THE GENERALIZED FINITE ELEMENT METHOD APPLIED TO FREE VIBRATION OF BEAMS

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Abstract. *This work deals with the hierarchical Generalized Finite Element Method (GFEM) applied to free vibration of beams. The variational problem of free vibration is formulated and the main aspects of GFEM are discussed. The Assumed Mode Method (AMM), the Composite Element Method (CEM) and the GFEM are presented. The AMM and the CEM are developed by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from closed form solutions of the Classical Theory of free vibration. These approaches result in hierarchical refinements. The GFEM is developed by enrichment of the conventional FEM solution space with knowledge about the differential equation being solved applying the Partition of Unity Method (PUM). The application of GFEM in vibration analysis of beams is investigated. The eigenvalues obtained by GFEM are compared with those obtained by analytical solution, by CEM and by h and p-versions of FEM. The numerical results show that GFEM proposed in this work has higher rates of convergence than the h-version of FEM and the CEM.*

Keywords*: generalized finite element method, free vibration, vibration analysis*

1. INTRODUCTION

Nowadays the vibration aspects of mechanical systems have significant importance in design processes, especially in automotive and aeronautical fields. The study of the dynamical behavior of structures is important to obtain an optimum design of them and to avoid vibration problems such as material fatigue, noise and human discomfort. In last years many researchers have developed vibration analysis methods.

The Finite Element Method (FEM) is commonly used in vibration analysis and its approximated solution can be improved using two refinement techniques: *h* and *p*-versions. The *h*-version consists of the refinement of element mesh, on the other side, the *p*-version may be understood as the increase in the number of form functions in the element domain without any change in the mesh (Ribeiro, 2001; Campion and Jarvis, 1996). The conventional *p*-version of FEM consists of increasing the polynomial degree in the solution.

Some enriched methods based on the FEM have been developed in last years. In the Spline Finite Element Method (SFEM), proposed by Leung and Au (1990), the displacement field to vibration analysis of beams and plates is described by B_3-spline form functions. The B_3-spline functions are computationally efficient and flexible to model different boundary conditions. Engels (1992) and, Ganesan and Engels (1992) presented the Assumed Mode Method (AMM) which is obtained adding to the FEM form functions set some interface restrained assumed modes. The Composite Element Method (CEM) (Zeng, 1998a,b,c) is obtained by enrichment of the conventional FEM local solution space with non-polynomial functions obtained from analytical solutions of simple vibration problems. This approach results in a hierarchical refinement called *c*-version. A modified CEM applied to analysis of beams was proposed by Lu and Law (2007). The use of products between polynomials and Fourier series instead of polynomials alone in the element shape functions is recommended by Leung and Chan (1998). They developed the Fourier *p*-element applied to the vibration analysis of bars, beams and plates.

The Generalized Finite Element Method (GFEM) was developed based on the ideas of the Partition of Unity Method (PUM) (Melenk and Babuska, 1996 and Babuska, Banerjee and Osborn, 2004). In the GFEM, the local solution spaces are formed for functions, not necessarily polynomial, that permits inclusion of a priori knowledge about fundamental solution of the governing differential equation. This procedure ensures good local approximation. The local spaces are grouped in the approximated solution space by the Partition of Unity Method that ensures good global approximation. Arndt, Machado and Scremin (2007) proposed a generalized finite element for vibration analysis of straight bars based on the enrichment function of CEM.

This work presents a first application of the GFEM in free vibration analysis of beams. The results are compared with the results obtained by *h* and *p* refinements of FEM and by CEM.

2. VARIATIONAL FORM OF THE FREE VIBRATION OF EULER-BERNOULLI BEAMS

The Euler-Bernoulli beam consists of a straight beam with lateral displacement (Fig. 1). The basic hypotheses are (Craig, 1981): (a) The *x* axis (neutral axis) of the beam undergoes no extension or contraction; (b) Cross sections perpendicular to the neutral axis in the undeformed beam remain perpendicular to the deformed neutral axis, that is, transverse shear deformation is neglected; (c) The material is linearly elastic and the beam is homogeneous at any cross section; (c) Normal stresses σ_y and σ_z are negligible compared to the axial stress σ_x ; and (d) The *xy*-plane is a principal plane.

