CHAOS-GALERKIN SOLUTION OF STOCHASTIC TIMOSHENKO BENDING PROBLEMS

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Abstract. *This paper presents a novel scheme of solution for the random displacement fields of uncertain Timoshenko beams. Uncertainties in material and geometrical properties are represented by parameterized stochastic processes, indexed in uniform random variables. The random response fields (transversal and angular displacements) are obtained in approximate form, using the Galerkin method and polynomials of the Askey-Wiener scheme. The space of approximate solutions is built using results of density between spaces of continuous functions and Sobolev spaces. The Chaos-Galerkin scheme developed herein is constructed by respecting the conditions for existence and uniqueness of the theoretical solution. The scheme is applied to an example problem involving random Young's modulus. Approximated Galerkin solutions are compared with results of Monte Carlo simulation, in terms of first and second order moments of the transversal and angular displacement responses, and in terms of cumulative distribution functions. Results show very fast convergence to the exact solutions, at excellent accuracies. It is shown that the Chaos-Galerkin solution accurately approximates not only the first and second order moments, but the complete probability distribution function of the displacement responses. The Chaos-Galerkin scheme developed herein is shown to be a theoretically sound and efficient method for the solution of stochastic problems in engineering.*

Keywords*: Timoshenko beam, Galerkin method, Askey-Wiener scheme, tensor product, parameterized stochastic processes, Monte Carlo simulation.*

1. INTRODUCTION

The analysis of stochastic engineering systems has received new impulse with use of finite element methods to obtain response statistics. Initially, finite element solutions were obtained via Monte Carlo simulation: samples of random system response were computed based on samples of random system parameters. Perturbation and Galerkin methods were used in this context (Araujo and Awruch, 1994). These methods allowed representation of uncertainty in system parameters by means of stochastic processes. Spanos and Ghanem (1989) presented a novel approach for the solution of stochastic mechanics problems, using the finite element method. The space of approximate solutions was built using the finite element method and chaos polynomials. These polynomials form a complete system in $L^2(\Omega,\mathcal{F},P)=\overline{\Psi}^{L^2(\Omega,\mathcal{F},P)}$, where $\Psi = span\left[\{\psi_i\}_{i=0}^{\infty}\right]$ is the space generated by the chaos polynomials and $(\Omega,\mathcal{F},P)$ is a probability space.

New theoretical insights into the problem were given by Babuska et al. (2005), who presented a stochastic version of the Lax-Milgram lemma. In this paper, the authors presented limitations to the modeling of uncertainty via Gaussian processes. They showed that for certain problems of mechanics, use of Gaussian processes can lead to loss of coercivity of the bi-linear form associated to the stochastic problem. This difficulty was indeed encountered in the study of Silva Jr. (2004), and resulted in non-convergence of the solution for the bending of plates with random parameters. This lack of convergence was due to the choice of a Gaussian process to represent the uncertainty in strictly non-positive parameters of the system. The lack of converge also affects solutions based on perturbation or simulation methods.

In a paper by Xiu and Karniadakis (2002) the Askey-Wiener scheme was presented. This scheme represents a family of polynomials which generate dense probability spaces with limited and unlimited support. The scheme increases the possibilities for modeling of uncertain system parameters. In recent years, large efforts are being made at representing uncertainty in stochastic systems via non-Gaussian processes.

The stochastic beam bending problem has been studied by several authors. Vanmarcke and Grigoriu (1983) studied the bending of Timoshenko beams with random shear modulus. Elishakoff et al. (1995) employed the theory of mean square calculus to construct a solution to the boundary value problem of bending with stochastic bending modulus. Ghanem and Spanos (1991) used the Galerkin method and the Karhunem-Loeve series to represent uncertainty in the bending modulus by means of a Gaussian process. Chakraborty and Sarkar (2000) used the Neumann series and Monte Carlo simulation to obtain statistical moments of the displacements of curved beams, with uncertainty in the elasticity modulus of the foundation. Although they present numerical solutions for stochastic beam problems, none of the papers referenced above address the matter of existence and uniqueness of the solutions.

In the present paper, the Galerkin method is used to obtain approximate solutions for the bending of Timoshenko beams with random parameters. A Chaos-Galerkin scheme is developed herein, which complies with the necessary conditions for existence and uniqueness of the solution. Uncertainties in material properties are represented by parameterized stochastic processes, indexed in uniform random variables. Polynomials of the Askey-Wiener scheme are used to represent the uncertainty in input parameters and to construct the approximate solution space.

The Timoshenko bending problem is presented in Section 2. Modeling of the uncertainty via parameterized stochastic processes is presented in Section 3. Section 4 presents the derivation of the abstract variational problem. Section 5 presents the approximate Galerkin formulation. The Chaos-Galerkin scheme developed herein is used in the numerical solution of an example problem in Section 6. Some conclusions are presented in Section 7.

2. BENDING OF TIMOSHENKO BEAMS

In this section, the strong and weak formulations of the problem of stochastic bending of Timoshenko beams are presented. The Lax-Milgram lemma is used to present a proof of existence and uniqueness of the solution. The strong form of the stochastic Timoshenko beam bending problem is given as,

$$\begin{cases} \frac{d}{dx}\left(\alpha \frac{d\phi}{dx}\right) + \beta\left(\frac{dw}{dx} - \phi\right) = 0; \\ \frac{d}{dx}\left(\beta\left(\frac{dw}{dx} - \phi\right)\right) = -f, \qquad \forall (x,\omega) \in (0,L)\times\Omega; \\ w(0,\omega) = w(L,\omega) = 0; \\ \phi(0,\omega) = \phi(L,\omega) = 0;, \qquad \forall \omega \in \Omega; \end{cases} \tag{1}$$

where $\alpha = E.I$ and $\beta = G.A$ are the bending and shear stiffness, respectively, Ω is the sample space, w is the transversal beam displacement, ϕ is the angular displacement and f is a load term. In order to guarantee existence and uniqueness of the solution, the following hypotheses are necessary:

$$\begin{aligned} &\text{H1}: \exists \underline{\alpha}, \overline{\alpha}, \underline{\beta}, \overline{\beta} \in \mathbb{R}^+ \setminus \{0\} : P\Big(\omega \in \Omega : \alpha(x,\omega) \in \left[\underline{\alpha}, \overline{\alpha}\right] \wedge \beta(x,\omega) \in \left[\underline{\beta}, \overline{\beta}\right], \forall x \in \left[0, L\right]\Big) = 1; \\ &\text{H2}: f \in L^2\left(\Omega, \mathcal{F}, P; L^2(0,L)\right). \end{aligned} \tag{2}$$

