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WHAT IS A VORTEX?

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Abstract. *Vorticity and vortex form an interesting duality. Vorticity has a mathematical definition but its physical interpretation is not well understood. On the other hand, a vortex is recognized from subjective and intuitive basis and does not have a rational definition. A mathematical definition of a vortex became an important issue in Fluid Mechanics specially after the recognition of the importance of vortical coherent structures on the turbulence dynamics. The birth, evolution, dissipation and death of a vortical coherent structure plays a crucial role on the understanding of turbulence as a phenomenon. Another interesting application is related to the drag reduction that occurs when a few parts per million of a polymer with high molecular weight and long chains is added to a Newtonian solvent, without changing its viscosity. The drag reduction is closely related to a dramatic reduction on the vortical coherent structures regions. A number of existent vortex definitions are discussed from the perspective of frame invariance, interpretation of vorticity, the role of the rate-of-rotation of the eigenvectors of the rate-of-strain tensor, Eulerian X Lagrangean approach and convected time derivatives. A new definition, that combine the basic features that a vortex definition should have, is proposed.*

Keywords: *Vortex; Coherent vortical structure; Frame invariance*

1. INTRODUCTION

Although the word *vortex* is frequently used when one wants to describe, understand, and explain flow patterns in fluid dynamic problems, the connection of this word to an entity which is unambiguously identified is still controversial. The mission of the fluid mechanicist which is involved in such identification is, therefore, to propose a mathematical definition of a concept which was constructed by its use, and not by a definition or a convention, during centuries of analysis of fluid mechanics problems.

Needless to say, the radical of the word vortex and the word vorticity are the same. The logical conclusion is that these two entities are closely related. This connection is an interesting start since vorticity is uniquely defined (mathematically). The only lack of agreement comes from an unimportant factor of 2. Although it can be thought as a skew-symmetric tensor or a vector, it is straightforward that one is referring to the same entity since a skew-symmetric tensor has only three independent components. The mathematical definition of vorticity, which will be explored later, associates vorticity with rate of rotation. This connection is a key one and helps to interpret what is been captured by this well defined quantity. It is generally accepted that a vortex is an entity where vorticity is *high*. The problem now turns to propose a mathematical condition that gives an appropriate quantification of the subjective word “high".

The intriguing question that someone who is invited to investigate the subject asks is “Why there is a controversy, or, in other words, why the previous definitions fail to be consensually accepted by the fluid mechanics community ?". The answers to these questions are not easy. A first attempt to do this is to discriminate the opposite poles that are considered by the different authors who have investigated this matter.

A first bi-polar strength present in the literature is the *CAUSE X MANIFESTATION* one. The approaches considered to identify a vortex can be, on one side, based on dynamics or force related quantities, entities related to the cause of the patterns of a flow. On the other side, the identification can be based on the manifestation, or the kinematics that is presented by the flow. It is worth mentioning, however, that even when the “cause" branch is used, the approach generally ends with some manifestation mathematic criterion.

A second opposition is the *LAGRANGEAN X EULERIAN* approaches. In fact, is not very clear in the literature if a vortex should be defined as a region in space which has certain instantaneous properties or a set of fluid particles that undergoes a particular trajectory in time.

Another unclear issue in the literature is related to kind of transformation of the frame of reference which leads the entities, necessary to the calculation of the considered criterion, invariant. Is the opposition *GALILEAN X GENERAL FRAME* invariance. There are authors that consider the criterion should be Galilean invariant while others are more restrictive and advocate the objectivity of the criterion, in other words, that the criterion should be invariant to any kind of rigid transformation.

The other controversial issue is related to the introduction of thresholds in the criterion. It is a *SUBJECTIVE XOBJECTIVE* bi-polarization. Although there are some advantages, associated to the flexibility of the criterion, since the threshold can depend on the problem considered, this flexibility implies a subjective criterion, i.e. depends on the value of the threshold inputed by the fluid mechanicist analyzing the problem.

