TOPOLOGY OPTIMIZATION FOR FLEXIBLE STRUCTURES

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Abstract. *A new approach is presented for relaxation-based topology optimization which does not require the homogenization assumption. In this approach the material distribution remains continuous throughout the design region and is represented using B-spline finite elements. No artificial restrictions are applied, and regions of intermediate density are penalized using the* SIMP *method. The generation of high-definition models are made possible by the refinement of the interface between solid and void with adaptive meshing techniques. This new technique is utilized in the generation of structures which satisfy prescribed stiffness requirements while minimizing mass. This is particularly important in aerospace applications where mass is the most prevalent figure of merit when analyzing deployment cost. Examples illustrate the difference in structures when: stiffness requirements are modified, multiple materials are considered and compliant mechanisms are desired.*

Keywords: *Topology optimization, B-splines, Adaptive mesh refinement, Mass minimization, Flexible structures*

1. Introduction

This work presents two extensions to the field of topological optimization enhancing its capability and applicability. These are: a B-spline material distribution algorithm and a minimum mass formulation. Both extensions are developed and illustrated by examples which show their effectiveness.

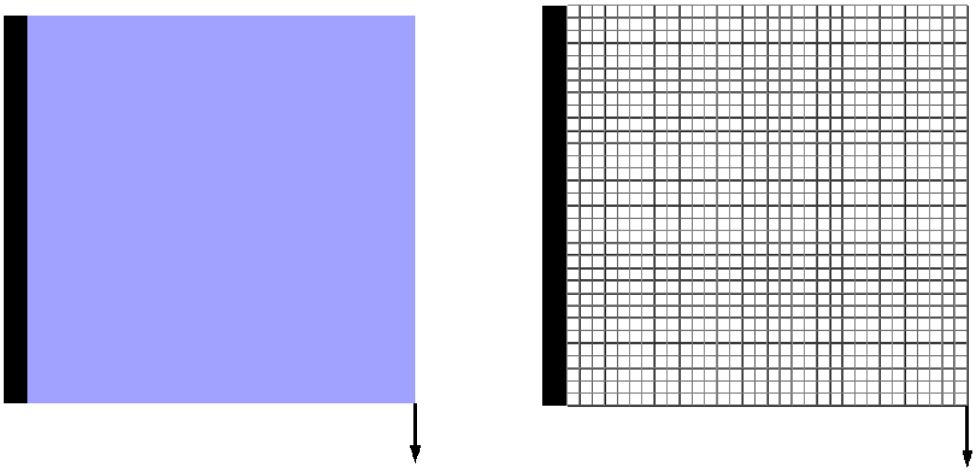
Relaxation-based topology optimization techniques are usually based on a homogenization of the material density. Homogenization results in what is referred to as a checkerboarding phenomenon in which the design density alternates between material and void; such a result is unrealistic. In order to eliminate checkerboarding, therefore producing realistic designs, additional criteria must be introduced. The most common approach is filtering which is a somewhat artificial approach to eliminate the numerically-introduced results. In the present work, the problem is redefined by imposing a B-spline smooth mass distribution which is inherently continuous and yields a formulation immune to checkerboarding. For the optimal topology this new approach is combined with an adaptive mesh refinement scheme to generate smooth high-fidelity designs.

Topological optimization problems are commonly formulated to find the stiffest structure subject to a volume constraint. Even when other design objectives are incorporated, the formulation has always been based on a volume constraint. However, from a practical design point of view, optimizing specific structural response characteristics for a fixed volume of material is rarely a useful approach. For example, in aerospace structures, mass is usually the critical design element. Launch cost is proportional to the vehicle mass, therefore increasing structural stiffness for a fixed volume is of little benefit. Similarly, in the case of structural design involving multiple materials, volume constraints may not be of practical significance. Thus, in the present work, the topological optimization problem is reformulation as a mass minimization problem with a constraint on the flexibility. Such an approach represents a more realistic problem for both single and multiple material design situations and results in very different structural topologies.