Hypothesis H1 ensures that the beam stiffness moduli are positive-defined and uniformly limited in probability (Babuska et al., 2005). Hypothesis H2 ensures that the stochastic load process has finite variance. These hypotheses are necessary for the application of the Lax-Milgram Lemma, which is used in the sequence to demonstrate the existence and uniqueness of the solution.

3. MODELLING OF THE UNCERTAINTY

In most engineering problems, complete statistical information on uncertain parameters is not available. Sometimes, the first and second order moments are the only information available: the probability distribution function is defined based on experience or heuristically.

In order to apply Galerkin's method, an explicit representation of the uncertainty is necessary. In this paper, uncertainties on material and geometrical properties are modeled via parameterized stochastic processes. These are defined as a linear combination of deterministic functions and random variables (Grigoriu, 1995):

$$\kappa(x,\omega) = \sum_{i=1}^{N} g_i(x)\,\xi_i(\omega) \quad , \tag{3}$$

where $\{g_i\}_{i=1}^N$ are deterministic functions and $\{\xi_i(\omega)\}_{i=1}^N$ are random variables with

$$\prec \xi_i, \xi_j \succ = \frac{1}{3}.\delta_{ij}, \forall (i,j) \in \{1,\ldots,N\}\times\{1,\ldots,N\} \tag{4}$$

In eq. (4), $\prec\cdot\succ$ is the mathematical expectation operator. Hence, the mean value of κ is: $\prec \kappa(\boldsymbol{x},\cdot) \succ = \mu_\kappa(\boldsymbol{x})$ and the variance of κ is $\prec \left[\kappa(\boldsymbol{x},\cdot) - \mu_\kappa(\boldsymbol{x})\right]^2 \succ = \sigma_\kappa^2(\boldsymbol{x})$. In this paper, polynomials of the Askey-Wiener scheme are used to represent parameter uncertainty and to construct the problems solution space.

3.1 The Askey-Wiener scheme

The Askey-Wiener scheme is a generalization of chaos polynomials, also known as Wiener-chaos. Chaos polynomials were proposed by Wiener (1938) to study statistical mechanics of gases. Xiu and Karniadakis (2003) have shown the close relationship between results presented by Wiener (1938) and Askey and Wilson (1985) for the representation of stochastic processes by orthogonal polynomials. Xiu and Karniadakis (2003) extended the studies of Ghanem and Spanos (1991) for polynomials belonging to the Askey-Wiener scheme. The Cameron-Martin theorem (1947) shows that Askey-Wiener polynomials form a base for a dense subspace of second order random variables $L^2(\Omega,\mathcal{F},P)$. As shown by Jason (1997), an element $X\in L^2(\Omega,\mathcal{F},P)$ can be represented as

$$X=\sum_{n=0}^{\infty}X_n\,. \tag{5}$$

Equation (5) represents an important result for the approximation theory applied to stochastic systems. Solution of a stochastic system is expressed as a non-linear function in terms random variables. This function is expanded in terms of chaos polynomials as,

$$u_i(\omega)=\sum_{j=1}^{\infty}u_{ij}\psi_j\left(\xi(\omega)\right). \tag{6}$$

The internal product between polynomials ψ_i and ψ_j in $L^2(\Omega,\mathcal{F},P)$ is defined as,

$$\left(\psi_i,\psi_j\right)_{L^2(\Omega,\mathcal{F},P)}=\int_{\mathbb{R}^N}\left(\psi_i.\psi_j\right)\left(\xi(\omega)\right)dP(\omega), \tag{7}$$

where dP is a probability measure. These polynomials form a complete ortho-normal system with respect to the probability measure dP.

4. NOTATION AND SPACE OF FUNCTIONS

In this section, some definitions and notations that will be used along the study are presented. The principle of causality says that, for problems with uncertainty in the source term or in system parameters, system response will necessarily show stochastic behavior. For these problems, the solution space should contain functions to represent this random behavior. In this study, the solution space is constructed via tensor product between Sobolev and probability spaces. This originates the so-called Stochastic Sobolev Spaces.

4.1 Stochastic Sobolev Spaces

The association between theory of probability, product space and Sobolev spaces originate what is known as stochastic Sobolev spaces. Theoretical and numerical solutions obtained in these spaces are based on the isomorphism between stochastic Sobolev spaces and Sobolev spaces defined in more complex measure spaces (Frauenfelder et al., 2005). The theoretical solution for the stochastic bending problem is defined in $V=L^2\left(\Omega,\mathcal{F},P;\left(H_0^1(0,L)\right)^2\right)$, given by,

$$V=\left\{(w,\phi):(0,L)\times\Omega\to\mathbb{R}^2\,\middle|\, w \text{ and } \phi \text{ are mensurable and } \int_\Omega\left\|w(\omega)\right\|^2_{H^1(0,L)}dP(\omega),\int_\Omega\left\|\phi(\omega)\right\|^2_{H^1(0,L)}dP(\omega)<+\infty\right\} \tag{8}$$

The set defined in Eq. (8) is a Hilbert space. For $\omega\in\Omega$ fixed, one has $\left(w(\cdot,\omega),\phi(\cdot,\omega)\right)\in H_0^1(0,L)\times H_0^1(0,L)$, whereas for $x\in(0,L)$ fixed, $w(x,\cdot),\phi(x,\cdot)\in L^2(\Omega,\mathcal{F},P)$. We define the tensorial product between $v\in L^2(\Omega,\mathcal{F},P)$ and $(\theta,\vartheta)\in H_0^1(0,L)\times H_0^1(0,L)$ as $(w,\phi)=(v.\theta,v.\vartheta)$.