The great majority of the investigations examine the importance of vorticity inside the velocity gradient. This approach is based on the fact that the velocity gradient is a sum of a symmetric (also called rate-of-strain tensor or rate-of-deformation tensor) and a skew-symmetric (also called vorticity tensor or rate-of-rotation tensor) parts. Therefore, one can do the mental exercise described next. Start with a fixed symmetric velocity gradient and increase, from zero, the skew-symmetric part until this skew-symmetric part achieves a threshold where it becomes important *in a certain sense* inside the velocity gradient tensor. The two most criteria used in the literature are based on this approach and their difference is the condition which identifies that the vorticity have become important.

On the other side, one can measure the role of vorticity inside the strain acceleration tensor. The idea is that for a material filament to have a rate of rotation which is comparable to the rate of linear deformation, means that the strain acceleration tensor is *somehow* not aligned with the rate-of-deformation tensor. This point will be explored later.

2. INADEQUACIES OF SOME INTUITIVE CRITERIA

The work of Jeong and Hussain (1995) is one of the first reviews on the subject of vortex identification. After this work it become established that some intuitive measures such as pressure minimum, closed or spiraling streamlines and pathlines, and isovorticity surface are inadequate in identifying vortices in a general flow.

2.1 Local pressure minimum

In steady inviscid planar flow occurs a cyclostrophic balance, i.e. the pressure force balances the centrifugal force. When this happens, pressure has a local minimum on the axis of the swirling motion, in a vortex. However, the requirement of a pressure minimum as a criterion for detecting a vortex in a general 3D flow is inadequate since pressure forces can balance with the unsteady irrotational term of inertia and/or the viscous forces.

2.2 Closed or spiraled streamlines and pathlines

In a vortex, because of the swirling motion, a common intuitive idea is to identify a vortex as a region where the streamlines and pathlines are closed or spiraled. The first problem of this definition is that this criterion is not Galilean invariant (and consequently not general frame invariant). Because of that, this definition will obscure two or more vortices moving at different speeds in any single reference frame. Another critical point is that a particle may not complete a full revolution around a vortex center during the lifetime of a vortex.

2.3 Vorticity magnitude

Although widely used, level of vorticity magnitude is not reasonable to detect a vortex. A shear flow between parallel plates, for example can exhibit high vorticity, without having a swirling motion. In many situations in laminar and turbulent flows, the vorticity is high near the wall and the wall must be excluded from a vortex core. Another well-known example is an axisymmetric vortex with a strong axial variation in vorticity in which the intensity of vorticity surface indicates segmented vortices, although there is only one continuous vortex column.

3. CLASSICAL CRITERIA FOR VORTEX IDENTIFICATION

The classical criteria for vortex identification, known as the Q-criterion (Hunt et al., 1988), the Δ-criterion (Chong et al., 1990), and λ_2-criterion (Jeong and Hussain, 1995) are the most widely used criteria in the literature. They have proved their superiority over the previous intuitive criteria described before.

3.1 Hunt et al. (1988) criterion

The criterion proposed by Hunt et al. (1988) is intrinsically related to a competition between vorticity and rate-of-strain where, in the case of a vortex, vorticity wins. Hunt et al. (1988) define a vortex as a connected region in space where

$$Q = \frac{1}{2}\left(|\mathbf{W}|^2 - |\mathbf{D}|^2\right) > 0 \qquad (1)$$

where $\mathbf{W}$ and $\mathbf{D}$ are, respectively the skew-symmetric and symmetric parts of the velocity gradient and the operator $(|\,|$ indicates the Euclidean norm of a tensor. Therefore, the competition between rate of rotation and rate of deformation is translated by the difference between the the Euclidean norm of each part, symmetric and skew-symmetric, of the velocity gradient. A vortex is identified where vorticity dominates the rate of deformation.

It is worth noticing that the Q-criterion is strictly related to the Vorticity number introduced by Truesdell (1953), defined as

$$N_K = \frac{|\mathbf{W}|}{|\mathbf{D}|} \qquad (2)$$

and interpreted as a "measure of the quality of the vorticity". $Q > 0$ is equivalent to $N_K > 1$.