2. Topology optimization

Topology optimization was developed by Bendsøe and Kikuchi (1988) to find the amount and distribution of material in the design domain so that the objective function is minimized. Therefore, finding the optimal topology corresponds to finding the optimal connectedness, shape and number of holes in a structure based on a given criterion. Originally used to find designs with minimum compliance for a given volume of material, this method has been found to be very versatile. However, a new formulation is proposed as there are numerous difficulties with the current method.

Topology optimization is in essence a method to solve a parameter distribution problem. The parameter under investigation in this case is the material. This is an integer programming problem, as any point in a resulting design may be either solid or void. This in itself is an ill-posed problem, as every point within a domain cannot be evaluated. Therefore, a number of specific points are chosen as design variables and the entire domain is interpolated from these points. Thus, the resulting design is dependant upon the number and distribution of these degrees of freedom.



(a) Design domain (b) Finite element grid

Figure 1. Design domain, applied force, boundary conditions and finite element grid used in sample problem.

Compliance is the inverse of stiffness; thus, the minimization of the compliance is analogous to the maximization of the stiffness. The discretized minimum compliance problem can be written as:

$$\min_{\chi} \quad \vec{U}^T[K(\chi)]\vec{U}$$
$$\text{subject to:} \begin{cases} \int_{\Omega} \bar{\rho}(\chi, x)dx \leq V_{max} \\ [K(\chi)]\vec{U} = \vec{F} \\ \bar{\rho}(\chi, x) = 0 \text{ or } 1 \end{cases} \tag{1}$$

$[K]$, $\vec{U}$ and $\vec{F}$ are the stiffness matrix, nodal displacement vector and force vector, respectively. The material properties are given as functions of the design variables χ and the spacial coordinates x. The normalized density $\bar{\rho}$ must be solid or void, and there is a constraint of V_{max} on the total volume of material present.

Throughout this work, the sample problem given in Fig. 1 will be used to illustrate the techniques discussed. A 32×32 square design domain is fully constrained on the left side, while a unit point load is applied at the lower right corner in the downward direction. The material properties are a Young's modulus of 1 and a Poisson's ration of 0.3. The constraint on the volume is 40%, and the compliance of the system is to be minimized. For clarity, only planar systems (two space dimensions) are given in this report, though the basic approach carries over into three dimensions.

2.1 Relaxation

Relaxation poses an integer programming problem as a continuous problem. Continuous optimization problems are easier to solve, as the sensitivities of the objective and constraint functions can be used in the optimization. The result of this relaxation is the following transformation of the constraint on the density:

$$\bar{\rho}(\chi, x) = 0 \text{ or } 1 \quad \rightarrow \quad 0 < \bar{\rho}_{min} \leq \bar{\rho}(\chi, x) \leq 1 \tag{2}$$

where $\bar{\rho}_{min}$ is a minimum normalized density that eliminate sensitivity singularities when the density goes to zero. The minimum normalized density used in the examples is $\bar{\rho}_{min} = 0.001$.

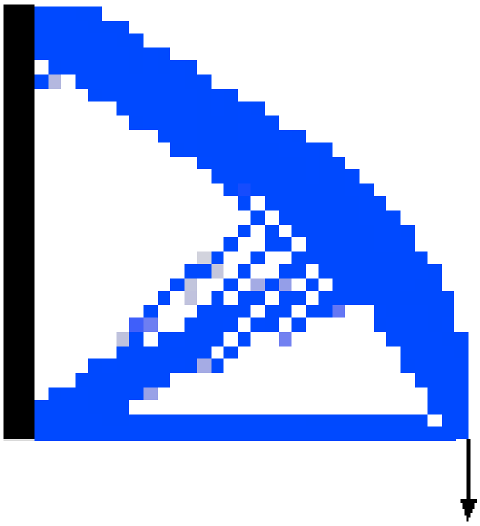
The normalized density is allowed to vary between $\bar{\rho}_{min}$ and 1, but for the final design it should be at one of the bounds. A method to penalize these intermediate densities needs to be utilized to arrive at solid-void designs. Bendsøe (1989) proposed a power-law penalized stiffness approach (also known as *SIMP* for *S*olid *I*sotropic *M*aterial with *P*enalization) as a versatile method to interpolate the elasticity tensor C_{ijkl}:

$$C_{ijkl}(\chi, x) = \bar{\rho}^{\eta}(\chi, x)C^0_{ijkl} \tag{3}$$

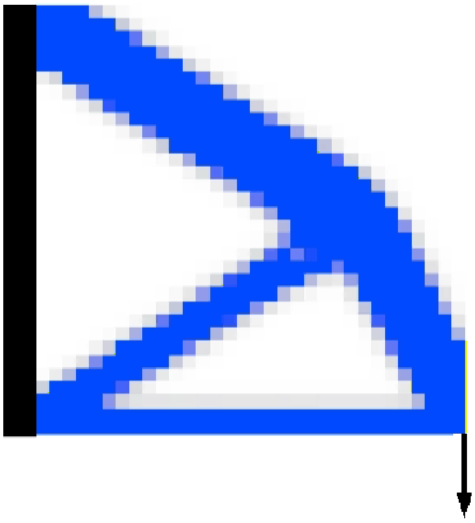
where C^0_{ijkl} is the elasticity tensor for the solid material.

2.2 Homogenization approach

Relaxation-based topology optimization techniques have until now been limited to the homogenization of design variables. Homogenization specifies that the design variables have one degree of freedom per cell; hence, they are constant within the element and discontinuous at the boundary. This causes *checkerboarding*: the optimal distribution alternates between regions of void and material as shown in Fig. 2a. Filtering techniques have been devised to alleviate this problem and improve on the smoothness of the design, but do not remove its cause, Fig. 2b.



(a) Resulting checkerboards



(b) Filtering used to suppress checkerboarding

Figure 2. Homogenization approach to topology optimization.

2.3 Continuous approach

There is nothing in Eq. 2 that requires homogenization; it only simplifies the problem and makes it more numerically efficient. A clear boundary is developed due to the discontinuity between elements, but results in solutions that physically do not model reality.

Any real structure will always differ from the models used to describe them. Tolerances specify how far the design can deviate from a norm. Therefore, modelling the change between solid and void need not require that the density field be discontinuous, but that the change be restricted to a small region. If the density remains continuous within the design domain, regions of intermediate density can be used as a tolerance on the design. Requiring that the final design be smooth, hence producible, implies that higher-order finite elements are required.

Higher-order elements have been used to represent the displacement (Bendsøe,1995), while the density remained discontinuous. Though they do give a better representation of the displacement field, the discontinuity in the density remains. Even though this approach improves the results; however, it does not exclude the formation of checkerboards or hinges.

Jog and Haber (1996) used bi-linear density elements, but found them to be numerically unstable. This is the only reference in the literature to using other types of finite elements in the interpolation of the density field. The reason for not using quadratic or higher order Lagrange and Legendre basis functions is that they do not remain positive over the entire domain of definition. Finite element theory requires that the basis functions sum to one everywhere within the domain, but have no restriction requiring them to be greater than zero. In the material distribution problem, a further restriction is that the density in the domain must always remain positive. There is no physical interpretation of a negative density and such a situation results in a non-positive definite stiffness matrix. Basis functions from the finite element method can be used to model the density field, with the extra restriction that they must remain positive.

This paper reformulates the distribution problem by using higher-order B-splines; Fig. 3 shows the quadratic and cubic variants. B-splines of degree p are piecewise polynomial functions that are $p-1$ times differentiable. That is, over a partitioned domain, the spline in each partition is a polynomial of degree p. Continuity is therefore intrinsic to the resulting distribution, with regions of intermediate density penalized using the *SIMP* method. No filtering or other restriction methods are employed. The normalized density field within an element is described by the following