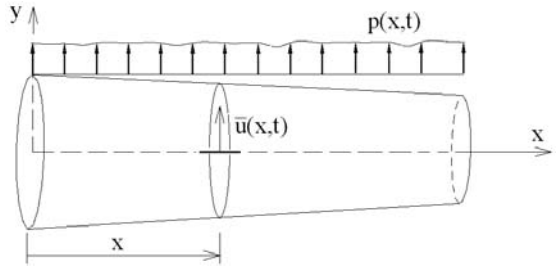

Figure 1. Straight beam with lateral displacement

The vibration of the beam is a time dependent problem. The PDE (Partial Differential Equation) that governs this problem may be written as: find the transversal displacement $\bar{u} = \bar{u}(x,t)$ which satisfies

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \bar{u}}{\partial x^2} \right) + \rho A \frac{\partial^2 \bar{u}}{\partial t^2} = p(x,t) \tag{1}$$

where *I* is the second moment of area, *A* is the cross section area, *E* is the Young modulus, ρ is the specific mass, p is the externally applied transversal force per unit length and *t* is the time. The solution $\bar{u} = \bar{u}(x,t)$ must satisfy boundary and initial conditions defined in the problem.

According to Carey and Oden (1984), the most popular way to obtain the variational form of a time dependent problem is consider the time *t* like a real parameter and develop a family of variational problems in *t*. This consists in selecting test functions $v = v(x)$, independent of *t*, and applying the weighted-residual method. If the finite element method is used to represent the spatial behavior of the solution, one obtains a system of ordinary differential equations of the degrees of freedom as functions of the parameter *t*. This approach is called semi discrete formulation of the problem.

Using the weighted-residual method, according to Carey and Oden (1984), to develop an integral statement of Eq.(1), the solution $\bar{u} = \bar{u}(x,t)$ must satisfy

$$\int_0^L \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \bar{u}}{\partial x^2} \right) v dx + \int_0^L \rho A \frac{\partial^2 \bar{u}}{\partial t^2} v dx = \int_0^L p(x,t) v dx \tag{2}$$

for admissible test functions $v = v(x)$ at any time $t \in (0,T]$.

Integrating Eq. (2) by parts twice, one obtains:

$$v \frac{\partial}{\partial x}\left(EI \frac{\partial^2 \bar{u}}{\partial x^2} \right)\Bigg]_0^L - \frac{\partial v}{\partial x}\left(EI \frac{\partial^2 \bar{u}}{\partial x^2} \right)\Bigg]_0^L + \int_0^L \left(EI \frac{\partial^2 \bar{u}}{\partial x^2} \right) \frac{\partial^2 v}{\partial x^2} dx + \int_0^L \rho A \frac{\partial^2 \bar{u}}{\partial t^2} v dx = \int_0^L p v dx \tag{3}$$

It is necessary to introduce the boundary and initial conditions to complete the problem. The admissible test functions $v = v(x)$ must satisfy the same boundary conditions of the solution $\bar{u}(x,t)$. For classical boundary conditions in boundaries $x = 0$ and $x = L$, the two first terms in Eq. (3) vanishes. According to Carey and Oden (1984), in vibration problems, one assumes periodic solutions $\bar{u}(x,t) = e^{i\omega t} u(x)$, where ω is the vibration

frequency. Thus, particularizing the problem to the case of free vibration of a straight Euler-Bernoulli beam where E and ρ are constants and $p(x,t)=0$, it becomes an eigenvalue problem with variational statement: find a pair (λ,u), with $u \in H^2(0,L)$ and $\lambda \in \mathbf{R}$, so that

$$E\int_0^L I\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}dx - \rho\lambda\int_0^L Auvdx = 0 \tag{4}$$

for admissible test functions $v \in H^2(0,L)$, where $\lambda = \omega^2$. The variational statement in Eq.(4) can also be written as:

$$B(u,v) = \lambda\ F(u,v) \qquad \forall v \in H^2 \tag{5}$$

where $B(u,v)$ and $F(u,v)$ are bilinear forms, obtained from:

$$B(u,v) = E\int_0^L I\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2}dx \tag{6}$$

$$F(u,v) = \rho\int_0^L Auvdx \tag{7}$$

In numerical methods, approximated finite dimensional subspaces $H^h \subset H^2(0,L)$ are chosen and the variational statement becomes: find $\lambda_h \in \mathbf{R}$ and $u_h \in H^h(0,L)$ so that

$$B(u_h,v_h) = \lambda_h F(u_h,v_h) \qquad \forall v_h \in H^h \tag{8}$$

The approximated solution $u_h(x)$ can be written, for a discrete system with N degrees of freedom, in the following form:

$$u_h(x) = \sum_{j=1}^{N} u_j\phi_j(x) \tag{9}$$

where ϕ_j are the global base functions of the approximated subspace H^h and u_j are the degrees of freedom. It will be shown that different approximated subspaces are proposed for the FEM, the AMM, the CEM and the GFEM. In general, according to Reddy (1986), the form functions are developed to master elements and then they are mapped to the real elements obtained from the mesh. In this work the master element domain is $\Omega_e(0,1)$.