In summary, the Q-criterion is not generally frame indifferent, since it is dependent on the vorticity. It is an Eulerian approach. It does not give a clear picture of how it can be extended to compressible flows, one can keep the same difference or work with the new second invariant. It has a non-subjective definition.

3.2 Chong et al. (1990) criterion

A second criterion was formulated by Chong et al. (1990). This criterion is based on the fact that, when vorticity vanishes, the eigenvalues and eigenvectors of the velocity gradient are (the same as the rate-of-strain) real, since the velocity gradient, in this case is symmetric. If we gradually increase the vorticity, there is a threshole which is eventually achieved, where there will be a real and two complex conjugates eigenvalues. Therefore, the importance of vorticity changes the nature of the eigenvalues of the velocity gradient and produce a rotation like behavior. The so-called $\Delta - criterion$ is given by a region where

$$\Delta = \frac{Q^3}{27} + \frac{III_L^2}{4} > 0 \tag{3}$$

where III_L is the third invariant (determinant) of the velocity gradient.

The Δ-vortex is a larger region than a Q-vortex, since $Q > 0 \Rightarrow \Delta > 0$. This also shows that, to produce complex eigenvalues, the vorticity intensity measured by its norm, may not overcome the rate-of-strain intensity with the same measurer.

3.3 Jeong and Hussain (1995) criterion

Another very important criterion in the literature was proposed by Jeong and Hussain (1995). This criterion is based on a pressure minimum at the vorticity plane. The gradient of the Navier-Stokes equation can be separated into a symmetric and skew-symmetric parts. The skew-symmetric part is related to the evolution of vorticity, while the symmetric part is connected to the evolution of the rate-of-strain. The symmetric part of the equation is given by

$$\dot{\mathbf{D}} - \nu\Delta\mathbf{D} + \mathbf{D}^2 + \mathbf{W}^2 = -\frac{1}{\rho}\mathbf{P} \tag{4}$$

where $\Delta\mathbf{D}$ is the laplacian of $\mathbf{D}$ and $\mathbf{P}$ is the pressure Hessian. According to Jeong and Hussain (1995), the principle of minimum pressure, can be corrected by discarding the terms related to unsteadiness of the flow and to viscous forces, this condition is satisfied, for an incompressible Newtonian fluid, when

$$\lambda_2^{\mathbf{D}^2+\mathbf{W}^2} < 0 \tag{5}$$

where $\lambda_2^{\mathbf{D}^2+\mathbf{W}^2}$ is the intermediate eigenvalue of tensor $\mathbf{D}^2 + \mathbf{W}^2$. It is interesting to notice that, although expressed by kinematic quantities, this criterion is based on dynamical arguments.

4. OTHER IMPORTANT CRITERIA

4.1 Tabor and Klapper (1994) criterion

Tabor and Klapper (1994) presented a systematic study on the stretching and alignment dynamics in general flows and came up with an interesting kinematic tensor very relevant to the present work. This tensor, $\mathbf{\Omega}$, measures the rate of rotation of the eigenvectors of $\mathbf{D}$, defined as

$$\mathbf{\Omega} = \dot{\mathbf{e}}_i^D \mathbf{e}_i^D. \tag{6}$$

where $\mathbf{e}_i^D$ is an eigenvectors of $\mathbf{D}$. They used, in that study, the Relative-rate-of-rotation tensor, $\overline{\mathbf{W}}$, the difference between the vorticity tensor and $\mathbf{\Omega}$. Therefore, a Q_D-criterion criterion can be constructed as

$$Q_D = \frac{1}{2}\left(|\mathbf{W} - \mathbf{\Omega}|^2 - |\mathbf{D}|^2\right) > 0 \tag{7}$$

It is worth noticing that the relation between Q and the vorticity number, N_K is analog to the relation between Q_D and the "stress-relieving" parameter, R_D, proposed by Astarita (1979), defined as

$$R_D = -\frac{\mathrm{tr}\overline{\mathbf{W}}^2}{\mathrm{tr}\mathbf{D}^2} \tag{8}$$

$Q_D > 0$ is equivalent to $R_D > 1$.