It is also necessary to redefine the differential operator for the space obtained via tensorial product. The operator $D_\omega^\alpha:V\to L^2(\Omega,\mathcal{F},P)\otimes\left(L^2(0,L)\right)^2$ (Matthies and Keese, 2005), acts over an element $w\in V$ the following way,

$$D_\omega^\alpha w:\left(\frac{d^\alpha v}{dx^\alpha}\right)(x).\theta(\omega). \tag{9}$$

where $\alpha \in \mathbb{N}$ and $\alpha \leq 2$. Now let $u,v \in V$, with $u=(w,\phi)$ and $v=(h,\upsilon)$, than V is a Hilbert space with internal product defined as:

$$(u,v)_V = \int_{\Omega}\Big[\big(D_\omega\phi(\omega),D_\omega\upsilon(\omega)\big)_{L^2} + \big((D_\omega w-\phi)(\omega),(D_\omega h-\upsilon)(\omega)\big)_{L^2}\Big]dP(\omega)\cdot \tag{10}$$

where $(\cdot,\cdot): L^2(0,L)\times L^2(0,L)\to\mathbb{R}$ is the internal product in $L^2(0,L)$. The internal product defined in Eq. (10) induces the V norm $\|u\|_V = (u,u)_V^{1/2}$, following Frauenfelder et al. (2005). The bilinear form $a: V\times V\to\mathbb{R}$ is defined as:

$$a(u,v) = \int_{\Omega}\Big[\big((\alpha.D_\omega\phi)(\omega),D_\omega\upsilon(\omega)\big)_{L^2} + \big((\beta.D_\omega w-\phi)(\omega),(D_\omega h-\upsilon)(\omega)\big)_{L^2}\Big]dP(\omega)\cdot \tag{11}$$

4.2 Abstract Variational Problem

The abstract variational problem, or weak form of the problem defined in Eq. (1) is given as,

$$\begin{cases}\text{Find } u\in V \text{ such that}\\ a(u,v)=\ell(v),\ \forall v\in V.\end{cases} \tag{12}$$

where $\ell: V\to V$ is a linear functional given by,

$$\ell(v) = \int_{\Omega}\int_0^L (f.v)(x,\omega)\,dx\,dP(\omega)\cdot \tag{13}$$

From hypothesis H1 and H2 (Eq. 2) and from the Lax-Milgram lemma, it is guaranteed that the problem defined in Eq. (12) has unique solution and continuous dependency on the data (Babuska et al., 2005; Brenner and Scott, 1994).
For the following numerical computations, it is more appropriate to express the abstract variational problem as,

$$\begin{cases}\text{Find } (w,\phi)\in V \text{ such that}\\[2ex] \int_{\Omega}\int_0^L \big(\beta.(D_\omega w-\phi).h\big)(x,\omega)\,dx\,dP(\omega) = \int_{\Omega}\int_0^L (f.h)(x,\omega)\,dx\,dP(\omega);\\ \int_{\Omega}\int_0^L (\alpha.D_\omega\phi.D_\omega\upsilon)(x,\omega)\,dx\,dP(\omega) = \int_{\Omega}\int_0^L \big(\beta.(D_\omega w-\phi).\upsilon\big)(x,\omega)\,dx\,dP(\omega),\ \forall(h,\upsilon)\in V.\end{cases} \tag{14}$$

This alternative formulation does not eliminate the coupling between the fields w and ϕ.

5. METHOD OF GALERKIN

The Galerkin method is used in this paper to build an approximate solution to the stochastic beam bending problems. It is proposed that approximated solutions have the following form:

$$\begin{cases} w(x,\omega) = \sum_{i=1}^{\infty} w_i\delta_i(x,\omega)\\ \phi(x,\omega) = \sum_{i=1}^{\infty} \phi_i\delta_i(x,\omega)\end{cases}, \tag{15}$$

where $w_i\in\mathbb{R},\forall i\in\mathbb{N}$ are coefficients to be determined and $\delta_i\in V$ are the test functions. Numerical solutions to the variational problem defined in Eq. (14) will be obtained. Hence, it becomes necessary to define spaces less abstract than those defined earlier, but without compromising the existence and uniqueness of the solution. From the theorem of Cameron-Martin (Cameron and Martin, 1947) one has $\overline{\mathcal{P}}^{L^2(\Omega,\mathcal{F},P)} = L^2(\Omega,\mathcal{F},P)$. Let $W = C_0(0,L)\cap C^1(0,L)$, with

$\overline{W}^{H_0^1(0,L)} \times \overline{W}^{H_0^1(0,L)} = \left(H_0^1(0,L)\right)^2$. Consider two complete orthogonal systems $\Phi = span\left[\{\varphi_i\}_{i=1}^{\infty}\right]$, with $\varphi_i \in W, \forall i \in \mathbb{N}$ and $\Psi = span\left[\{\psi_i\}_{i=1}^{\infty}\right]$, such that, $\overline{\Psi}^{L^2(\Omega,\mathcal{F},P)} = L^2(\Omega,\mathcal{F},P)$. Now define the tensorial product between Φ and Ψ as:

$$\left(\varphi \otimes \psi\right)_i (x,\omega) = \varphi_j(x).\psi_k(\omega), \text{ with } j,k \in \mathbb{N}. \tag{16}$$