3. FINITE ELEMENT METHOD

The conventional FEM uses polynomials form functions and the approximated solution can be improved using two refinement techniques: *h* and *p*-versions. The *h*-version consists of the refinement of the mesh, that is, to improve the total number of elements without changing the element family. The conventional *p*-version of FEM consists of increasing the polynomial degree in the solution without change in the mesh.

3.1. *h* refinement

Taking the two node uniform beam element (Fig. 2) with two degrees of freedom per each node, the approximated solution in the element domain can be defined as:

$$u_h^e(\xi) = \psi_1^e(\xi)u_1 + \psi_2^e(\xi)\theta_1 + \psi_3^e(\xi)u_2 + \psi_4^e(\xi)\theta_2 \tag{10}$$

or in matrix form:

$$u_h^e(\xi) = \mathbf{N}^T \mathbf{q} \tag{11}$$

where $\xi = \frac{x}{L_e}$, $\mathbf{N}^T = \left[\psi_1^e \quad \psi_2^e \quad \psi_3^e \quad \psi_4^e\right]$, $\mathbf{q}^T = \left[u_1 \quad \theta_1 \quad u_2 \quad \theta_2\right]$, L_e is the element length, u_1 and u_2 are the nodal displacements and, θ_1 and θ_2 are the nodal rotations.

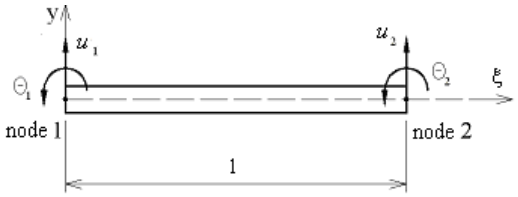

Figure 2. Beam element

Using Hermitian polynomials as local form functions, one obtains:

$$\mathbf{N}^T = \left[1 - 3\xi^2 + 2\xi^3 \quad L_e\left(\xi - 2\xi^2 + \xi^3\right) \quad 3\xi^2 - 2\xi^3 \quad L_e\left(\xi^3 - \xi^2\right)\right] \tag{12}$$

The conventional *h*-version of FEM consists in increasing the number of elements in the model to improve the approximated solution.

3.2. *p* hierarchical refinement

According to Ribeiro (2001), Zienkiewicz et al. (1982) and, Carey and Oden (1984), in a *p* refinement, if a form functions family is of order *p* and it constitutes a subset of other family with order *p+1*, then this refinement is called "hierarchical". The hierarchical form functions were introduced by Zienkiewicz, Irons, Scott and Campbell in 1971, according to the work of Zienkiewicz et al. (1982).

The application of hierarchical methods in the solution of structural vibration problems allows that stiffness and mass matrices previously calculated can be maintained and just the matrices coefficients related to the new form function must be calculated. This feature reduces the computational effort necessary to build the matrices in each step of the refinement.

It is important to stand out that the new degrees of freedom introduced by the adding of new form functions in a hierarchical refinement do not corresponds to the physical quantities generally related to the elementary degrees of freedom. Just the degrees of freedom that define the mesh stay related to the physical quantities of interest, which can correspond, for instance, to the nodal displacements and rotations.

The hierarchical *p* refinement of FEM can be obtained adding a new node between the original element nodes of the two node beam element (Fig. 2). The local form functions in Eq. (12) are maintained and just the new local form functions of superior order related to the new node are added. By this approach the new form functions in a three node element obtained from two node element are:

$$\psi_5^e = 16\xi^4 - 32\xi^3 + 16\xi^2 \tag{13}$$

$$\psi_6^e = L_e\left(16\xi^5 - 40\xi^4 + 32\xi^3 - 8\xi^2\right) \tag{14}$$

Two new degrees of freedom u_3 and θ_3 appear and they do not correspond to the nodal displacement or rotation. Adding new nodes in the element domain and taking the previous form functions, one obtains new local bases hierarchically superiors.

4. THE ASSUMED MODE METHOD (AMM) AND THE COMPOSITE ELEMENT METHOD (CEM)

Other methods can be obtained using the FEM conventional element with local form functions enriched by adding non-polynomial functions related to the closed form solutions from classical theory. Weaver Junior and Loh (1985)

used analytical solution as form functions to lateral displacements in the analysis of local vibration mode shapes of trusses.