4.2 Kida and Miura (1998) criterion

The criterion proposed by Kida and Miura (1998) follows the same principle considered by Jeong and Hussain (1995). The difference lies on the fact that the pressure minimum is calculated at the plane defined by the eigenvector correspondent to the smallest eigenvalue of the pressure Hessian. The vortex core is defined as a region where the skew-symmetric part of the velocity gradient projected on this plane overcomes its symmetric part.

4.3 Zhou et al. (1999)

The so-called λ_{ci}-criterion was introduced by Zhou et al. (1999). It is based on the Δ-criterion of Chong et al. (1990). When $\Delta > 0$, the velocity gradient has two complex eigenvalues $\lambda_{cr} \pm i\lambda_{ci}$. The imaginary part λ_{ci} is identified as the *swirling strength* of the vortex. The criterion consists of a $\lambda_{ci}^2 > \delta$, where δ is a threshold generally chosen as percentage of its maximum value. When $\delta = 0$, $\Delta > 0$ and $\lambda_{ci}^2 > \delta$ are equivalent.

4.4 Cucitore et al. (1999) criterion

Maybe the first non-local criterion to identify a vortex structure was proposed by Cucitore et al. (1999). The main argument is that in a *structure* that is formed in space there should be an inter-dependence among the particles that compose this structure, and, therefore, there should be a non-local relation. In this connection, Cucitore et al. (1999) introduce a Galilean invariant measurer of the relative distance between pair of particles and define a vortex as a set of particles where this relative distance is *small*. The quantity used to quantify such relative distance was

$$\mathbf{R}(\mathbf{x}, t) = \frac{\left| \int_0^t \mathbf{u}_a(\tau) d\tau - \int_0^t \mathbf{u}_b(\tau) d\tau \right|}{\int_0^t \left| \mathbf{u}_a(\tau) - \mathbf{u}_b(\tau) \right| d\tau} \tag{9}$$

4.5 Haller (2005) criterion

Another criterion based on a non-local vortex definition was presented by Haller (2005). He considered a vortex as a set of fluid trajectories that avoid the so-called hyperbolic domain, a domain defined as a region in space where the fluid defies, *in a certain sense*, the trend suggested by the rate-of-strain. To define the hyperbolic domain Haller (2005) uses (half of) the second Rivlin-Ericksen tensor, $\mathbf{A}_2$, the covariant convected time derivative of the rate of deformation tensor, $\mathbf{A}_1 = 2\mathbf{D}$, defined as

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1 \mathbf{L} + \mathbf{L}^T \mathbf{A}_1 \tag{10}$$

where $\mathbf{L}$ is the transpose of the velocity gradient. For flows where the first invariant of $\mathbf{D}$ vanishes ($I_D = 0$, isochoric flows) and the third invariant of $\mathbf{D}$ is a non-zero quantity ($III_D \neq 0$), he defines an elliptical cone on the basis of the eigenvectors of $\mathbf{D}$, ($\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$) as

$$d\xi_3^2 = a\, d\xi_1^2 + (1-a) d\xi_2^2 \tag{11}$$

where $d\boldsymbol{\xi} = d\xi_1 \mathbf{e}_1 + d\xi_2 \mathbf{e}_2 + d\xi_3 \mathbf{e}_3$ is an infinitesimal vector and a is the ratio between the greatest and smallest eigenvalues of $\mathbf{D}$. The hyperbolic domain is a region in space where the second Rivlin-Ericksen tensor is positive definite in the elliptical cone defined by Eq.(11).

4.6 Chakraborty et al. (2005) criterion

Chakraborty et al. (2005) proposed a further step on the analysis of Zhou et al. (1999) by adding to the swirling strength criterion, an *inverse spiraling compactness*, measured by the ratio $\frac{\lambda_{cr}}{\lambda_{ci}}$. This ratio can be seen as local version of the non-local quantity introduced by Cucitore et al. (1999).

