OPTIMIZATION OF A STOCHASTIC DYNAMICAL SYSTEM

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Abstract. *This paper analyzes a simple stochastic dynamical system and a target performance to be achieved by the system, in other words, the optimization algorithm seeks the optimal parameter values to achieve a pre-defined performance. However, some parameters of the system are modeled as random variables, thus, requiring the use of stochastic optimization. To model the uncertainties of the system the parametric approach is used and the probability density functions are derived using the Maximum Entropy Principle. To take into account the uncertainties of the dynamical system in the optimization process, a multi-objective optimization of some statistical characteristics of a distance between the response of the system and the target performance is proposed. The global and bounded Nelder-Mead optimization algorithm is employed to optimize the stochastic function. The results showed that when the uncertainties are considered, the optimum design is different from the deterministic optimization and that the robust optimization is a very useful tool to deal with uncertainties in dynamical systems.*

Keywords: *stochastic dynamics, robust optimization, stochastic optimization, target performance*

1. INTRODUCTION

Optimization has become a very important tool in several fields, especially in engineering design. Although deterministic optimization methods have been widely applied, it is difficult to find examples of systems to be optimized, in any field, that do not include some level of uncertainty, for example, on its parameters, geometry, boundary conditions, or even in the very model being used.

When one takes into account uncertainties in the design optimization process, it is named robust design optimization (RDO). Several forms of objective function have been proposed in the RDO, for instance: minimization of the mean and/or variance of the response of the system under consideration. Thus, the robust optimal design concentrates the probability distribution of the response near to its mean. Stochastic programming (Kall and Wallace, 1994), Taguchi methodology (Phadke, 1989), and optimization methods (Chen et al., 1999) have been applied to solve RDO. As the robust optimization may have two or more objective functions (*e.g.*, minimize the mean and the variance), its formulation is in the form of a multi-objective optimization problem. To deal with multi-objective optimization, among others, the weighted sum, compromise approach, and the preference aggregation methods have been employed, (Beyer and Sendhoff, 2007).

The application of the RDO in dynamical systems is recent (Zang et al., 2005; Capiez-Lernout and Soize, 2008). In this paper, we are concerned with the RDO of a simple dynamical system with only two-degrees-of-freedom in order to focus in the robust optimization. A target performance optimization is proposed, in other words, we seek the set of system parameters that lead to a system response as close as possible to a performance defined *a priori*. Here, the uncertainties on the stiffness of such system are taken into account by modeling them as random variables. To do so, the parametric approach is used and the probability density functions are derived using the Maximum Entropy Principle (Shannon, 1948; Jaynes, 1957a; Jaynes, 1957b). Then, the objective function of the RDO problem is constructed as the minimization of the mean and the variance of the difference between the system performance and the target one. The Global and Bounded Nelder-Mead (GBNM) algorithm is employed as optimizer due to its ability to handle non-convex and multimodal functions (Luersen and Le Riche, 2004).

This paper is organized as follows. The deterministic dynamical problem is presented in Section 2 and the probabilistic model is presented in Section 3. The robust optimization problem is defined in Section 4 and the optimization algorithm is explained in Section 5. Finally, the numerical results are presented in Section 6 and the concluding remarks are given in Section 7.

2. DETERMINISTIC MODEL

A simple dynamical system is considered in order to focus on the robust optimization strategy used. The optimization procedure proposed is easily extended to more complex dynamical problems, but it is interesting to note that, even for a simple system, the RDO presents results that are not trivial. Figure 1 shows the two degrees of freedom dynamical system that is analyzed.

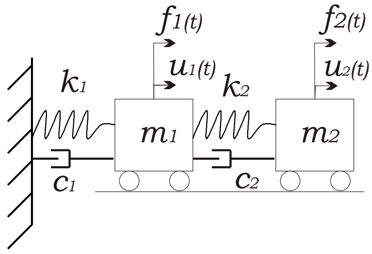

Figure 1. Two d.o.f. system used.