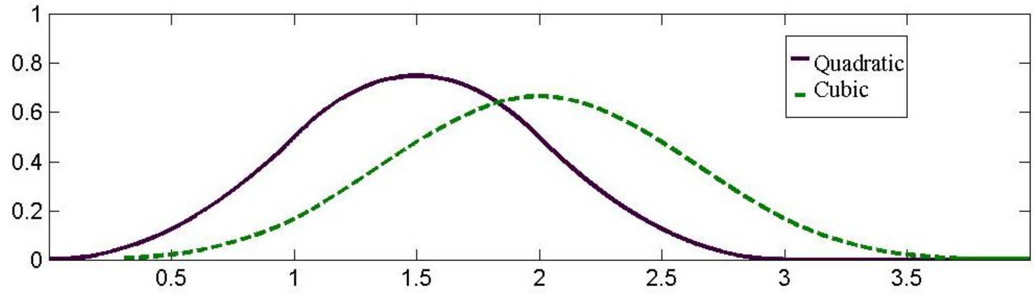
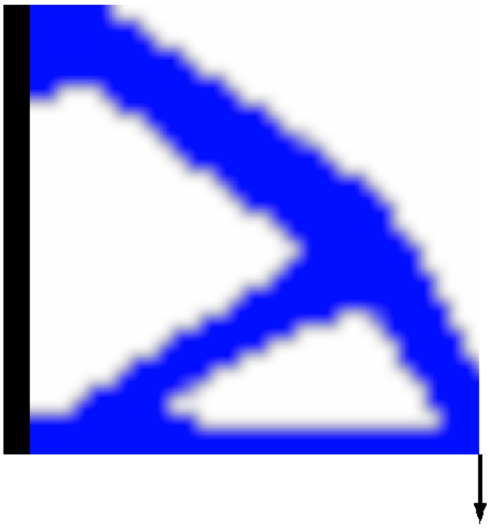
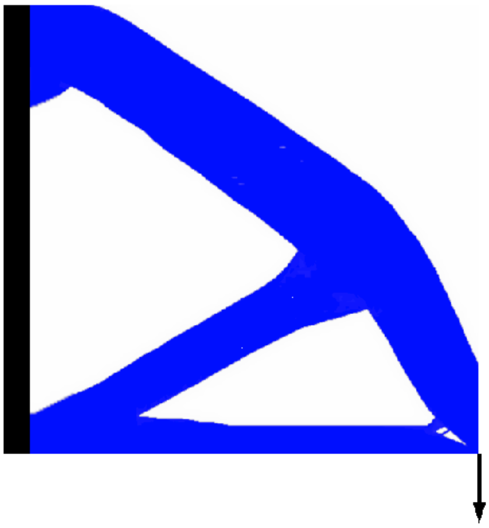


Figure 3. Quadratic and cubic B-splines.



(a) Results using original mesh



(b) Results after four mesh refinements

Figure 4. Topology optimization using quadratic B-splines.

summation:

$$\bar{\rho}(\chi, x_e) = \sum_{\alpha} \chi_\alpha \tilde{b}_\alpha(x_e) \quad (4)$$

where $\tilde{b}_\alpha$ are the non-zero local B-spline basis functions for the element.

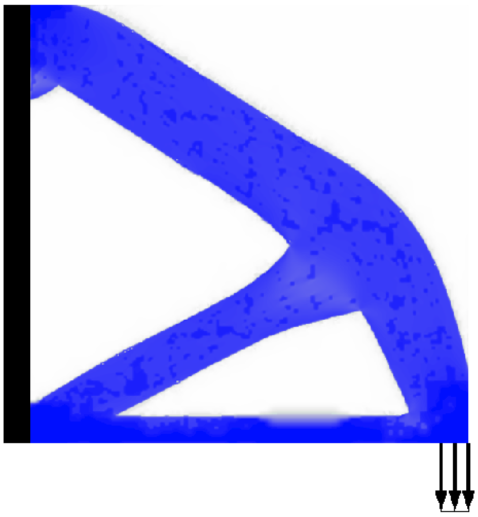
Using adaptive meshing techniques in the refinement of the interface between solid and void allows for the generation of high-fidelity models. A residual error estimate for the displacement in a homogeneous medium was developed by Kelly et al. (1983) and Gago et al. (1983). For the topology optimization problem, this estimate was used to calculate the magnitude of the density change in each local cell. Large values for this estimate indicate that the cell belongs to the transition region between solid and void. Cells can then be ranked and a specified fraction refined. Figure 4 gives the resulting optimal topology for the original mesh and the optimal topology after four mesh refinements.