The Assumed Mode Method (AMM) is based on the idea presented by Craig (1981) that the displacement field can be written as a linear combination of functions representing assumed vibration modes. The AMM for free vibration analysis of bars, beams and frames proposed by Engels (1992) and, Ganesan and Engels (1992) describe the displacement field of an element by:

$$u_h^e(\xi) = u_I(\xi) + u_{AM}(\xi) \tag{15}$$

where u_I and u_{AM} are interface and assumed mode displacement fields, respectively. The interface corresponds to the element nodes in structures formed by one dimensional components. The assumed modes must be linearly independent, sufficiently differentiable and at least satisfy the geometric boundary conditions of the element.

The first part of Eq. (15) is a quasistatic displacement due to the displacement of the interface. For beam elements, the interface displacement field is the conventional cubic FEM approximated displacement field (Eqs. 10-12). The second part of Eq. (17) represents the remainder of the total displacement measured by an absolute observer. The assumed mode displacement vanishes at the element interface and can be expressed as a linear combination of interface restrained assumed modes. There are several sets of assumed modes that can be used. In fact, the assumed modes can be any functions that vanish at the element interface. For example, the restrained normal vibration modes are assumed modes that are obtained by analytical solution of the free vibration problem of the element with all nodal displacements restrained.

The AMM approximated solution proposed by Engels (1992), in the element domain of a beam, is obtained by:

$$u_h^e = \mathbf{N}^T\mathbf{q} + \boldsymbol{\Phi}^T\overline{\mathbf{q}} \tag{16}$$

where $\mathbf{q}^T = \begin{bmatrix} u_1 & \theta_1 & u_2 & \theta_2 \end{bmatrix}$ is the same of Eq. (11), the vector **N** contains the FEM form functions obtained in Eq. (12) and the vectors $\boldsymbol{\Phi}$ and $\overline{\mathbf{q}}$ are obtained by:

$$\overline{\boldsymbol{\Phi}}^{\mathrm{T}}(\xi) = \begin{bmatrix} F_1 & F_2 & \dots & F_r & \dots & F_n \end{bmatrix} \tag{17}$$

$$\overline{\mathbf{q}}^T = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \tag{18}$$

$$F_r = \frac{1}{\sqrt{\rho A L \alpha_r^2}}\left\{\sinh(\lambda_r\xi) - \sin(\lambda_r\xi) - \frac{\sinh(\lambda_r) - \sin(\lambda_r)}{\cosh(\lambda_r) - \cos(\lambda_r)}\left[\cosh(\lambda_r\xi) - \cos(\lambda_r\xi)\right]\right\} \tag{19}$$

where c_i are the coefficients that multiply the mass normalized restrained normal vibration modes F_r, and λ_r are the eigenvalues associated to the solution of a clamped-clamped beam obtained by the following characteristic equation

$$\cos(\lambda_r)\cosh(\lambda_r) - 1 = 0 \tag{20}$$

In order to avoid the round-off errors associated with the hyperbolic functions when the argument becomes large, Ganesan and Engels (1992) used the restrained normal vibration modes in the alternative form proposed by Gartner and Olgac (1982). In this sense, the assumed mode form functions can be written in the following form:

$$F_r = \frac{1}{\sqrt{\rho A L}}\left[\cos(\lambda_r\xi) - \frac{1 + (-1)^r e^{-\lambda_r}}{1 - (-1)^r e^{-\lambda_r}}\sin(\lambda_r\xi) - \frac{e^{-\lambda_r\xi} - (-1)^r e^{-\lambda_r(1-\xi)}}{1 - (-1)^r e^{-\lambda_r}}\right] \tag{21}$$

$$\cos(\lambda_r) - \frac{2e^{-\lambda_r}}{1 + e^{-2\lambda_r}} = 0 \tag{22}$$

Zeng (1998a,b,c) developed elements of trusses, Euler-Bernoulli beams and frames using a similar approach to free vibration analysis. In the work of Zeng (1998a,b,c) this technique was called Composite Element Method (CEM).

The beam element of CEM differs to AMM just by using the enrichment form functions of Eq. (19) in the non normalized form

$$F_r = \sin(\lambda_r \xi) - \sinh(\lambda_r \xi) - \frac{\sin \lambda_r - \sinh \lambda_r}{\cos \lambda_r - \cosh \lambda_r}\left[\cos(\lambda_r \xi) - \cosh(\lambda_r \xi)\right] \tag{23}$$

The new degrees of freedom related to the enriched form functions do not have direct physical meaning and they were called "*c degrees of freedom*" by Zeng (1998a,b,c) in the Composite Element Method (CEM). The enrichment proposed by AMM and CEM are similar and produces hierarchical models and better results than those obtained from *h*-version of FEM (Arndt, Machado and Hecke, 2002 and 2003). The hierarchical refinement produced when the number of analytical functions in the approximated solution is increased was called "*c refinement*" by Zeng (1998a,b,c).

