \epsilon^2\Phi_2 + \cdots, \tag{13}$$
$$\eta(x) = \epsilon\eta_1 + \epsilon^2\eta_2 + \cdots, \tag{14}$$

valid for the small wave approximation in which ϵ is a small parameter. The functions η_i $(i = 1, 2, \cdots)$ depends on x, while Φ_i depends on x and η.

Substituting Eq. (13) into Laplace's equation and extracting the ϵ and ϵ^2 terms give,

$$\nabla^2\Phi_1 = 0, \qquad \nabla^2\Phi_2 = 0, \qquad \text{in } \mathcal{D} - \bigcup_{i=1}^{n} \mathbf{x}_{\mathbf{s}_i}, \tag{15}$$

while at the far field,

$$\Phi_1 \sim 0, \qquad \Phi_2 \sim 0 \qquad \text{as } |x| \to \infty, \qquad \text{for } -\infty < y \leq 0. \tag{16}$$

For any ϵ, each Φ_i must itself be harmonic.

By substituting the Eq. (13) and (14), the boundary conditions (11) and (12) take the following form,

$$\left[\epsilon\frac{\partial\Phi_1}{\partial x} + \epsilon^2\frac{\partial\Phi_2}{\partial x} + \cdots\right]\left[\epsilon\frac{\partial\eta_1}{\partial x} + \epsilon^2\frac{\partial\eta_2}{\partial x} + \cdots\right] = \epsilon\frac{\partial\Phi_1}{\partial y} + \epsilon^2\frac{\partial\Phi_2}{\partial y} + \cdots,$$
$$\frac{1}{2}\left[\left(\epsilon\frac{\partial\Phi_1}{\partial x} + \epsilon^2\frac{\partial\Phi_2}{\partial x} + \cdots\right)^2 + \left(\epsilon\frac{\partial\Phi_1}{\partial y} + \epsilon^2\frac{\partial\Phi_2}{\partial y} + \cdots\right)^2\right] + g\left(\epsilon\eta_1 + \epsilon^2\eta_2 + \cdots\right) = 0.$$

Extracting the ϵ and ϵ^2 terms of the boundary conditions above give,

$$\frac{\partial\Phi_1}{\partial y} = 0, \tag{17}$$
$$\eta_1 = 0, \tag{18}$$
$$\frac{\partial\Phi_2}{\partial y} = \frac{\partial\Phi_1}{\partial x}\frac{\partial\eta_1}{\partial x}, \tag{19}$$
$$\eta_2 = -\frac{1}{2g}\left(\frac{\partial\Phi_1}{\partial x}\right)^2, \tag{20}$$

which are valid for any submerged distribution of vortices.

4. FULLY NONLINEAR BOUNDARY-INTEGRAL SOLVER

The present boundary value problem is solved using an adapted version of the fully nonlinear potential flow solver developed by Dold & Peregrine (1986) (more fully described in Dold 1992). The method consists of applying a boundary-integral method to a free surface flow problem, which reduces significantly the computational demand for the calculation of the fluid motion since only surface properties are evaluated. The solution method is based on solving an integral equation that arises from Cauchy's integral theorem for functions of a complex variable. The original numerical scheme is modified for the inclusion of singularities.

Basically the method of solution consists of the following stages. Initially the full potential Φ is known on the surface for each time step. The potential ϕ_s due to the singularities is also defined and subtracted from the surface value of Φ such that the remaining surface wave potential ϕ_w, which has no singularities in the fluid domain, can be used with Cauchy's integral theorem to calculate the velocity $\nabla\phi_w$ on the free surface. Then the potential ϕ_s is added back in and corresponding "total" velocities are evaluated. The free surface is stepped in time using a truncated Taylor series. Such stages are repeated until either the final time is reached, or the algorithm breaks down.

4.1 Vortex model

Assuming that the singular points are a vortex couple with strength k, occupying respectively the positions z_1 $(= x_1 + iy_1)$ and z_2 $(= x_2 + iy_2)$, then the complex potential induced by those vortices in a transformed ζ-plane is given by (Batchelor 1967),

$$\omega(\zeta) = -ik\log\left(\frac{\zeta - \zeta_1}{\zeta - \zeta_2}\right), \tag{21}$$

where $\zeta_1 \left(= e^{y_1+ix_1}\right)$ and $\zeta_2 \left(= e^{y_2+ix_2}\right)$ represent the corresponding positions of the vortex couple in the ζ-plane.