To simplify the notation, we will use $\delta_i = \left(\varphi \otimes \psi\right)_i$. Since approximated numerical solutions are constructed, the solution space has finite dimensions. This implies truncation of the complete orthogonal systems Φ and Ψ. Hence one has $\Phi_m = span\left[\{\varphi_i\}_{i=1}^{m}\right]$ and $\Psi_n = span\left[\{\psi_i\}_{i=1}^{n}\right]$, which results in $V_M = \Phi_m \otimes \Psi_n$. With the above definitions and results, it is proposed that numerical solutions are obtained from truncation of the series expressed in Eq. (15) at the M^{th} term:

$$\begin{cases} w_M(x,\omega) = \sum_{i=1}^{M} w_i \delta_i(x,\omega) \\ \phi_M(x,\omega) = \sum_{i=1}^{M} \phi_i \delta_i(x,\omega) \end{cases}. \tag{17}$$

Substituting Eq. (17) in Eq. (14), one arrives at the approximated variational problem:

$$\begin{cases} \text{Find } (w_i,\phi_i) \in \mathbb{R}^{2.M} \text{ such that} \\ \sum_{i=1}^{M}\left\{\left[\int_\Omega\int_0^L (\beta.D_\omega\delta_i.\delta_j)(x,\omega)\,dxdP(\omega)\right]w_i - \left[\int_\Omega\int_0^L (\beta.\delta_i.\delta_j)(x,\omega)\,dxdP(\omega)\right]\phi_i\right\} = \int_\Omega\int_0^L (f.\delta_j)(x,\omega)\,dxdP(\omega); \\ \sum_{i=1}^{M}\left[\int_\Omega\int_0^L (\alpha.D_\omega\delta_i.D_\omega\delta_j + \beta.\delta_i.\delta_j)(x,\omega)\,dxdP(\omega)\right]\phi_i = \sum_{i=1}^{M}\left[\int_\Omega\int_0^L (\beta.D_\omega\delta_i.\delta_j)(x,\omega)\,dxdP(\omega)\right]w_i, \forall \delta_j \in V \end{cases} \tag{18}$$

The approximated variational problem (Eq. 18) consists in finding the coefficients of the linear combination expressed in Eq. (17). Using a vector-matrix representation, the system of linear algebraic equations defined in Eq. (18) can be written as

$$\begin{cases} \text{Find } \mathbf{w},\phi \in \mathbb{R}^M \text{ such that} \\ \mathbf{Aw} + \mathbf{B}\phi = \mathbf{F}; \\ \mathbf{Cw} = \mathbf{D}\phi; \end{cases} \tag{19}$$

where $\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{M}_M(\mathbb{R})$. The solution to the linear system given in Eq. (19) is:

$$\begin{cases} \phi = \left(\mathbf{AC}^{-1}\mathbf{D} + \mathbf{B}\right)^{-1}\mathbf{F}; \\ \mathbf{w} = \mathbf{C}^{-1}\mathbf{D}\left(\mathbf{AC}^{-1}\mathbf{D} + \mathbf{B}\right)^{-1}\mathbf{F}. \end{cases} \tag{20}$$

Elements of the matrix are:

$$\begin{cases} \mathbf{A} = \left[a_{ij}\right]_{M\times M}, \ a_{ij} = \int_\Omega\int_0^L \left(\beta.D_\omega\delta_i.\delta_j\right)(x,\omega)\,dxdP(\omega); \\ \mathbf{B} = \left[b_{ij}\right]_{M\times M}, \ b_{ij} = -\int_\Omega\int_0^L \left(\beta.\delta_i.\delta_j\right)(x,\omega)\,dxdP(\omega); \\ \mathbf{C} = \left[c_{ij}\right]_{M\times M}, \ c_{ij} = \int_\Omega\int_0^L \left(\alpha.D_\omega\delta_i.D_\omega\delta_j + \beta.\delta_i.\delta_j\right)(x,\omega)\,dxdP(\omega); \\ \mathbf{D} = \left[d_{ij}\right]_{M\times M}, \ d_{ij} = \int_\Omega\int_0^L \left(\beta.D_\omega\delta_i.\delta_j\right)(x,\omega)\,dxdP(\omega). \end{cases} \tag{21}$$

The load vector is given by

$$\mathbf{F} = \{f_i\}_{i=1}^{M}, \quad f_i = \int_{\Omega}\int_{0}^{L} \left(f.\delta_j \right)(x,\omega) dx dP(\omega) \cdot \tag{22}$$

It is important to observe that conversion of the continuous problem (Eq. 14) into the discretized form of Eq. (19) resulted in the de-coupling of the transversal and angular displacement random fields.

6. NUMERICAL RESULTS

In this section, a numerical example of the stochastic Timoshenko beam bending problem is presented. Uncertainty in the Young's modulus is modeled by parameterized stochastic processes. The beam is clamped at both ends, the span (L) equals half meter, the cross-section is rectangular with $b = 1/25$ m and $h = 1/20$ m, and the beam is subject to an uniform distributed load of $f(x) = 100$ KPa.m, $\forall x \in [0,L]$.

The first and second order statistical moments of the approximated Galerkin solutions are compared with the same moments evaluated via Monte Carlo simulation. To evaluate the error of approximated solutions, relative error functions in expected value and variance are defined as:

$$\begin{cases} e_{\mu_w}(x) = \left|(\mu_w - \hat{\mu}_w)(x)\right| \\ e_{\mu_\phi}(x) = \left|(\mu_\phi - \hat{\mu}_\phi)(x)\right|, \forall \ x \in [0,L]; \end{cases} \quad \wedge \quad \begin{cases} e_{\sigma_w^2}(x) = \left|(\sigma_w^2 - \hat{\sigma}_w^2)(x)\right| \\ e_{\sigma_\phi^2}(x) = \left|(\sigma_\phi^2 - \hat{\sigma}_\phi^2)(x)\right|, \forall \ x \in [0,L]; \end{cases} \tag{23}$$

where (μ_w, μ_ϕ) and $(\sigma_w^2, \sigma_\phi^2)$, are the Galerkin-based expected value and variance, respectively, and $(\hat{\mu}_w, \hat{\mu}_\phi)$ and $(\hat{\sigma}_w^2, \hat{\sigma}_\phi^2)$ are the Monte Carlo estimates of the same moments.