5. THE GENERALIZED FINITE ELEMENT METHOD (GFEM)

The Generalized Finite Element Method (GFEM) is a Galerkin method which main goal is the construction of an approximation using local knowledge of the differential equation to be solved, that ensures good local and global results. The GFEM was initially named Partition of Unity Finite Element Method (PUFEM) by Melenk and Babuska (1996). The local enrichment in the approximated space is incorporated by the partition of unity approach. The classical FEM is a special case of the GFEM (Babuska, Banerjee and Osborn, 2004).

Let $\{\eta_i\}$ be a partition of unity subordinate to the cover $\{\Omega_i\}$ of domain Ω satisfying:

$$\text{supp}(\eta_i) = \{x \in \Omega \ | \eta_i(x) \neq 0\} \subset closure(\Omega_i), \quad \forall i \tag{24}$$

$$\sum_i \eta_i \equiv 1 \text{ on } \Omega \tag{25}$$

If a local function space $S_i \subset \mathrm{H}(\Omega_i \cap \Omega)$ can approximate *u* on each patch $\Omega_i \cap \Omega$, then the GFEM global space *S* and the approximate solution u_h on Ω are obtained by:

$$S := \sum_i \eta_i \ S_i = \left\{ \sum_i \eta_i \ s_i^j \ \middle| s_i^j \in S_i \right\} \subset H(\Omega) \tag{26}$$

$$u_h(x) = \sum_i \sum_{s_i^j \in S_i} \eta_i s_i^j(x) \ a_{ij} \tag{27}$$

where a_{ij} are the degrees of freedom.

In the GFEM proposed in this work the cover $\{\Omega_i\}$ corresponds to the finite element mesh and the patches Ω_i are the subdomais of Ω obtained by union of the elements that share node x_i. For the element with two degrees of freedom (Fig. 2) the partition of unity functions in the subdomain $\Omega_i = (x_{i-1}, x_{i+1})$ are the linear FEM local form functions:

$$\eta_i = \begin{cases} 1 + \dfrac{x - x_i}{x_i - x_{i-1}} & \text{if } x \in (x_{i-1}, x_i) \\ 1 - \dfrac{x - x_i}{x_{i+1} - x_i} & \text{if } x \in (x_i, x_{i+1}) \end{cases} \tag{28}$$

In order to guarantee the global space regularity and allow introduction of the boundary conditions by the conventional finite element procedures, the approximated local space on subdomain $\Omega_i = (x_{i-1}, x_{i+1})$ takes the form:

$$S_i = span\{ \ \phi_1 \quad \phi_2 \quad \gamma_{1j} \quad \gamma_{2j} \quad \ldots \}, \ j = 1,2,\ldots,n_l \tag{29}$$

$$\phi_1 = \begin{cases} 3\dfrac{x-x_{i-1}}{x_i-x_{i-1}} - 2\left(\dfrac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2 & if \quad x \in (x_{i-1}, x_i) \\ 1+\dfrac{x-x_i}{x_{i+1}-x_i} - 2\left(\dfrac{x-x_i}{x_{i+1}-x_i}\right)^2 & if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{30}$$

$$\phi_2 = \begin{cases} \left[\left(\dfrac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2 - \dfrac{x-x_{i-1}}{x_i-x_{i-1}}\right](x_i-x_{i-1}) & if \quad x \in (x_{i-1}, x_i) \\ \left[\dfrac{x-x_i}{x_{i+1}-x_i} - \left(\dfrac{x-x_i}{x_{i+1}-x_i}\right)^2\right](x_{i+1}-x_i) & if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{31}$$

$$\gamma_{1j} = \begin{cases} \cos(\lambda_j z_1) - \dfrac{1+(-1)^j e^{-\lambda_j}}{1-(-1)^j e^{-\lambda_j}}\sin(\lambda_j z_1) - \dfrac{e^{-\lambda_j z_1} - (-1)^j e^{-\lambda_j(1-z_1)}}{1-(-1)^j e^{-\lambda_j}} & if \quad x \in (x_{i-1}, x_i) \\ 0 \quad if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{32}$$