3. Mass minimization

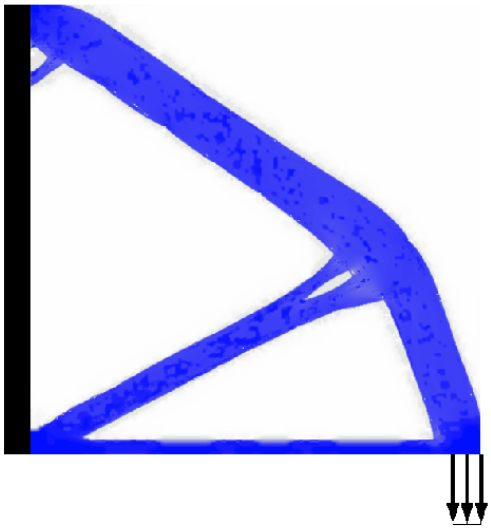
For a mass minimization problem, the required stiffness of the structure needs to be specified as a constraint. The stiffness can be given as the maximum deflection allowed under a given load. That is, the integral of the displacement u in some direction x along a specified surface Γ_1 is to be less than prescribed displacement u_{max} multiplied by the area. This displacement constraint is expressed as:

$$\int_{\Gamma_1} w_i u_i ds \leq u_{max} \Gamma_1 \quad (5)$$

where w_i are weights that determine the direction of the displacement and $\sum w_i = 1$.



(a) Displacement limit of 20



(b) Displacement limit of 32

Figure 5. Minimum mass objective function

The problem shown in Fig. 1 is revisited as a mass minimization problem with a distributed load applied and a displacement constraint at this location. Different optimal structures were found when the displacement limit was modified, as shown in Figure 5.

3.1 Multiple materials

Any significantly complex structure will be composed of multiple materials because different locations within the structure can benefit from different material properties. Sigmund (2001) developed an interpolation method for two materials and void, which has been extended here to include as many materials as desired. The normalized density field of each of the n_m materials is related to the design variables as follows:

$$\bar{\rho}^m(\chi, x_e) = \sum_{\alpha} \tilde{b}_\alpha(x) \begin{cases} \chi_{1\alpha}\chi_{(m+1)\alpha} \prod_{i=2}^{m} (1 - \chi_{i\alpha}) & \text{if} \quad m \neq n_m \\ \chi_{1\alpha} \prod_{i=2}^{m} (1 - \chi_{i\alpha}) & \text{if} \quad m = n_m \end{cases} \tag{6}$$

where $\chi_{i\alpha}$ is design variable i associated with the basis functions $\tilde{b}_\alpha$.

The mass of a certain material is the product of the volume of the normalized density and the density ρ^m of the solid material. This allows the calculation of the total mass m of the structure and the elasticity tensor as follows:

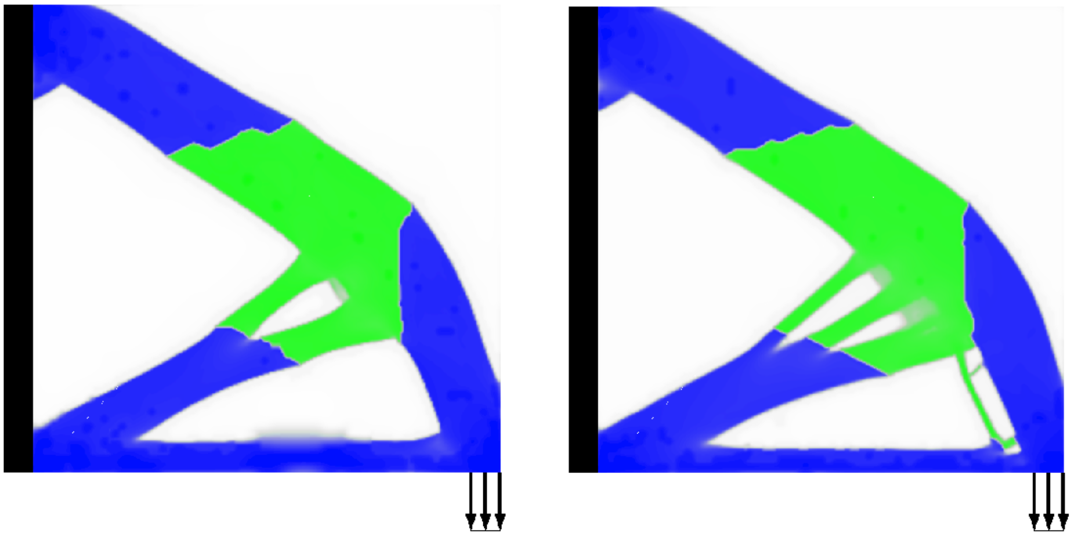
$$m = \sum_{m=1}^{n_m} \rho^m \int_\Omega \bar{\rho}^m(\chi, x) dx \tag{7}$$

$$C_{ijkl}(\chi, x) = \sum_{m=1}^{n_m} (\bar{\rho}^m(\chi, x))^\eta C_{ijkl}^m \tag{8}$$