This linear system has two natural frequencies and two normal modes. The dynamics of the system is given by:

$$[M]\ddot{\mathbf{u}}(t) + [C]\dot{\mathbf{u}}(t) + [K]\mathbf{u}(t) = \mathbf{f}(t)\,, \tag{1}$$

with initial conditions $\mathbf{u}(0) = \mathbf{u}_0$ and $\dot{\mathbf{u}}(0) = \mathbf{v}_0$, in which

$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}; \quad [C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}; \tag{2}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{3}$$

and

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}; \quad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}. \tag{4}$$

The mass, damping, and stiffness matrices, denoted by $[M]$, $[C]$, and $[K]$, are real symmetric positive-definite. The external force is represented by the vector $\mathbf{f} = (f_1, f_2)^T$, where f_1 and f_2 are the forces applied on the masses m_1 and m_2. The displacements of the masses are denoted by u_1 and u_2, which are the components of the vector $\mathbf{u}$. Let $\mathbf{f}(\mathbf{t}) = (f_1(t), 0)^T$ be the input force applied on the system, and let $\mathbf{u}(\mathbf{t}) = (u_1(t), u_2(t))^T$ be the corresponding output. Let $\widehat{f}_1$ be the Fourier transform of f_1, $\widehat{u}_1$ be the Fourier transform of u_1, and $\widehat{u}_2$ be the Fourier transform of u_2. In the frequency domain Eq. (1) is written as:

$$\begin{pmatrix} \widehat{u}_1(\omega) \\ \widehat{u}_2(\omega) \end{pmatrix} = \left(-\omega^2[M] + i\omega[C] + [K]\right)^{-1} \begin{pmatrix} \widehat{f}_1 \\ 0 \end{pmatrix}, \tag{5}$$

where $i = \sqrt{-1}$ and ω is the frequency. In this paper, the deterministic frequency response function of interest is denoted by h and is defined as

$$h(\omega) = \left| \frac{\widehat{u}_2(\omega)}{\widehat{f}_1(\omega)} \right|. \tag{6}$$

3. PROBABILISTIC MODEL

We consider a design problem that the rigidity of the two springs are uncertain, say they were taken from a lot. As the manufacturing process is not perfect, there are uncertainties in the values of the stiffnesses and they may differ from the nominal value. A parametric probabilistic approach has been employed to model the uncertainties on the stiffnesses which have mean values $\underline{k}_1$ and $\underline{k}_2$, their nominal values, and associated random variables K_1 and K_2 (the capital letter is used for the random variables).

The choice of the probability distribution function is crucial since all the stochastic simulations depend on it. The Maximum Entropy Principle (Shannon, 1948; Jaynes, 1957a; Jaynes, 1957b) has been used to construct the probability density function of the random variables in a way that only the known information is used. This information is: (1) the stiffness is always positive, $K_1 > 0$ and $K_2 > 0$, (2) the mean values are known ($E\{K_1\} = \underline{k}_1$ and $E\{K_2\} = \underline{k}_2$), and (3) $E\{K_1^{-2}\} = c_1 < +\infty$ and $E\{K_2^{-2}\} = c_2 < +\infty$, so that the response of the system is a second order random variable (this means that from the measured displacement and the stiffness the force could be computed, that is the inverse problem is well posed).

It follows that K_1 and K_2 are Gamma random variables: $K_1 \sim$ Gamma$(\underline{k}_1, \delta_{K_1})$ and $K_2 \sim$ Gamma$(\underline{k}_2, \delta_{K_2})$, where $\delta_K = \sigma_K/\underline{k}$ is the coefficient of variation and σ_K is the standard deviation. In terms of probability density functions:

$$p_{K_1}(k_1) = \mathbb{1}_{]0,+\infty[}(k_1)\frac{1}{\underline{k}_1}\left(\frac{1}{\delta_{K_1}^2}\right)^{\frac{1}{\delta_{K_1}^2}}\frac{1}{\Gamma\left(1/\delta_{K_1}^2\right)}\left(\frac{k_1}{\underline{k}_1}\right)^{\frac{1}{\delta_{K_1}^2}-1}\exp\left(-\frac{k_1}{\delta_{K_1}^2\underline{k}_1}\right), \tag{7}$$

and

$$p_{K_2}(k_2) = \mathbb{1}_{]0,+\infty[}(k_2)\frac{1}{\underline{k}_2}\left(\frac{1}{\delta_{K_2}^2}\right)^{\frac{1}{\delta_{K_2}^2}}\frac{1}{\Gamma\left(1/\delta_{K_2}^2\right)}\left(\frac{k_2}{\underline{k}_2}\right)^{\frac{1}{\delta_{K_2}^2}-1}\exp\left(-\frac{k_2}{\delta_{K_2}^2\underline{k}_2}\right). \tag{8}$$