$$z_1 = \frac{x-x_{i-1}}{x_i-x_{i-1}} \tag{33}$$

$$\gamma_{2j} = \begin{cases} 0 \quad if \quad x \in (x_{i-1}, x_i) \\ \cos(\lambda_j z_2) - \dfrac{1+(-1)^j e^{-\lambda_j}}{1-(-1)^j e^{-\lambda_j}}\sin(\lambda_j z_2) - \dfrac{e^{-\lambda_j z_2} - (-1)^j e^{-\lambda_j(1-z_2)}}{1-(-1)^j e^{-\lambda_j}} & if \quad x \in (x_i, x_{i+1}) \end{cases} \tag{34}$$

$$z_2 = \frac{x-x_i}{x_{i+1}-x_i} \tag{35}$$

where λ_j are the eigenvalues obtained by Eq. (22) and n_l is the number of enrichment levels.

6. APPLICATION

The free vibration of an uniform clamped-free beam in lateral motion, with length *L*, second moment of area *I*, elasticity modulus *E*, mass density ρ and cross section area *A*, is analyzed in order to demonstrate the application of the proposed method. The analytical natural frequencies (ω_r) of this beam are obtained by the solution of the equation:

$$\cos(\kappa_r L)\cosh(\kappa_r L) + 1 = 0 \quad , r = 1,2,\ldots \tag{36}$$

$$\kappa_r = \sqrt[4]{\frac{\omega_r{}^2 \rho A}{EI}} \tag{37}$$

To check the efficiency of the proposed GFEM the results were compared to those obtained by *h*-version of FEM, by *p*-version of FEM and by *c* refinement of CEM. The eigenvalue $\chi_r = \kappa_r . L$ is used to compare the analytical solution with the approximated ones. In the analyses by *p*-version of FEM, CEM and GFEM, the beam was described geometrically by one element and the successive refinements were obtained increasing the number of form functions.

Figures 3 to 6 present the evolution of relative error for the eight earliest eigenvalues in logarithmic scale. The total number of degrees of freedom corresponds to the effective number of them after introduction of boundary conditions.