In this paper, a family of Legendre polynomials is used to construct space Ψ_n, defined in two independent, uniform random variables ($n_{rv} = 2$). Numerical solutions are obtained for $m = 3$ $(m = \dim\langle\Phi_m\rangle)$ and for different orders of chaos polynomials, with $p \in \{1,2,3\}$. The size of the chaos polynomial basis (Ψ_n) becomes $n \in \{3,6,10\}$, since $n = (p + n_{rv})! / (p!.n_{rv}!)$. This results in numerical solutions with $M \in \{9,18,30\}$ coefficients to determinate. Numerical results were obtained in a personal computer, running a MATLAB computational code.

The materials Young's modulus is modeled as a parameterized stochastic process,

$$E(x,\omega) = \mu_E + \sqrt{3}.\sigma_E \left[\xi_1(\omega) cos\left(\tfrac{x}{L}\right) + \xi_2(\omega) sen\left(\tfrac{x}{L}\right) \right], \tag{24}$$

where μ_E is the mean value, σ_E is the standard deviation and $\{\xi_1, \xi_2\}$ are orthogonal uniform random variables. Numerical solutions are obtained for $\sigma_E = (\frac{1}{10}).\mu_E$. The covariance function of Young's modulus is obtained in exact form from Eq. (24) and from the orthogonality property of random variables $\{\xi_1, \xi_2\}$. The covariance function is a strictly stationary random process.

Results obtained via Monte Carlo simulation are used as reference and are shown first. Figure 1 shows all the ten thousand samples or realizations (N_S=10.000) of the transversal and angular displacements of the beam.

Figure 2a shows the expected value of random transversal displacements obtained via Monte Carlo simulation, and for the Galerkin solutions with $p \in \{1,2,3\}$. Figure 2b shows the absolute error in expected values, obtained by comparing Galerkin results with simulation. It is observed that approximated solutions converge very fast, and monotonically, to the exact solution. A good approximation of the expected value is already obtained for *p=1*, with a relative error of the order of 0.3%.

Similar results are shown in Figure 3 in terms of the expected value of angular displacements. Figure 3a shows the expected value obtained via simulation, and via Galerkin method for $p \in \{1,2,3\}$. Figure 3b shows the absolute error in expected value for the angular displacements. It is observed that convergence of Galerkin solutions to the exact result is monotonous and very fast. Even for *p=1*, the absolute error is already very small.

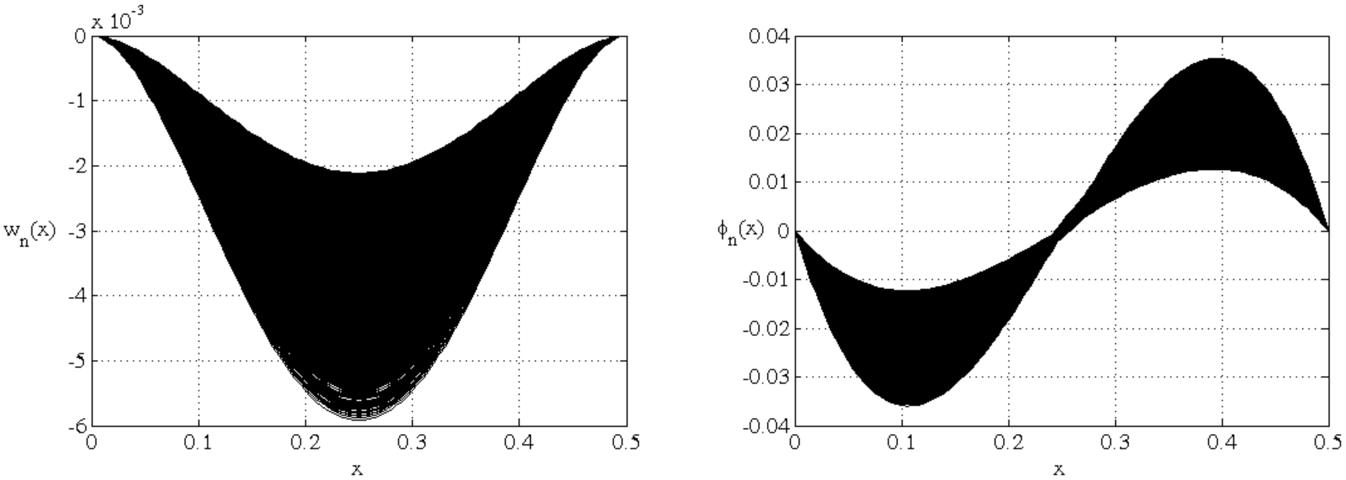

Figure 1: Ten thousand samples of the transversal and angular displacements of the beam.