which are Gamma probability density functions. $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-1}dt$ is the Gamma function defined for $z > 0$. The stiffness random matrix has the following form:

$$[\mathbf{K}] = \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}, \tag{9}$$

where the boldface is used to represent a random matrix. The Monte Carlo Method (Rubinstein, 1981) is used to generate the random variables K_1 and K_2. The stochastic dynamical equation is written as:

$$\begin{pmatrix} \widehat{U}_1(\omega) \\ \widehat{U}_2(\omega) \end{pmatrix} = \left(-\omega^2[M] + i\omega[C] + [\mathbf{K}]\right)^{-1}\begin{pmatrix} \widehat{f}_1 \\ 0 \end{pmatrix}, \tag{10}$$

and the random frequency response function H is the randomization of h

$$H(\omega) = \left|\frac{\widehat{U}_2(\omega)}{\widehat{f}_1(\omega)}\right|. \tag{11}$$

4. ROBUST DESIGN OPTIMIZATION

Here, we wish to obtain a structural design for which the response is as close as possible to a pre-defined target performance. Thus, we aim at minimizing the distance between the system response and the target performance. Such distance d in a deterministic problem would be defined as:

$$d(\mathbf{s}) = \left(\frac{p_1(\mathbf{s}) - t_1}{t_1}\right)^2 + \left(\frac{p_2(\mathbf{s}) - t_2}{t_2}\right)^2, \tag{12}$$

where $\mathbf{s}$ is the variable design vector $\mathbf{s} = (k_1, k_2)$, t_1 and t_2 are the target peaks in the frequency bands $\mathbb{B}_1$ and $\mathbb{B}_2$, respectively. For $i = 1$ and 2, p_i is the peak given by $p_i = \max(h_{\mathbb{B}_i})$ where $h_{\mathbb{B}_i}$ is the response of the system in the frequency band $\mathbb{B}_i$. Thus, d measures how close the performance of the system (peaks p_1 and p_2) is to the target performance (peaks t_1 and t_2). Figure 2 shows a frequency response function with two targets t_1 in $\mathbb{B}_1$ and t_2 in $\mathbb{B}_2$.

In the associated stochastic problem the peaks will be random variables P_1 and P_2 and, consequently, the distance is also a random variable D. So,

$$D(\underline{\mathbf{s}}) = \left(\frac{P_1(\underline{\mathbf{s}}) - t_1}{t_1}\right)^2 + \left(\frac{P_2(\underline{\mathbf{s}}) - t_2}{t_2}\right)^2, \tag{13}$$

where $\underline{\mathbf{s}}$ is the mean value of the variable design vector $\underline{\mathbf{s}} = (\underline{k}_1, \underline{k}_2)$. For $i = 1$ and 2, P_i is the peak given by $P_i = \max(H_{\mathbb{B}_i})$ where $H_{\mathbb{B}_i}$ is the response of the system in the frequency band $\mathbb{B}_i$. Thus, D has mean $E\{D\}$ and variance var$\{D\}$.

As already commented, the aim of this paper is to pursue the RDO of a dynamical system, in other words, the system is optimized taking into account uncertainties. To accomplish this, we propose to minimize some statistical characteristics of D (Eq.(13)). First, the stochastic optimization problem is posed as:

$$\underset{\underline{\mathbf{s}} \in \mathcal{C}_{adm}}{\text{minimize}}\ J(\underline{\mathbf{s}}), \tag{14}$$

where J is comprised by some statistical characteristics of the distance D and the feasible set is defined by $\mathcal{C}_{adm} = \{\underline{\mathbf{s}} = (\underline{k}_1, \underline{k}_2); k_{1min} \leq \underline{k}_1 \leq k_{1max}, k_{2min} \leq \underline{k}_2 \leq k_{2max}\}$. To take into account the uncertainties of the parameters,