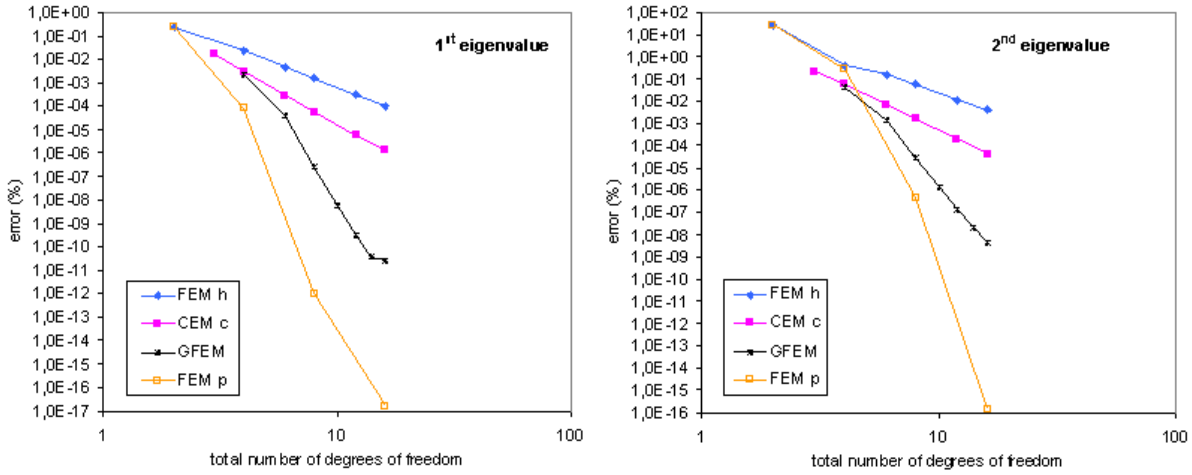

Figure 3. Relative error (%) for the 1st and 2nd beam eigenvalues

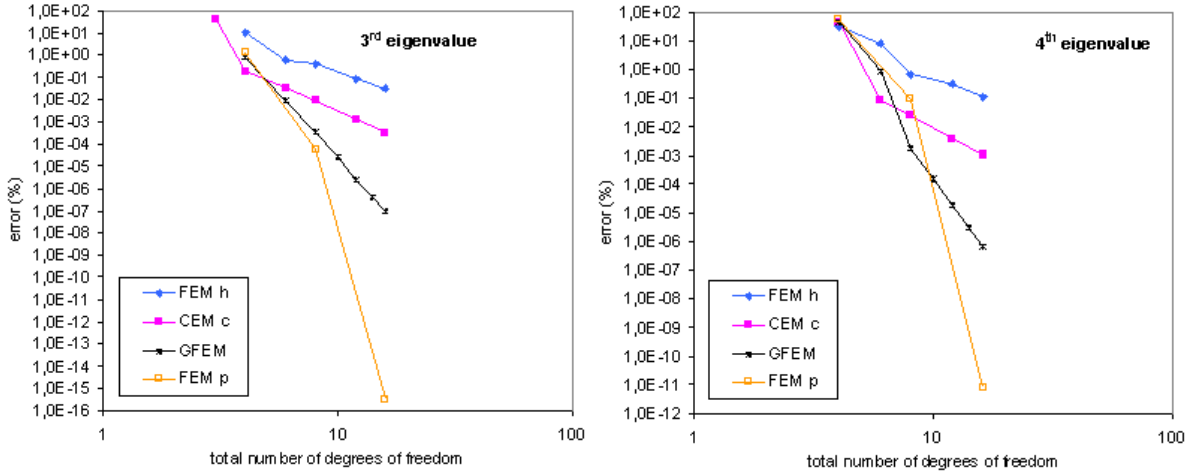

Figure 4. Relative error (%) for the 3rd and 4th beam eigenvalues

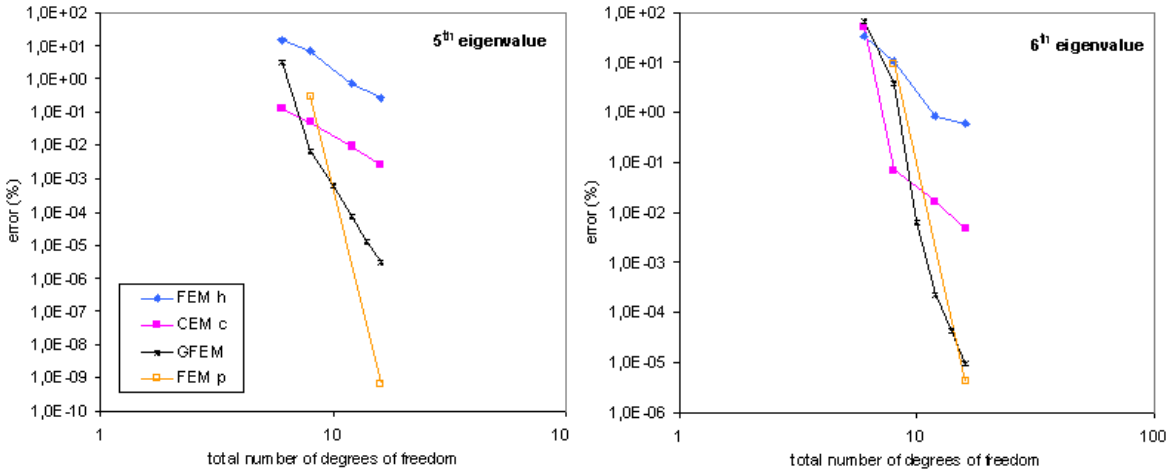

Figure 5. Relative error (%) for the 5th and 6th beam eigenvalues

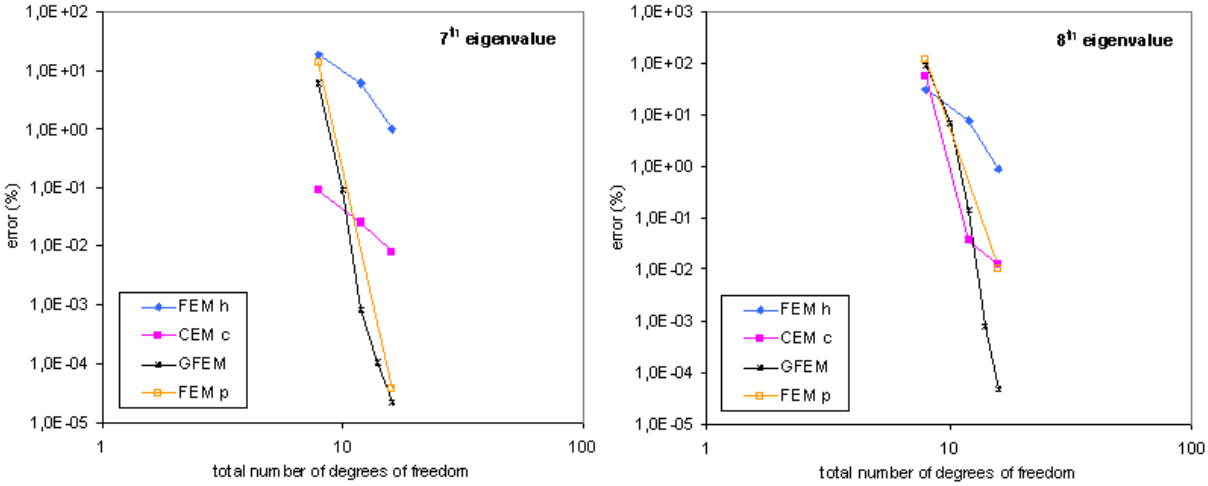

Figure 6. Relative error (%) for the 7th and 8th beam eigenvalues

Analyzing the results obtained for the clamped-free beam, one observes that the results from the *p* refinements of the FEM and the proposed GFEM show convergence rates greater than the *h* refinement of the FEM and the *c* refinement of CEM. The hierarchical *p* refinement of the FEM has greater accuracy than the GFEM for the first six eigenvalues. Otherwise, the GFEM showed better precision than the *p* version of the FEM for eigenvalues with order higher than six.

Table 1 shows the relative error for the first eight eigenvalues $\chi_r = \kappa_r.L$ obtained by different methods in comparison to the analytical solution. The *h*-version of the FEM solution was obtained with 50 elements, that is, 100 effective degrees of freedom. The CEM solution was obtained with one element and 58 enrichment functions that corresponds to two nodal degrees of freedom and 58 *c* degrees of freedom, resulting in 60 effective degrees of freedom. The problem was also solved by the *p* refinement of the FEM with one hierarchical nine node element that corresponds to 16 effective degrees of freedom. The GFEM solution was obtained with one element and 7 enrichment levels ($n_l = 7$) that corresponds to two nodal degrees of freedom and 14 field degrees of freedom resulting in 16 effective degrees of freedom.

Table 1. Results to free vibration of uniform clamped-free beam.

Eigenvalue	*h* FEM ndof = 100	*c* CEM ndof(1) = 60	*p* FEM ndof(1) = 16	GFEM ndof(1) = 16
	error (%)	error (%)	error (%)	error (%)
1	4,623 e-7	1,753 e-7	1,675 e-17	2,773 e-11
2	2,622 e-6	1,164 e-8	1,517 e-16	4,199 e-9
3	2,112 e-5	3,702 e-7	3,204 e-16	8,848 e-8
4	8,101 e-5	1,676 e-6	8,191 e-12	6,592 e-7
5	2,211 e-4	4,710 e-6	6,734 e-10	2,899 e-6
6	4,927 e-4	1,067 e-5	4,120 e-6	9,179 e-6