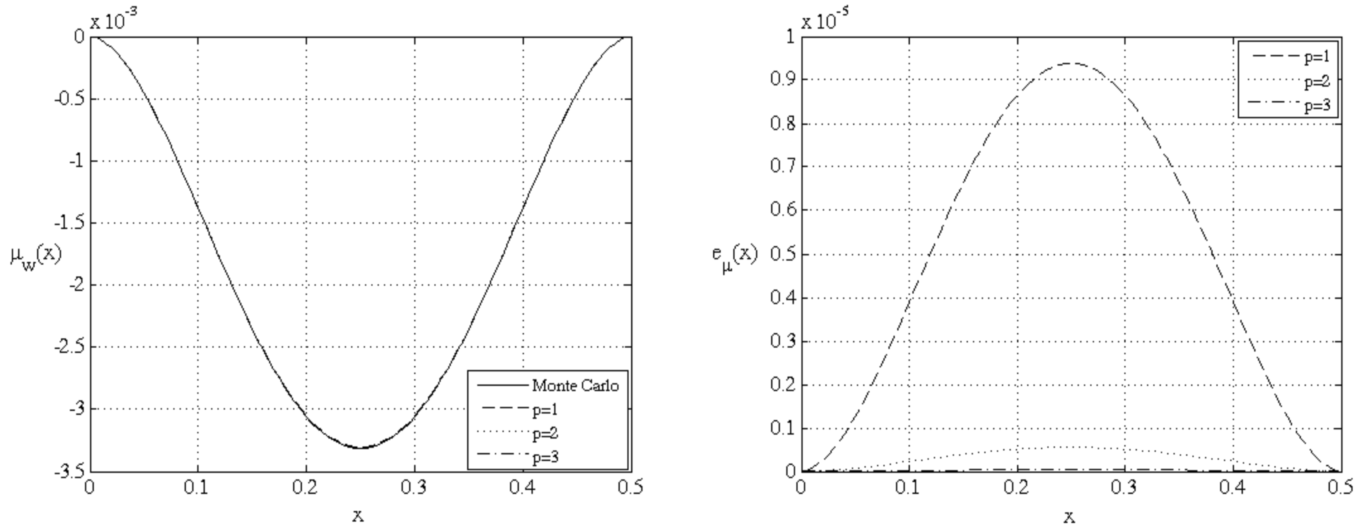

Figure 2: a) Expected value of transversal displacement; b) Absolute error in expected value.

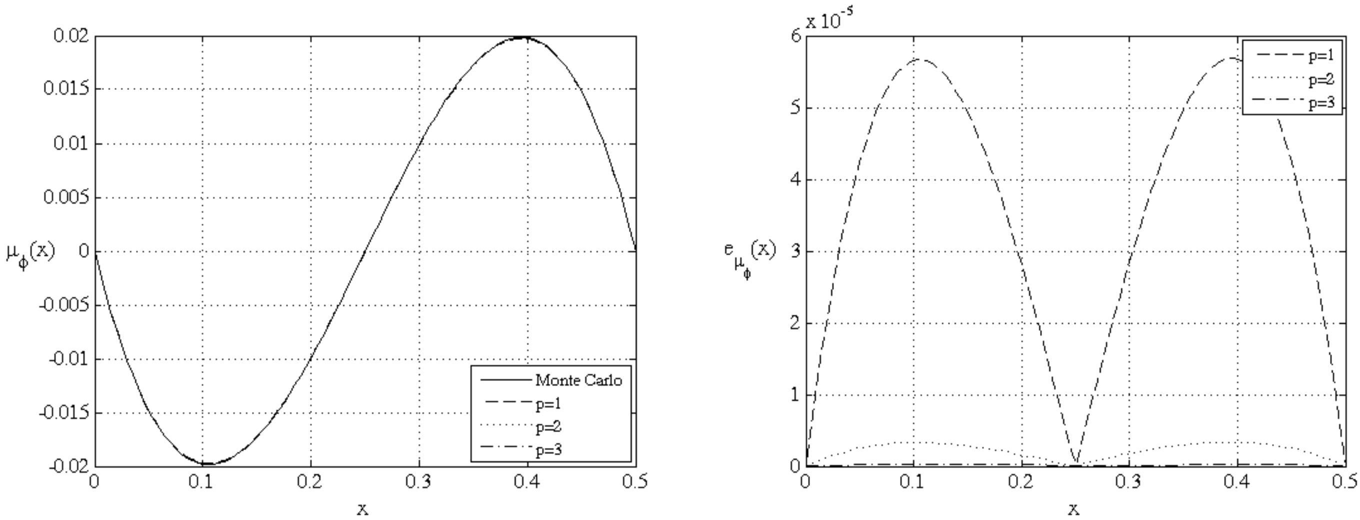

Figure 3: a) Expected value of angular displacements; b) Absolute error in expected value.

Figure 4 shows the covariance functions of transversal displacements (4a) and angular displacement (4b) random fields. There are no observable differences between the Galerkin and Monte Carlo estimates of these functions.

Convergence of Galerkin solutions in variance of transversal displacements is presented in Figure 5. Figure 5a shows the variance obtained via simulation and via Galerkin method for $p \in \{1,2,3\}$. Figure 5b shows the absolute error in variance. It is observed that convergence is fast and monotonous, and that good approximation of the variance is already obtained for *p=2*.

Similar results are presented in Figure 6, for the variance of angular displacements. Figure 6a shows the variance obtained via simulation, and via Galerkin method for $p \in \{1,2,3\}$. Figure 6b shows the absolute error in variance of angular displacements. It is observed that convergence of Galerkin solutions to the exact result is monotonous and very fast. For *p=2*, the absolute error in variance is already very small.

The fast and monotonous convergence rate of Galerkin solutions, as observed, is not limited to the first and second moments of the random displacement fields. Figure 7 shows that this excellent convergence rate is also obtained for the probability distribution functions of the random displacement responses. Figure 7 shows histograms (above) and cumulative distribution functions (below) of the random variables transversal displacement $w_{\left(\frac{L}{2}\right)} = w\left(\frac{L}{2},\omega\right)$ (left) and angular displacement $\phi_{\left(\frac{L}{5}\right)} = \phi\left(\frac{L}{5},\omega\right)$ (right), for a Galerkin solution with *p=3*. It is observed that the Galerkin solutions converge to the exact probability distribution function of the random response.