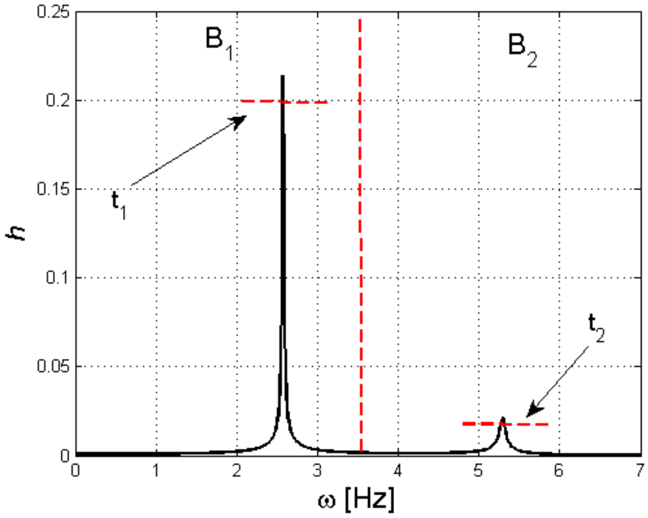

Figure 2. Frequency response function with targets t_1 and t_2.

we propose to minimize simultaneously the mean and the variance of D ending up in a multi-objective optimization problem similar to what was done in (Zang et al., 2005; Capiez-Lernout and Soize, 2008). The first step to solve the multi-objective optimization problem is to find the Utopia points by minimizing individually the mean and the variance as single objective functions. The Utopia points are denoted by E^* and var*. Since there are only two objective functions, they may be combined into a single objective function using the weighted sum method. Therefore, the objective function becomes:

$$J(\underline{\mathbf{s}}) = \alpha \frac{E\{D(\underline{\mathbf{s}})\}}{E^*} + (1-\alpha)\frac{\mathrm{var}\{D(\underline{\mathbf{s}})\}}{\mathrm{var}^*}\,, \tag{15}$$

where $\alpha \in [0, 1]$ is the weighting factor. Then, J is minimized for different values of α between 0 and 1 in order to construct the Pareto frontier obtaining trade offs between the two objectives of the problem. The function to be minimized is non-convex and multimodal as will be shown in the numerical analysis (Section 8). Thus, in order to get good results, the use of a global optimizer becomes mandatory. The globalized and bounded Nelder-Mead algorithm (Luersen and Le Riche, 2004) is employed here and it is described in the sequel.

5. OPTIMIZATION ALGORITHM

As the function under analysis is non-convex and multimodal, the utilization of a global optimization algorithm is required. In this framework, stochastic or hybrid stochastic/deterministic methods are often used. The simplest approach is a random search, where a new point is randomly generated and examined. It is kept if its performance is better than the previous iteration, if not it is rejected and the old point is kept. Of course, this procedure leads to a very high computational cost. Thus, several classes of global optimization algorithms have been developed to perform the search in a more efficient way. One of them is the coupling of global and local optimization algorithms. For instance, any local optimizer can be turned into a global one by restarting the search randomly. The Global and Bounded Nelder-Mead (GBNM) optimization algorithm does it in a interesting way. The restart procedure uses an adaptive probability density constructed using the memory of past local searches. The algorithm is fully described in Luersen and Le Riche (2004) and Luersen et al. (2004). Here, the main parts of the algorithm are detailed, especially its probabilistic restart.

The local search of the GBNM is performed by the Nelder-Mead algorithm, which is a classic zero order method that is based on the comparison of function values at the $n+1$ vertices of a simplex. Some modifications have been implemented in the GBNM such as the handling of constraints by penalization, bounds through projection and the degeneracy of the simplex. The stopping criteria of the local searches are when the simplex is flat, small or degenerated. When one of such criteria is achieved, the search is restarted, which is described in the sequel.

The probability of having sampled a point $\underline{\mathbf{s}}$ in the GBNM is described by a Gaussian-Parzen-window approach (Duda, 2001):

$$f(\underline{\mathbf{s}}) = \frac{1}{N}\sum_{i=1}^{N} f_i(\underline{\mathbf{s}})\,, \tag{16}$$

where N is the number of points $\underline{s}_{(i)}$ already sampled. Such points come from the memory kept from the previous local searches, being, in the present version of the algorithm, all the starting points and local optima already found. $f_i(\underline{s})$ is the normal multidimensional probability density function given by:

$$f_i(\underline{s}) = \frac{1}{(2\Pi)^{n/2} det(\Sigma)^{1/2}} \times \exp\left(-\frac{1}{2}(\underline{s} - \underline{s}_{(i)})^T \Sigma^{-1} (\underline{s} - \underline{s}_{(i)})\right) \tag{17}$$

where n is the problem dimension and Σ is the covariance matrix:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} \tag{18}$$

and variances are estimated by the relation:

$$\sigma_j^2 = \beta \left(s_j^{max} - s_j^{min}\right)^2 \tag{19}$$

where β is a positive parameter that controls the length of the Gaussians, and s_j^{max} and s_j^{min} are the bounds of the j^{th} variable ($j = 1, 2$ in our case). To keep the method simple, such variances are kept constant during the optimization. After every local minimum is found, N points are randomly sampled $(\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_N)$ to restart the next search and the one that minimizes Eq.(16) is selected. The stopping criterion of the global optimization is the maximum number of function evaluations n_{max} defined *a priori* by the user.