Figure 6 shows the resulting topologies for the multiple material problem. The blue (darker) material was described in Section 2; however, the green (lighter) with a modulus of elasticity 0.9 and a Poisson's ratio of 0.1 is more effective in shear. Both materials have the same density; therefore, material placement is dependant upon the specific material requirements at that location. The *steps* at the material interface seen in the Fig. 6a are a result of the original design variable distribution, which are greatly reduced in Fig. 6b when a much denser mesh is started with.

3.2 Compliant mechanism

Compliant mechanisms are a type of flexible structure that are used to transmit force and motion. They do not rely on joints connecting rigid members, instead the structure deforms in service. The desired load characteristics are then



(a) Using original mesh and 3 refinements

(b) Starting mesh 4 times as dense with 2 refinements

Figure 6. Multiple material minimum mass objective function

matched to the drive system characteristics by the compliant mechanism.

When designing a mechanism, a certain output displacement and force is deemed adequate. An acceptable design requires the mechanism to achieve these performance specifications. Both of these requirements are constraints on the problem, hence should not be part of the objective function. This allows for the specification of another property as the objective and, for the same reasons outlined at the beginning of this section, mass minimization will be used.

For mechanism design, a specification of both the input force provided and the output force required are needed. The maximum displacement at the input port and a minimum displacement at the output port are defined as constraints. The resulting optimal structure must satisfy these requirements and be as lightweight as possible. There is one last requirement for compliant mechanisms: the structure must provide a certain stiffness at the output. This stops the formation of hinge-like structures and result in a distributed compliance mechanism, and it will provide a restoring force so that the structure will return to its undeformed state when the loads are removed. This can be easily accomplished by specifying another load case that will only be applied at the output port, and a maximum deflection (minimum stiffness) allowed.

Sigmund (1997) described a displacement inverter problem using the homogenization method for topology optimization. A similar problem using the continuous method and a mass minimization will be explained here. The input force is shown on the left hand side of Fig. 7, while on the right is the output force (2% of the input force). A one-to-one displacement inversion is desired, with a displacement limit of one third of its length. The second load case consists of a force 20% of the input force applied to the output, and the same displacement limit imposed. Figure 8 shows the optimal topology.