From the circle theorem (Milne-Thomson 1962) the complex potential of the flow induced by the pair of point vortices and their reflected images in the ζ-plane is given by,

$$\omega_s(\zeta) = -ik\log\left[\left(\frac{\zeta-\zeta_1}{\frac{1}{\zeta}-\overline{\zeta_1}}\right)\left(\frac{\frac{1}{\zeta}-\overline{\zeta_2}}{\zeta-\zeta_2}\right)\right], \tag{22}$$

where $\overline{\zeta_1}$ and $\overline{\zeta_2}$ are the complex conjugate of ζ_1 and ζ_2. Note that the reflection of the vortices onto the free surface represents a convenient choice for deep water only. They are placed outside the body of the fluid and used to approximate the complex potential within the fluid. For an unbounded domain with a bed, a vertically periodic set of vortices reflected onto the bed is more convenient.

Then the velocity potential ϕ_s of a pair of counter-rotating vortices and its corresponding image can be expressed in the transformed plane by,

$$\phi_s(\zeta) = -k\,\Re\left\{i\log\left[\left(\frac{\zeta-\zeta_1}{\zeta-\zeta_2}\right)\left(\frac{\frac{1}{\zeta}-\overline{\zeta_2}}{\frac{1}{\zeta}-\overline{\zeta_1}}\right)\right]\right\}. \tag{23}$$

The first term inside the logarithm represents the contribution of the two vortices to the system, while the second term refers to their reflected images. In our examples the point vortices are prescribed to be at fixed positions in time. The free surface moves under the influence of the eddy couple and gravity since our first aim is to analyse the contribution of not advected vortices beneath a free surface. The total circulation Γ (positive and equal to $2\pi k$, in the case of a single counter-clockwise point vortex) vanishes around the pair of counter-rotating vortices. The contribution of the eddy couple to the "total" velocity $\mathbf{u}$ is then given by $\nabla\phi_s$.

The stream function ψ_s induced by the singularities is obtained by simply taking the imaginary part of the complex potential ω_s. Figure 2 shows the streamlines plotted in the z-plane for a periodic line of counter-rotating vortices in deep water. The free surface $\mathcal{F}$ is represented by the linear steady solution derived in the next section, with no vertical exaggeration. Two counter-rotating vortices per period, equally spaced in the fluid domain, are showed in this case.

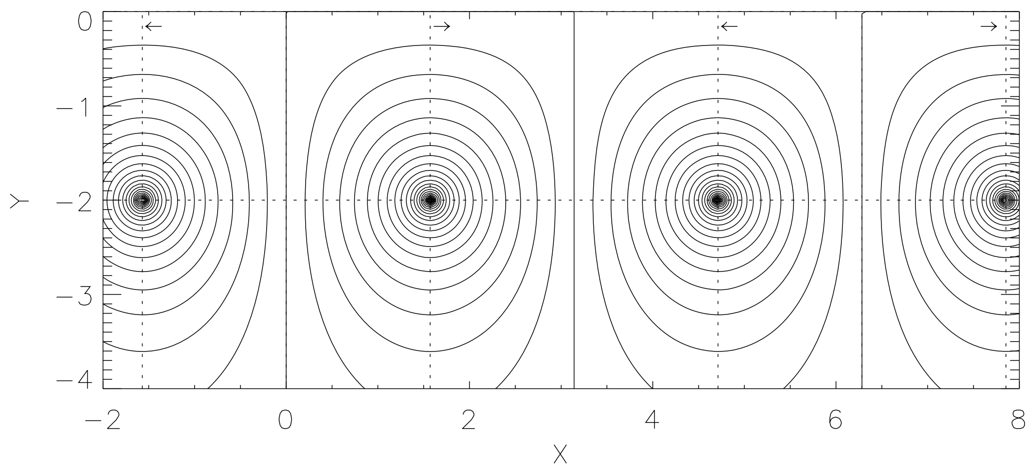

Figure 2. Streamlines obtained for a periodic distribution of counter-rotating vortices in deep water. $Fr = 1.0 \times 10^{-2}$, $Fr_s = 1.8 \times 10^{-3}$. Depressions at the linear steady free surface cannot be seen due to its small slope.

