STOCHASTIC APPROACH TO HEAT CONDUCTION PROBLEM USING THE GALERKIN METHOD

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Abstract. *In this paper a one dimensional steady state heat conduction problem with the uncertainty introduced through the thermal conductivity is modeled and solved using Galerkin method. The problem of the uncertainty in the thermal conductivity is modeled as random variable and random field. The functional space, which was used to obtain the numerical solutions, is defined by the space generated by functions of the deterministic problem and the polynomials chaos. The first and the second statistic moments from the Galerkin solution are compared with the Monte Carlo solution giving good results.*

Keywords: *Heat Transfer, Galerkin Method, Polynomial chaos, Stochastic solution, Monte Carlo simulation*

1. Introduction

The modeling and analysis of the thermal behavior of a mechanical system is a very important step on the standard design methodology. For these activities the simulation of the structural performance is important for the design and forecasting of the thermal behavior of the system. The numerical heat transfer solutions have been increased a lot during the last decades, due to the development of new analysis methods and the growing in the computational capacity. These advances stimulate scientists and engineers to look for more powerful and accurate methods in order to solve more complex problems. However, the development in the description of the constitutive models and improvements in consistency and robustness of the mathematical models, are not enough to preview the behavior of the uncertainties present on the physical problems, mainly because they are not considered during the problem modeling. The use of probabilistic modeling in continuous mechanical problems is increasing during the last years. The Galerkin Method applied to stochastic systems was presented firstly by Spanos and Ghanem (1981).

The conduction heat transfer with uncertainties about the materials properties has been studied. Hien and Kleiber (1997) studied the transient heat transfer conduction using the perturbation technique; Kaminski and Hien (1999) applied the perturbation method to transient heat transfer conduction in composition materials; Xiu and Karniadakis (2003) applied the Galerkin method to study the transient heat transfer conduction; Emery (2004) solved the transient heat transfer conduction using different methods. The present paper includes the uncertainty of the random variable during the modeling.

In this paper the steady state conduction heat transfer problem is modeled with uncertainties introduced through the thermal conductivity. Two approached solutions are obtained to problems where uncertainty in the thermal conductivity is modeled as a random variable or as a random field. In the first case, a lognormal random variable is used. In the second case, a random field is used. The Karhunem-Loeve (KL) expansion is applied when uncertainty about the thermal conductivity were modeled as a random Gaussian field. The Monte Carlo simulation is used to evaluate the Galerkin method performance. It is worth mentioning that the technique presented in this paper can be applied in many areas of the engineering, physics and mathematics fields.

2. The stochastic heat equation

The steady state stochastic conduction heat transfer equation is given by

$$\begin{cases} \nabla\bullet(\kappa\nabla u)(x,\theta) = f(x), & \forall (x,\theta)\in\Omega\times\Theta \\ u(x) = h(x), & \forall x\in\partial\Omega \end{cases} \quad (1)$$

where u, κ and f are temperature field, thermal conductivity and source term, respectively, Ω is a closed domain, θ is a random variable and Θ is a probability space. The numerical solutions presented in this paper consider two study cases related to the uncertainty in the thermal conductivity, namely: random variable and random field. For the first case, random variable, we use a lognormal variable. For the random field case, a Gaussian random field is applied. The first and second statistical moments of the numerical solutions obtained with the Galerkin method are compared with the statistical moments from the Monte Carlo Simulation (MCS).

3. Karhunem-Loeve Expansion

When the uncertainty is modeled as random field, it becomes necessary to discretize the problem domain. The Karhunen-Loeve (KL) expansion proposes that the random field κ, to be represented in a form of a convergent expansion in quadratic average defined as

$$\kappa(x,\theta) = \kappa_0(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\varphi_i(x)\xi_i(\theta) \tag{2}$$

where ξ, λ and ϕ are the orthonormal random variables, the eigenvalues and the eigenfunctions, respectively. κ_0 is the expected value of the thermal conductivity. The eigenvalues and the eigenfunctions used in the KL expansion are originated from the proper value problem defined by Eq. (3),

$$\int_{\Omega} C_\kappa(x,y)\overline{\varphi}_i(y)\,dy = \lambda_i\varphi_i(x), \qquad \forall x \in \Omega \tag{3}$$

A KL expansion inconvenience is the necessity of the eigenvalues problem solutions expressed in Eq. (3). There are analytical methods to determine the eigenvalues and eigenfunctions for some kinds of covariance functions defined in simple domains. When it is not possible to apply the analytical solution, Ghanem and Spanos (1991) proposed the Galerkin technique, in order to estimate a solution for the proper value problem. In this work, when the uncertainty is modeled as a Gaussian random field for the covariance function, eigenfunctions of the proper value problem are previously known.

4. Random variable representation in terms of polynomials chaos

The polynomials chaos which are also known as Wiener-Chaos, are multi-dimensional Hermite polynomials, in standard Gaussian random variables (Wiener, 1947). Through the Cameron-Martin theorem (Cameron and Martin, 1947), these polynomials form a base, for a dense subspace of the second order random variables, $L^2(\Theta,\mathcal{F},P)$. Let $H \subseteq L^2(\Theta,\mathcal{F},P)$ a Gaussian separable Hilbert space and $\mathcal{P}_n(H)$ the vectorial space of all polynomial of order smaller than *n*.