7	9,595 e-4	1,940 e-5	3,694 e-5	2,284 e-5
8	1,697 e-3	3,302 e-5	1,033 e-2	4,602 e-5

(1): ndof = number of effective degrees of freedom after introduction of boundary conditions.

Analyzing the results in Tab. 1, one observes that the results from the GFEM are more accurate than those obtained by *h*-version of FEM and *c*-version of CEM with superior number of degrees of freedom. The hierarchical *p* refinement of FEM with 16 degrees of freedom is more accurate than GFEM with same number of degrees of freedom just for the first six eigenvalues.

7. CONCLUSION

This work presents the variational form to the free vibration problem of straight Euler-Bernoulli beams with classical boundary conditions.

The *h* and *p*-versions of FEM, the AMM and the CEM were presented. In the Assumed Mode Method (AMM) and the Composite Element Method (CEM), the local form functions of a FEM conventional element are enriched by adding non-polynomial functions obtained from closed form solutions of the classical theory. These approaches produce

hierarchical refinements. In the proposed Generalized Finite Element Method (GFEM), the AMM enrichment functions are added to the FEM form functions by the partition of unity approach. This technique produces a hierarchical *p* refinement and allows a direct introduction of boundary conditions like in FEM.

To compare these methods, some eigenvalues of free vibration of a clamped-free beam were calculated. The proposed GFEM was compared to *h* and *p* refinements of FEM, and to the *c* refinement of CEM. The results have shown that the proposed GFEM presents convergence rates greater than those obtained from CEM and *h* refinement of FEM for all eigenvalues obtained. The GFEM results for eigenvalues with order higher than six were better than those obtained by the *p* refinement of FEM. Moreover the hierarchically superiors form functions of the proposed GFEM are easier to obtain compared to *p*-version of FEM.

New GFEM enrichment functions and the application of this method in vibration analysis of frames, plates and shells will be investigated in future works.

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