5. FULLY NONLINEAR RESULTS

The form of the free surface motion depends on the initial condition chosen to complete the boundary value problem defined in section 2. Any smooth initial data is permissible in the computations. However it must be consistent with the imposed boundary conditions of the problem. In addition the distance between successive discretisation points should vary smoothly with distance along the free surface. The initial condition must then have more points in regions where the total velocity potential Φ changes rapidly with respect to arclength along the free surface. In our model the underlying flow induced by the singularity distribution is "switched on" at $t = 0$. Due to this impulsive initial motion, an initially flat free surface would be disturbed in the region immediately above the singularity, with gravity and, in a smaller scale, surface tension acting as the restoring forces. Waves may then be formed as a response of this interaction.

Figure 3 shows an initially flat free surface disturbed by the motion of a periodic line of counter-rotating vortices. Initially, two depressions are formed abruptly above the vortices. In this case, surface tension effects are neglected. Gravity is then the only restoring force in our model, leading to the formation of a set of waves that propagates in the $+x$ and $-x$ direction. Eventually these waves reach regions where the surface current is adverse and sufficiently strong to block their group velocity, increasing their wave steepness and leading to wave breaking. This can be observed in the final stages of Fig. 3.

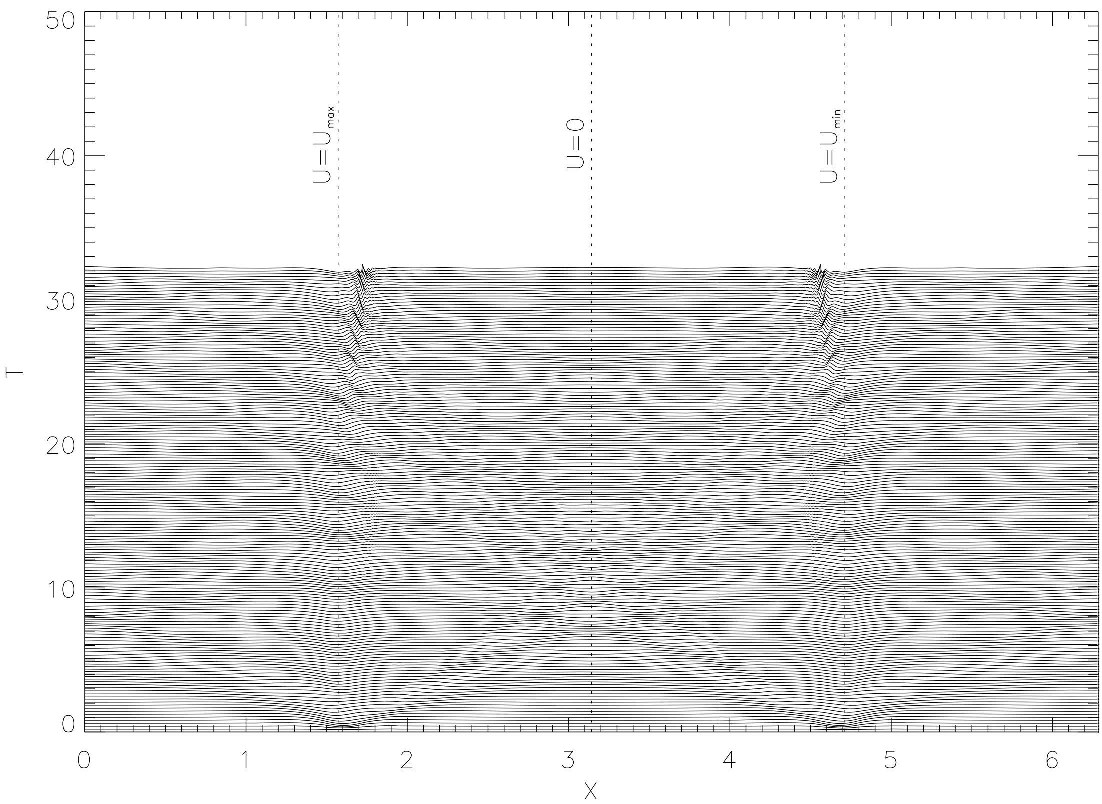

Figure 3. Disturbances generated by a pair of counter-rotating vortices at an initially flat free surface. $t_{breaking} = 32.2$, $Fr = 0.08$, $Fr_s = 1.8 \times 10^{-3}$, no smoothing used. Vertical exaggeration $100 : 1$.