4.7 Zhang and Choudhury (2006) criterion

A special criterion for compressible flows was developed by Zhang and Choudhury (2006). It is based on the so-called *eigen helicity density*, H_e, given by

$$H_e = \mathbf{n}_{swirl} \cdot \mathbf{w} \tag{12}$$

where $\mathbf{w}$ is the vorticity vector and $\mathbf{n}_{swirl} = -\frac{i}{2}(\mathbf{e}_1 \times \mathbf{e}_2)$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are the complex eigenvectors correspondent to the complex eigenvalues of the velocity gradient.

5. What vorticity represents?

The vorticity tensor, $\mathbf{W}$, is defined as the skew-symmetric part of the velocity gradient

$$\mathbf{W} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right), \tag{13}$$

and the vorticity vector, $\mathbf{v}$ as

$$\mathbf{w} = \frac{1}{2}\nabla \times \mathbf{v}. \tag{14}$$

A vorticity component, $W_{ij} \equiv -w_k$ (where ijk form a positive triad) is given by

$$W_{ij} = -w_k = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right). \tag{15}$$

The skew-symmetry of the tensor implies that $i = j \Rightarrow W_{ij} = 0$. For $i \neq j$, the term $\frac{\partial v_i}{\partial x_j}$ represents the instantaneous rotation rate of a filament aligned with x_j direction towards the x_i direction. Analogously, $-\frac{\partial v_j}{\partial x_i}$ represents the instantaneous rotation rate of a filament aligned with x_i direction towards the $-x_i$ direction. Since these two rotation rates are *in the same direction*, and, the vorticity component at a plane remains unchanged with rotations of the axis in that plane, this component is interpreted as the arithmetic mean of the rate of rotations of two filaments mutually orthogonal. Therefore, vorticity is associated to mean rate of rotation, to a measure of the rate of rotation in a certain plane.

It is worth observing that, another interpretation, very relevant to the present work, can be considered. The question is the following: are there material filaments that rotate with the angular velocity? The answer to this question is yes. And the filaments where this happens are exactly the material filaments which are aligned with the eigenvectors of $\mathbf{D}$. This fact markedly separates, from the vorticity perspective, flows where the eigenvectors of $\mathbf{D}$ have a rate of rotation from other flows where these eigenvectors do not change the rotation with time. The problem is that the same flow seen by two observers that are rotating relative to each other, will measure different vorticities. In other words, vorticity is a non-objective quantity. On the other hand, the rate-of-rotation of a material filament relative to the rate of rotation of the eigenvectors of $\mathbf{D}$ are independent of the observer. **?** conducted an investigation in the 2D context where the relative rate of rotation is identified as a natural measurer of the rate of rotation of the fluid. This result indicates that the relative-rate-of-rotation tensor, also called “efective-rate-of-rotation" (Hua and Klein, 1999) or “absolute vorticity" (Gatski and Jongen, 2000), do play an important role on fluid mechanics.

6. General basis for vortex identification

A criterion to identify a vortex is also related to the partition of the flow into sub-domains associated to rotation, filamentation and the boundaries between these. Another way of describing this idea is to called these sub-domains: elliptical, hyperbolical, and parabolical.

Maybe the study developed by Astarita (1979) is the first one that considers the relative rate of rotation as a fundamental quantity. He classified motions with a “stress-relieving" parameter, R_D, to classify flows based on the competition between the relative-rate-of-rotation tensor and the rate-of-strain tensor as follows

$$R_D = -\frac{\mathrm{tr}\overline{\boldsymbol{W}}^2}{\mathrm{tr}\boldsymbol{D}^2} \tag{16}$$

Astarita (1979) defined some basis for flow classification criteria and criticized previous works on the subject. The main properties a flow classifier should have are

- Local. The parameter should be calculated at a point.
- Objective. The parameter should be indifferent under changes of reference frames.
- Generally applicable. The parameter should be applicable to any kind of flow, and not to particular ones.

Following Thompson and Souza Mendes (2005), the persistence of straining tensor is defined as

$$\mathbf{P} = \boldsymbol{D}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{D} \tag{17}$$

where the overline indicates that vorticity is computed relative to the rotation of the eigenvectors of $\mathbf{D}$.