6. NUMERICAL ANALYSIS

In this section some results are discussed. Section 6.1 shows the surface generated by d Eq. (12) considering the deterministic problem. The convergence of the Monte Carlo simulations to evaluate the mean and variance of D Eq. (13) are in Section 6.2. In Section 6.3 the robust optimization problem is solved and the results are compared to the ones of the deterministic optimization.

The data used in the simulations are: $m_1 = 1.5$ kg, $m_2 = 0.75$ kg, $c_1 = 0.5$ N.s/m, $c_2 = 0.05$ N.s/m, $900 \leq \underline{k}_1 \leq$ 1100 N/m, $130 \leq \underline{k}_2 \leq 170$ N/m, $t_1 = 0.2$ m/N, $t_2 = 0.02$ m/N, $\mathbb{B}_1 = [0, 3.5]$ Hz, $\mathbb{B}_2 = [3.5, 7]$ Hz, $\delta_{K_1} = 0.025$, $\delta_{K_2} = 0.025$. The parameters used in the GBNM are shown in Table 1.

Table 1. Parameters used for the optimization algorithm.

Parameter	value
N	10
n_{max}	5000
simplex size	5
β	0.01

6.1 Surface generated by d

Figure 3 shows an approximation of the surface generated employing Eq.(12) in function of $\mathbf{s} = (k_1, k_2)$.

It is noticed that, for the target chosen, we get a very complicated surface with many local minima. This point must be emphasized, note that the system considered for the analysis is very simple, nevertheless it turns out that the optimization problem is non-convex and multimodal.

6.2 Convergence of the stochastic solution

Monte Carlo simulations are employed to compute the mean and variance at each point of the robust optimization. The typical convergence curves for the estimator of the mean and variance of D are shown in Figures 4 and 5.

In all the numerical experiments, the sample size used is 2000.

6.3 Robust Design Optimization

Before presenting the results of the RDO, a comment has to be made regarding the computational cost of the optimization. As shown in section 6.2, each point evaluated by the GBNM in the RDO requires a sample of 2000 analysis,

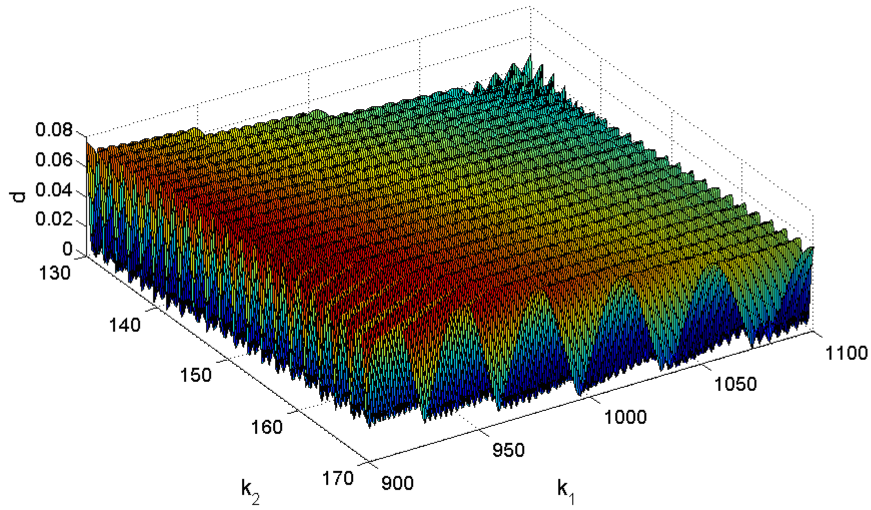

Figure 3. Surface generated by J using D_1.