5.1 Inclusion of damping in the nonlinear calculations

Figure 3 shows that the disturbances generated by the impulsive initial motion of the underlying flow at a initially flat free surface are significant and sufficient to lead to wave breaking. These initial disturbances can be damped by the numerical scheme by introducing a damping term to the second harmonic of the Taylor time series, leading to a stationary surface in finite time,

$$\Phi(t + \Delta t) = \Phi(t) + \frac{D\Phi}{Dt}\Delta t + \frac{1}{2}\frac{D^2\Phi}{Dt^2}\Delta t^2 + \frac{\delta}{2}\frac{\partial^2\Phi}{\partial t^2}\Delta t^2 + O(\Delta t^3), \quad (24)$$

where δ is the damping factor and Φ is the full velocity potential.

Figure 4 shows the resulted stacked free surface for $\delta = 1.5$. The impulsive motion still produces waves which are now numerically dissipated by the scheme. As a consequence breaking does not occur, with the free surface tending to reach a steady profile.

6. SUMMARY

The linear steady free surface solution obtained for a periodic line of counter-rotating vortices is illustrated by the continuous lines in Fig. 5. Two depressions are formed immediately above the vortices, which approach the linear steady solution given by Novikov (1981). The depth of the depression reaches its maximum where the maximum and minimum surface currents are imposed. For bigger Froude numbers these depressions become deeper.

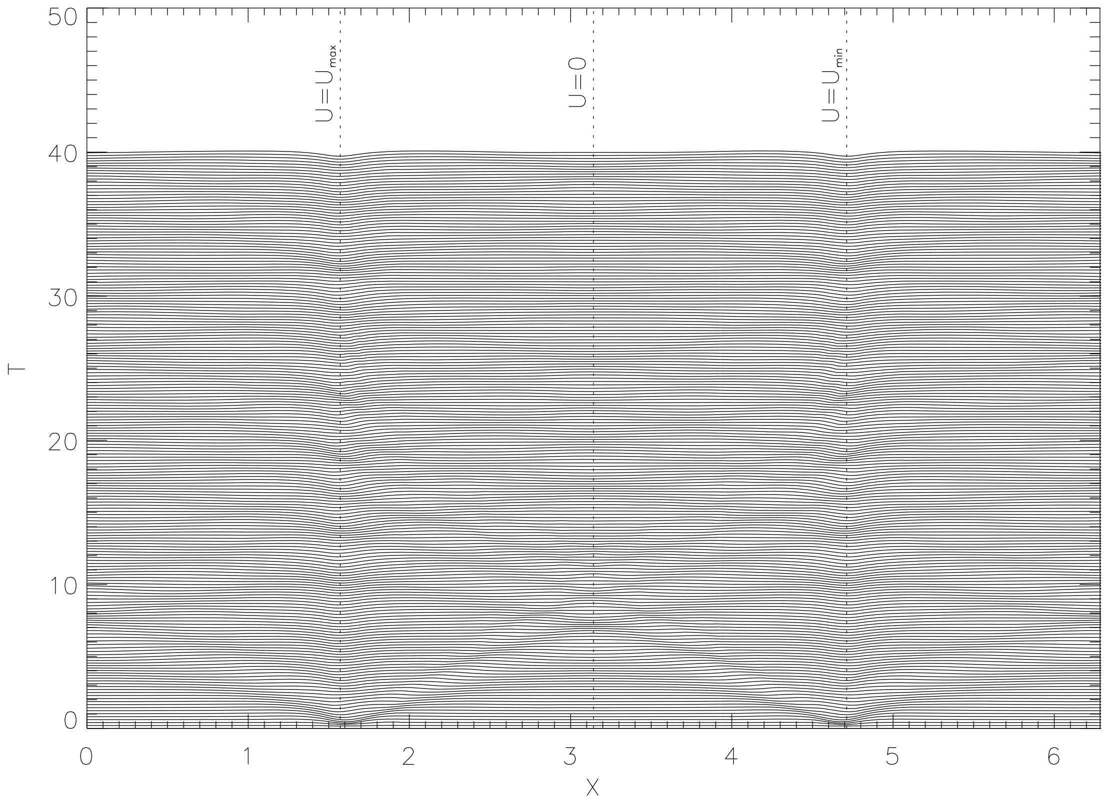

Figure 4. Disturbances generated by a pair of counter-rotating vortices at an initially flat free surface. $\delta = 1.5$. No breaking observed. $Fr = 0.08$, $Fr_s = 1.8 \times 10^{-3}$, no smoothing used. Vertical exaggeration $100 : 1$.