In Figure 7, the Monte Carlo histograms are obtained directly from the ten thousand samples of the beams displacement responses. The Galerkin histograms (and cumulative distribution function) are obtained by sampling random variables $\{\xi_1, \xi_2\}$ of the chaos polynomial basis. Ten thousand samples are used.

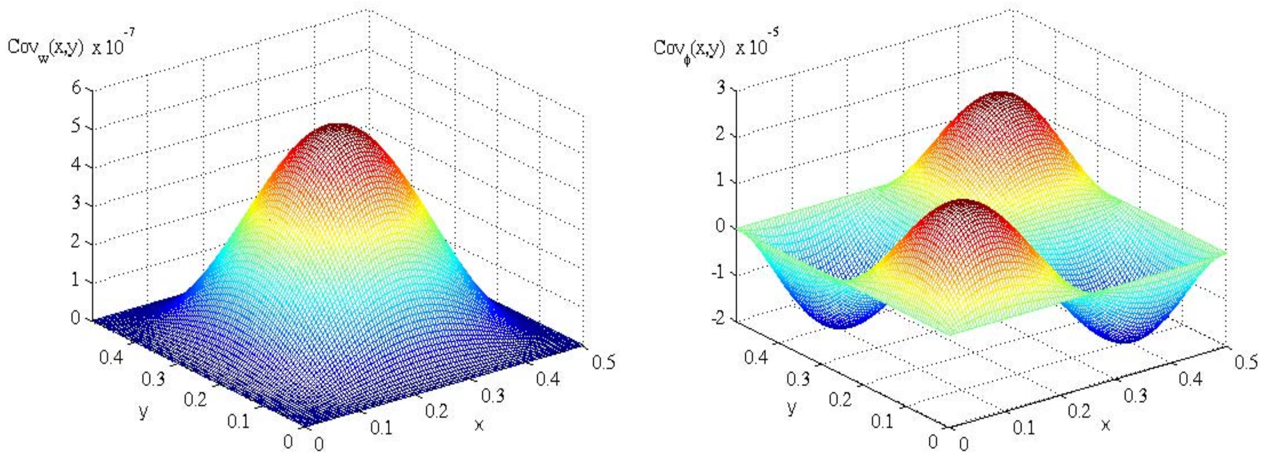

Figure 4: Covariance functions of a) transversal and b) angular displacements.

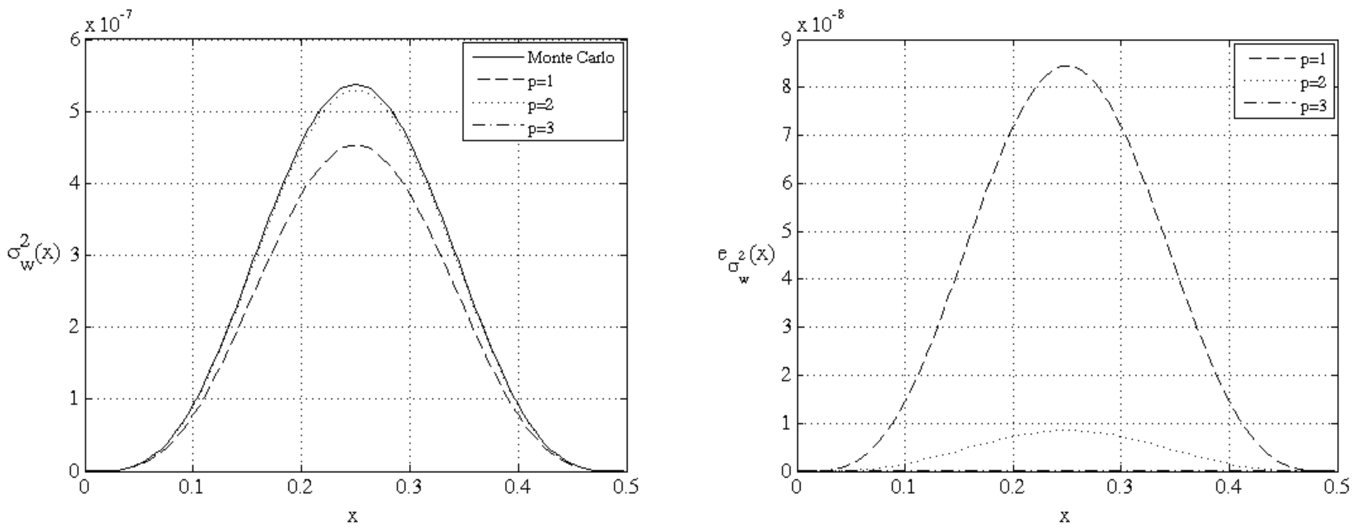

Figure 5: a) Variance of transversal displacements; b) Absolute error in variance.