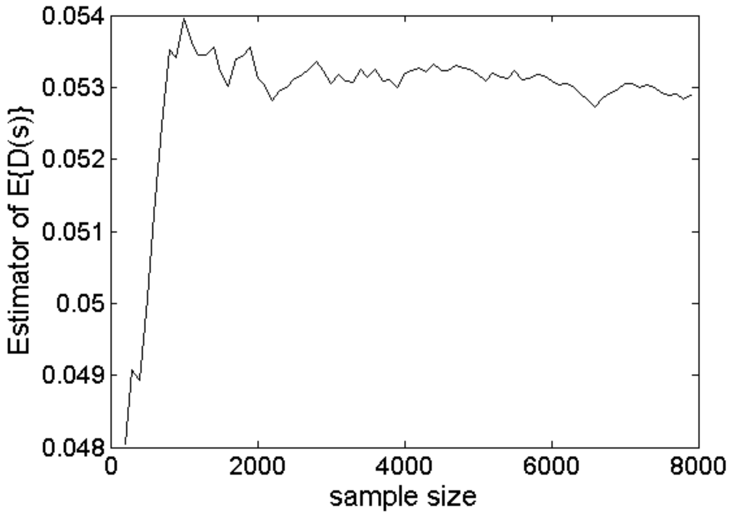

Figure 4. Convergence of the mean.

which costs 85.70 seconds in a Intel Pentium M 1.6GHz processor. Thus, each point in the Pareto frontier (Figure 6) took approximately 5 days to be computed.

Figure 6 shows the results in the objective space, formed by the normalized values of $E\{D(\underline{s})\}$ versus var$\{D(\underline{s})\}$ for each α considered. The trade-off between the mean and the variance can clearly be observed.

The results of the robust optimization using different values of α is shown in Table 2. It can be seen that when the value of α changes, different optimal results are obtained. The higher the α is, the better the mean value of the response is and the worse the variance of response is. Then, it is up to the designer to choose his prefered trade-off between the mean and the variance of the response for the structure under analysis.

A good feature of the GBNM algorithm is that it provides several local optima of the optimization problem. Such feature is explored in Figure 7, where the five best local optima of four different situations are shown: deterministic problem and robust problems for $\alpha = [0.0, 0.5, 1.0]$. Note that the bigger symbol of each situation gives the best design found, except for the deterministic case, where, of course, the response of all the five designs coincide. It can be seen that when the uncertainties of the system are considered, the optimum design changes, even in the case where only the mean of the target function is minimized ($\alpha = 1.0$). To see how different the deterministic and the robust optimization results are, Table 3 shows the mean and variance values of the five best local minima of the deterministic optimization. Table 3 shows clearly that the mean and variance of the deterministic optima are much higher than the optimum results of the robust optimization (Table 2), especially when the variance is considered. One sees then, when the uncertainties are

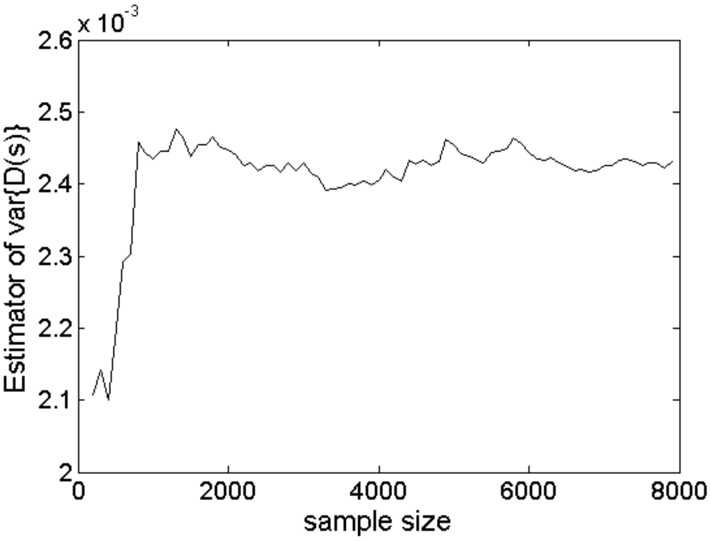

Figure 5. Convergence of the variance.