The persistence-of-straining parameter is defined as a dimensionless intensity of the persistence-of-straining tensor and is given by

$$\mathcal{R} \equiv \frac{\sqrt{\frac{1}{2}\mathrm{tr}\left[(\boldsymbol{D}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{D})^2\right]}}{\mathrm{tr}\left[\boldsymbol{D}^2\right]} \tag{18}$$

For the understanding and interpretation of $\mathbf{P}$ and $\mathcal{R}$ we refer the reader to the papers Thompson and Souza Mendes (2005), Thompson and Souza Mendes (2005b) and Thompson (2008).

7. The anisotropic criterion

7.1 Measurement of the diagonal-dominance in a matrix

It is very common, in many physical and mathematical situations, the identification of the necessity to compare the diagonal components of a matrix with its off-diagonal ones. One simple idea is to measure this competition by an overall ratio index. A parameter which has in the numerator and the denominator, the intensities of one and other sides of this balance: diagonal and off-diagonal components. Let $\mathbf{A}$ be a general square matrix. This matrix can be splitted into two other ones, one with the diagonal components of $\mathbf{A}$, $\mathbf{A}_{diag}$, and the other with its off-diagonal components, $\mathbf{A}_{off-diag}$, such that $\mathbf{A} = \mathbf{A}_{diag} + \mathbf{A}_{off-diag}$, or

$$[\boldsymbol{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} + \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{bmatrix} \tag{19}$$

The intensity of a second order tensor $\mathbf{A}$ (e.g. (Truesdell and Noll, 1965)) is given by $\|\mathbf{A}\| = \sqrt{\mathrm{tr}\mathbf{A}\cdot\mathbf{A}^{\mathrm{T}}}$. Therefore, an index that can measure how *off-diagonal* is matrix $\mathbf{A}$ is

$$Index_I = \frac{\|\mathbf{A_{off-diag}}\|}{\|\mathbf{A_{diag}}\|} \tag{20}$$

Here, we have developed two methods for an anisotropic comparison between the diagonal and off-diagonal components of a matrix. By anisotropic we mean that these two methods provide diferent values depending on the direction we evaluate this comparison. The first method which will be called here *line-method* is to compare, in the diagonal components of the tensor $\mathbf{A}^2$, the part of each component that comes from the diagonal and off-diagonal component of tensor $\mathbf{A}$. For a symmetric tensor, to produce a line-index, we made this comparison by the following formulae

$$Index_l(i) = \frac{(A_{ii})^2}{[\mathbf{A}^2]_{ii}} \tag{21}$$

The version for a non-symmetric tensor is given by

$$Index_l(i) = \frac{(A_{ii})^2}{(A_{ii})^2 + |A_{ij}A_{ji}| + |A_{ik}A_{ki}|} \tag{22}$$

The second method, called here *surface-method* provides an anisotropic comparison evaluating a projection of tensor $\mathbf{A}$ into a domain of one order below. In the case of a tensor of second order, we will have three square matrices $2X2$ whith two diagonal components and two off-diagonal ones. In this case we compute the trace of the square of the sub-matrices formed and see which part comes from the diagonal and off-diagonal componets of $\mathbf{A}$. The normalized surface-index, $index_s$ is given by

$$Index_s(k) = \frac{(A_{ii} - A_{jj})^2}{\Delta_{A_k}} \tag{23}$$

where $\mathbf{A}_k$ is the projection of tensor $\mathbf{A}$ out of the $k-subspace$. $\mathbf{A}_k$ is given by the linear operation

$$\mathbf{A}_k = \widetilde{\boldsymbol{\psi}}_k : \mathbf{A} \tag{24}$$

where $\widetilde{\boldsymbol{\psi}}_k$ is a fourth order tensor given by (no sum)

$$\widetilde{\boldsymbol{\psi}}_k = \mathbf{e}_i\mathbf{e}_i\mathbf{e}_i\mathbf{e}_i + \mathbf{e}_i\mathbf{e}_j\mathbf{e}_j\mathbf{e}_i + \mathbf{e}_j\mathbf{e}_i\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_j\mathbf{e}_j\mathbf{e}_j \tag{25}$$

where $\mathbf{e}_i$, $\mathbf{e}_j$, and $\mathbf{e}_k$ form a positive triad of orthonormal vectors which constitutes the basis where $\mathbf{A}$ was written. The quantity Δ_{A_k} is the discriminant of characteristic equation of $\mathbf{A}_k$. It can be shown that, $0 \leq Index_s(k) \leq 1$ and that, as

$$Index_s(k) \rightarrow 1 \Rightarrow \mathbf{A}_k \rightarrow \begin{bmatrix} A_{ii} & 0 \\ 0 & A_{jj} \end{bmatrix} \tag{26}$$