A comparison between the linearised steady solutions and the damped nonlinear results is shown in the same Fig. 5. The damped nonlinear results converges to the linearised steady solution for a vortex flow. For bigger Froude numbers the damping factor δ is increased such that a faster convergence of the nonlinear results is achieved. After a certain time, a good agreement was found between the linear and nonlinear results.

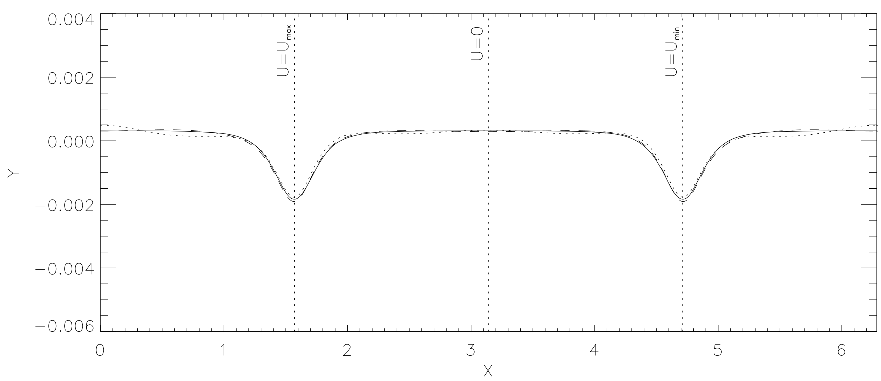

Figure 5. The linear steady free surface solution (——) and nonlinear numerical results due to an eddy couple; $Fr = 0.06$, $\delta = 1.5$, $t = 79.2$ $(\cdots\cdots)$ and $t = 119.2$ (– – –). Vertical exaggeration $250 : 1$.

7. ACKNOWLEDGEMENTS

The authors acknowledge the financial support through CNPq, the national research and development council (contract number 62.0018/2003-8-PADCT III / FAPERJ).

8. REFERENCES

Barnes, T.C.D., Brocchini, M., Peregrine, D.H. & Stansby, P.K., 1996, "Modelling post-wave breaking turbulence and vorticity", Proc. 25th Internat. Conf. on Coastal Engng., Orlando, ASCE, v. 1, pp. 186-199.

Batchelor, G.K., 1967, "An Introduction to Fluid Dynamics", Cambridge University Press, 615p.

Dold, J.W., 1992, "An Efficient Surface-Integral Algorithm Applied to Unsteady Gravity Waves", Journal of Computational Physics, v. 103, pp. 90-115.

Dold, J.W. & Peregrine, D.H., 1986, "An Efficient Boundary-Integral Method for Steep Unsteady Water Waves", in Numer. Meth. for Fluid Dynamics II, Eds. K.W. Morton & M.J. Baines, Oxford University Press, pp.671-679.

Marcus, D.L. & Berger, S.A., 1989, "The interaction between a counter-rotating vortex pair in vertical ascent and a free surface", Physics of Fluids A, v. 1, pp. 1988-2000.

Milne-Thomson, L.M., 1962, "Theoretical Hydrodynamics", Macmillan & Co Ltd.

Novikov, YE.A., 1981, "Generation of surface waves by discrete vortices", Izvestiya, Atmospheric and Oceanic Physics, v. 17, pp. 709-714.

Telste, J.G., 1989, "Potential flow about two counter-rotating vortices approaching a free surface", Journal of Fluid Mechanics, v. 201, pp. 259-278.

Tong, R.P., 1991, "Unsteady Flow with a Free Surface: A Study of Numerical Approximations in the Boundary Integral Method", PhD thesis, University of Bristol, U.K.

Tyvand, P.A., 1991, "Motion of a vortex near a free surface", Journal of Fluid Mechanics, v. 225, pp. 673-686.

Willmarth, W.W., Tryggvason, G., Hirsa, A. & Yu, D., 1989, "Vortex pair generation and interaction with a free surface", Physics of Fluids A, v. 1, pp. 170-172.

9. Responsibility notice

The authors are the only responsible for the printed material included in this paper.