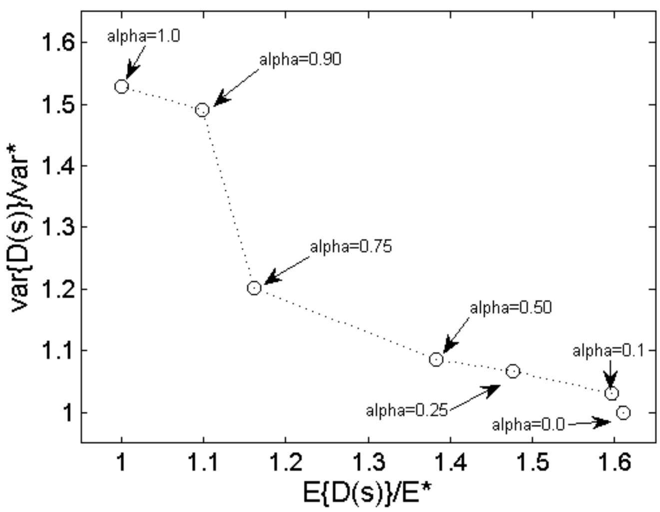

Figure 6. Pareto Front.

not considered, the deterministic optimization results are poor. Moreover, the results show that the robust optimization is a very useful tool to deal with uncertainties in dynamical systems.

Table 2. Results of the multi-objective optimization.

α	$E\{K_1\}$	$E\{K_2\}$	$E\{D(\underline{s})\}$	var$\{D(\underline{s})\}$
0.00	1096.27	139.72	0.0481	0.2528 $\times 10^{-3}$
0.10	1097.11	140.33	0.0477	0.2690 $\times 10^{-3}$
0.25	1090.04	141.29	0.0441	0.2890 $\times 10^{-3}$
0.50	1094.90	144.81	0.0413	0.2993 $\times 10^{-3}$
0.75	1099.96	151.57	0.0347	0.3648 $\times 10^{-3}$
0.90	1097.04	158.62	0.0328	0.5617 $\times 10^{-3}$
1.00	1100.00	161.64	0.0299	0.5905 $\times 10^{-3}$

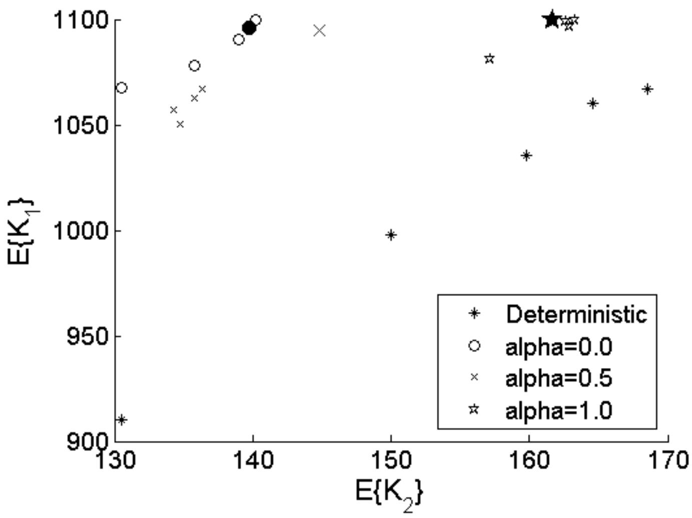

Figure 7. Five best local minima found for the deterministic case and for the RDO with different α values.

Table 3. Mean and variance of the deterministic optima.

$E\{K_1\}$	$E\{K_2\}$	$E\{D(\underline{s})\}$	var$\{D(\underline{s})\}$
998.08	150.00	0.0458	0.0017
1060.18	164.55	0.0402	0.0012
1066.78	168.56	0.0405	0.0014
910.07	130.52	0.0585	0.0027
1035.36	159.79	0.0443	0.0016

7. CONCLUDING REMARKS

This paper dealt with the target performance optimization of a simple stochastic dynamical system subject to uncertainties in two parameters. To consider such uncertainties of the system, the parametric approach was used and the probability density functions were derived using the Maximum Entropy Principle. To take into account the uncertainties of the dynamical system in the optimization, a multi-objective optimization of some statistical characteristics of a distance between the response of the system and the target performance was proposed and solved using the weighted sum approach. The GBNM algorithm was employed in the optimization due to its ability to handle non-convex and multimodal functions. The results showed that: (i) even for a very simple system, the optimization problem can be complicated, (ii) the GBNM has successfully dealt with the optimization of the stochastic function; (iii) when the uncertainties are considered, the robust optimum design is different from the deterministic optimization optimum, and (iv) the robust optimization is a very useful tool to deal with uncertainties in dynamical systems.

8. ACKNOWLEDGEMENTS

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