7.2 Particular matrices considered

In the case of the velocity gradient, we have that

$$[\boldsymbol{L}] = \begin{bmatrix} \lambda_1^D & -\bar{w}_3 & \bar{w}_2 \\ \bar{w}_3 & \lambda_2^D & -\bar{w}_1 \\ -\bar{w}_2 & \bar{w}_1 & \lambda_3^D \end{bmatrix} = \begin{bmatrix} \lambda_1^D & 0 & 0 \\ 0 & \lambda_2^D & 0 \\ 0 & 0 & \lambda_3^D \end{bmatrix} + \begin{bmatrix} 0 & -\bar{w}_3 & \bar{w}_2 \\ \bar{w}_3 & 0 & -\bar{w}_1 \\ -\bar{w}_2 & \bar{w}_1 & 0 \end{bmatrix} \quad (27)$$

Following Haller (2005) and Thompson (2008) we use the matrix associated with the second Rivlin-Ericksen tensor, Eq.(10), written on the basis of the eigenvectors of the first Rivlin-Ericksen tensor, as our base-matrix. The acceleration of a deformation is measured by this quantity. Therefore, $\mathbf{A}_2$ can be written as

$$[\boldsymbol{A}_2] = \begin{bmatrix} \lambda_1^2 + \dot{\lambda}_1 & (\lambda_2 - \lambda_1)\bar{w}_3 & (\lambda_1 - \lambda_3)\bar{w}_2 \\ \cdot & \lambda_2^2 + \dot{\lambda}_2 & (\lambda_3 - \lambda_2)\bar{w}_1 \\ \cdot & \cdot & \lambda_3^2 + \dot{\lambda}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \dot{\lambda}_1 & 0 & 0 \\ 0 & \lambda_2^2 + \dot{\lambda}_2 & 0 \\ 0 & 0 & \lambda_3^2 + \dot{\lambda}_3 \end{bmatrix} +$$

$$\begin{bmatrix} 0 & (\lambda_2 - \lambda_1)\bar{w}_3 & (\lambda_1 - \lambda_3)\bar{w}_2 \\ \cdot & 0 & (\lambda_3 - \lambda_2)\bar{w}_1 \\ \cdot & \cdot & 0 \end{bmatrix} \quad (28)$$

and the indexes can be constructed in the same fashion as was done generically with tensor $\mathbf{A}$ in the previous section. In other words, we produce 6 different indices to classify motions, 3 line indices and three surface indices, with respect to the directions of the eigenvectors of $\mathbf{D}$.

The other possibility considered is to choose a specific direction, for example the relative vorticity vector, $\overline{\mathbf{w}}$, and execute the analysis described above in the plane defined by this direction. For this purpose, it is necessary to find the projections of tensors $\mathbf{A_2}$ and $\mathbf{D}$ in this plane, $\mathbf{A_{2\overline{w}}}$ and $\mathbf{D_{\overline{w}}}$, respectively, calculate the eigenvectors of $\mathbf{D_{\overline{w}}}$, and write $\mathbf{A_{2\overline{w}}}$ in this basis.

8. FINAL REMARKS

We have presented a theoretical analysis to capture directional tendencies of stretching material elements. These directional quantities are able to delineate coherent structures that are present in turbulent flows. Besides that they are strongly related to flow-type classification criteria, giving an anisotropic version of previous criteria in the literature. The theoretical entities introduced are applied in a accompanied paper. This study opens a systematic procedure, not only to identify, but also, to classify a vortex.

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11. Responsibility notice

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