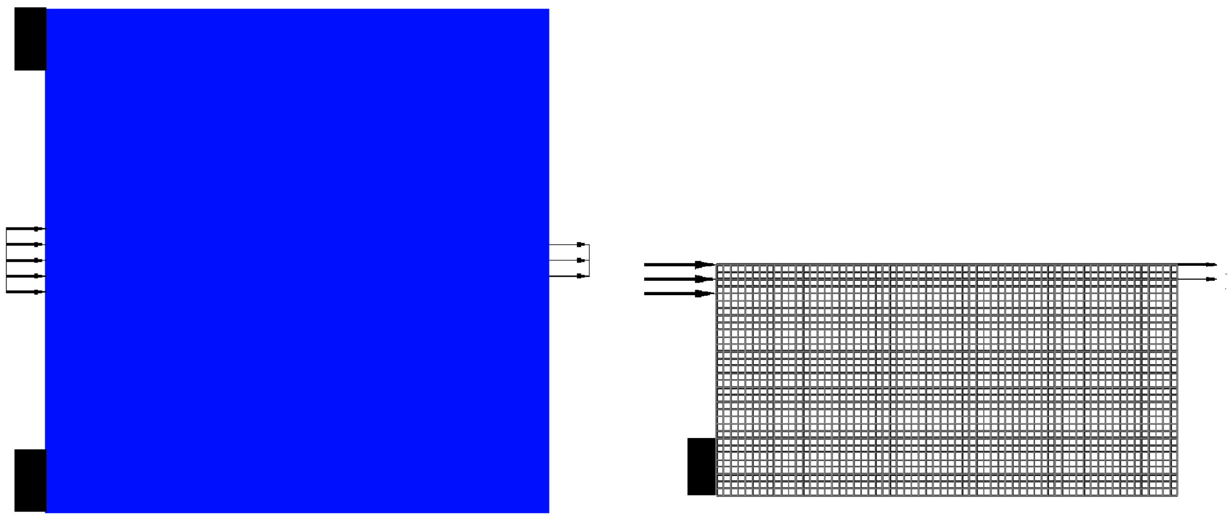
4. Conclusion

A very precise approach to topological optimization which yields manufacturable structures without imposing artificial constraints has been developed. Based on straight-forward and realistic hypotheses, the new approach combines local mesh refinement and B-spline interpolation of the material density to achieve this goal.

This procedure was applied to a number of design problems involving mass minimization. This allows for the deflections under different loading conditions to be prescribed in the optimization and results in lightweight structures which are only as rigid as necessary. This is a more natural formulation for engineering optimization problems than applying a constraint on the volume.

5. Acknowledgements

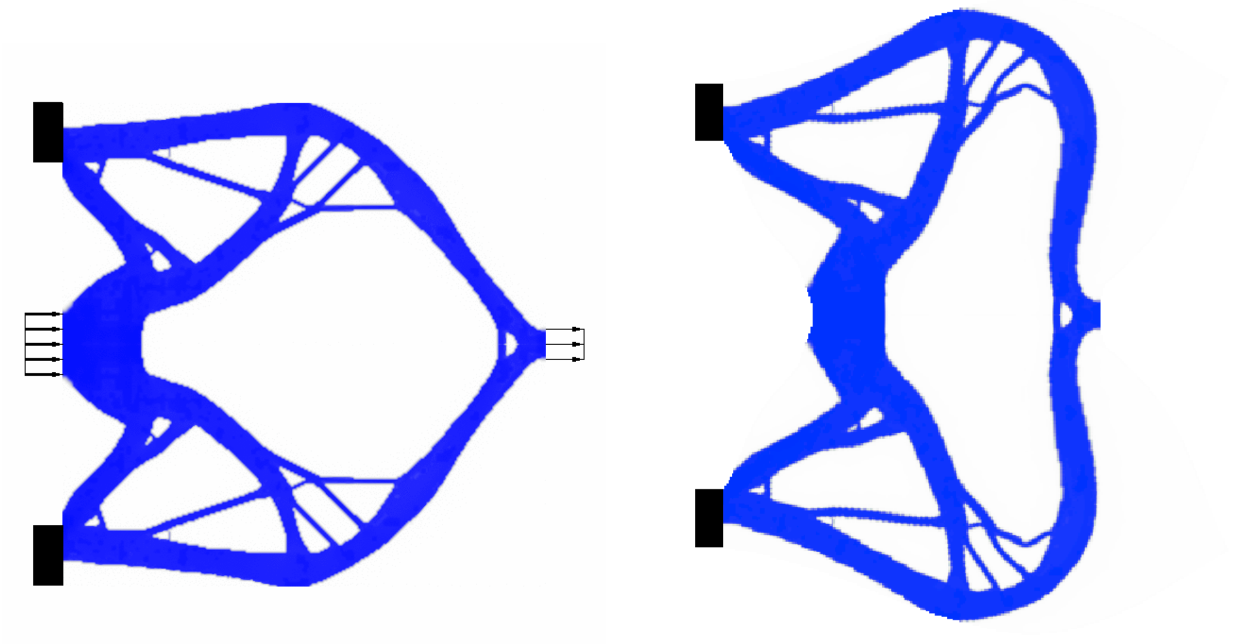
The computer code developed uses the `deal.II` open source finite element library (Bangerth, Hartmann and Kanschat, 2005).



(a) Full design space

(b) Symmetry used to reduce size of mesh

Figure 7. Design space for a mechanical inverter.



(a) Optimal topology

(b) Deformed topology

Figure 8. Optimal mechanical inverter for the minimization of mass.

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