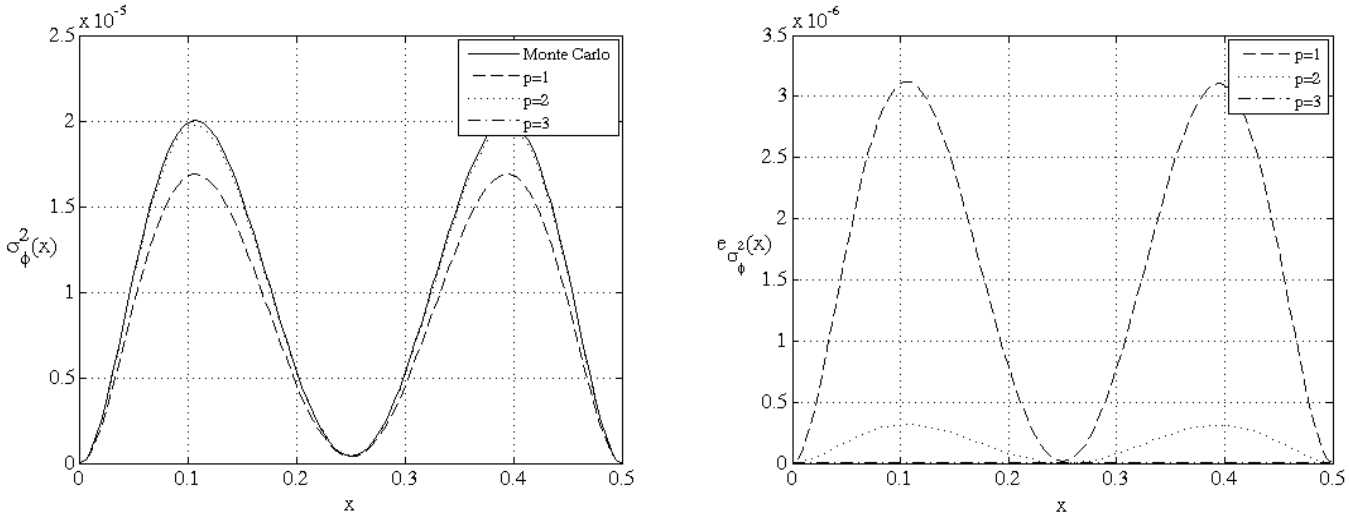

Figure 6: a) Variance of angular displacements; b) Absolute error in variance.

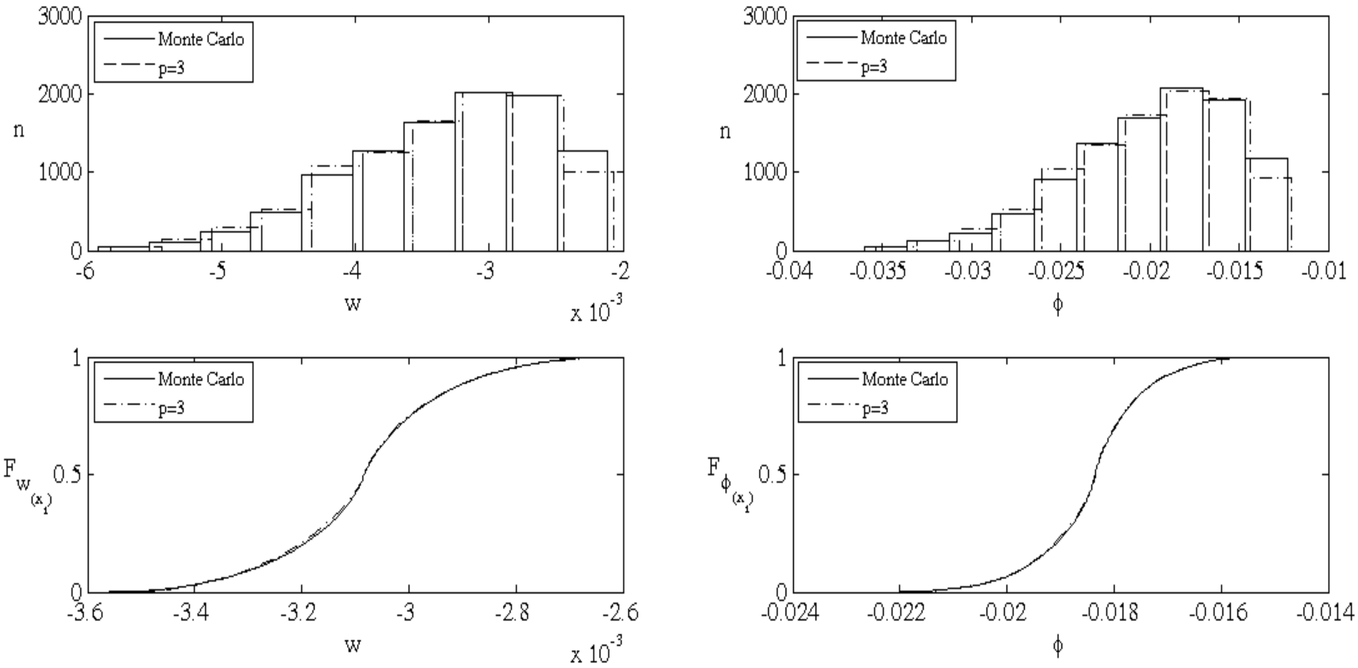

Figure 7: Histograms (above) and cumulative distribution functions (below) of the random variables transversal displacement $w_{\left(\frac{L}{2}\right)}$ (left) and angular displacement $\phi_{\left(\frac{L}{5}\right)}$ (right).

7. CONCLUSIONS

In this paper, theoretical and practical results for bending of stochastic Timoshenko beams were presented. The Lax-Milgram lemma was used for a theoretical study about the existence and uniqueness of the solution. A Chaos-Galerkin scheme was developed for the solution of the problem. The scheme involves representation of the uncertainties in input parameters an in random displacement fields as parameterized stochastic processes, indexed in uniform random variables. A family of Legendre polynomials, derived from the Askey-Wiener scheme, was used to construct the solution space. The developed scheme respects the conditions for existence and uniqueness of the solution, and was shown to provide very fast convergence to the exact solutions.

The Chaos-Galerkin scheme developed herein was applied in the solution of an example problem, involving a beam with random Young's modulus. The Chaos-Galerkin scheme presented very fast convergence rates to the exact transversal and angular displacement fields. The mean value of both fields was accurately predicted with a chaos basis of polynomials of first order (p=1). For the variance of both fields, a higher polynomial order was required. However, accurate results were already obtained for *p*=2. The convergence rate of the developed scheme is so good, that the complete probability distribution functions of transversal and angular displacement fields were accurately represented by a Chaos-Galerkin solution using third-order polynomials (*p*=3).

The Chaos-Galerkin scheme developed herein was shown to provide very fast convergence to the exact solution of stochastic Timoshenko beam bending problems. The scheme is a theoretically sound and efficient method for the solution of stochastic problems in engineering.

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