$$\mathcal{P}_n(H) = \left\{ p\left(\{\xi_i\}_{i=1}^{M}\right) : p \text{ is polynomials of degree} \le n; \xi_i \in H, \forall i = 1,\ldots,M; M < \infty \right\} \tag{4}$$

and

$$H^{:n:} = \overline{\mathcal{P}}_n(H) \cap \overline{\mathcal{P}}_{n-1}(H)^{\perp} \tag{5}$$

where $\overline{\mathcal{P}}_n$ is the closure of $\mathcal{P}_n$. The space $H^{:n:}$ is known as chaos homogeneous of order *n*, for $n = 0$ we have $H^{:0:} = \overline{\mathcal{P}}_0(H)$, the space of constants and $H^{:-1:} = \overline{\mathcal{P}}_{-1}(H) = \{0\}$. The space $\mathcal{P}(H) = \bigcup_n \mathcal{P}_n(H)$ is a dense subspace in $L^p(\Theta,\mathcal{F},P)$ with $p \in \mathbb{N}$. Particularly, $L^2(\Theta,\mathcal{F},P)$ can be decomposed as (Jason, 1987)

$$L^2 = \bigoplus_{0}^{\infty} H^{:n:} \tag{6}$$

Equation (6) expresses an orthogonal decomposition for $L^2(\Theta,\mathcal{F},P)$ known as Wiener-Chaos decomposition, or chaos decomposition. One of the applications of the decomposition is the representation of the element $X \in L^2(\Theta,\mathcal{F},P)$ using the expansion

$$X = \sum_{n=0}^{\infty} X_n, \qquad X_n \in H^{:n:} \tag{7}$$

Equation (7) represents an important theoretical result of the approximation theory applied to stochastic systems.

The solution of a stochastic system will be expressed as a non-linear function in terms of standard Gaussian random variables. This function is expanded using polynomials chaos in the following way,

$$u_i(\theta) = \sum_{p\ge 0} \sum_{n_1+\ldots+n_r=p} \sum_{i_1,\ldots,i_r} u_{i_1\ldots i_r}^{j_1\ldots j_r} \Gamma_p\left(\xi_{i_1}(\theta),\ldots,\xi_{i_r}(\theta)\right) \tag{8}$$

where Γ_p is the polynomial chaos of order p and $u_{i_1 \dots i_r}^{j_1 \dots j_r}$ are the polynomial coefficients, and the superscript refers to the number of occurrences $\xi_{i_k}(\theta)$. The chaos polynomial of order p consists of a Hermite Polynomial, in standard Gaussian random variables $\left\{\xi_{i_k}(\theta)\right\}_{k=1}^{r}$, with order not exceeding p. Introducing an injective mapping in the set of scripts $\left\{i_k\right\}_{k=1}^{r}$ and $\left\{j_k\right\}_{k=1}^{r}$, Eq. (8) can be represented by

$$u_i(\theta) = \sum_{j=1}^{\infty} u_{ij} \psi_j\left(\xi(\theta)\right) \tag{9}$$

The internal product between the polynomials ψ_i e ψ_j is defined as

$$\left\langle \psi_i, \psi_j \right\rangle = \int_{\Theta} \psi_i(\xi) \psi_j(\xi) e^{-\frac{\|\xi\|^2}{2}} \boldsymbol{d}\boldsymbol{\xi}(\theta) \tag{10}$$

These polynomials generate a complete orthonormal system with the following properties

$$\psi_0 = 1, \qquad \left\langle \psi_i, \psi_j \right\rangle = \delta_{ij}, \quad \forall i, j \in \mathbb{N} \tag{11}$$

Considering the objective is to obtain a random temperature field approached solutions, instead of working with the polynomials chaos expansion, an expansion partial sum will be used in Eq. (8). In order to approach a second-order random variable, $u \in L^2(\Theta, \mathcal{F}, P)$, with $\dim(H) = M$, that is $\boldsymbol{\xi} = \left\{\xi_k(\theta)\right\}_{k=1}^{M}$, and using the chaos polynomial of order p, the number of chaos polynomials which will be used is given by,

$$P = 1 + \sum_{\zeta=1}^{p} \frac{1}{\zeta!} \prod_{\vartheta=0}^{\zeta-1} (M + \vartheta) \tag{12}$$

Equation (12) proves that the partial terms number of the expansion sum defined in Eq. (9), can be a strong function of the H dimension and of the order demanded during the approximation by polynomials chaos.

5. Problem formulation of the stochastic heat conduction using the Galerkin method

A lognormal random variable is obtained by applying the exponential operator to the Gaussian random variable v

$$\kappa(\theta) = e^{v(\theta)} \tag{13}$$

The Gaussian random variable v is defined as

$$v(\theta) = v_0 + \sigma_v \xi(\theta) \tag{14}$$

where v_0 is the expected value, σ_v is the standard deviation of the random variable and ξ is a Gaussian random variable.

$$\kappa = e^{v_0 + \sigma_v \xi} \tag{15}$$

Substituting Eq. (13) in Eq. (1) results in the following relation

$$\begin{cases} e^{v_0} \nabla \bullet \left(e^{\sigma_v \xi(\theta)} \nabla u \right) = f(x), & \forall (x, \theta) \in \Omega \times \Theta \\ u(x) = 0, & \forall\, x \in \partial\Omega \end{cases} \tag{16}$$

In order to obtain an approached solution to the problem of Eq. (16), we can define $V_m \triangleq \left\langle \phi_1, \dots, \phi_m \right\rangle$, $\dim(V_m) = m$, such that $V_m \subset V$, where $V \triangleq \left\{ \phi \in C^2(\Omega) \middle| \phi(x) = 0, \forall x \in \partial\Omega \right\}$. The approached solutions can be proposed to be obtained in the following form

$$u_m(x, \theta) = \sum_{i=1}^{m} u_i(\theta) \phi_i(x) \tag{17}$$

where $\left\{u_i, \phi_i\right\}_{i=1}^{m}$ are the coordinates functions and V_m are the base functions. Substituting Eq. (17) in Eq. (16) results in

$$\varepsilon_m(\theta) = e^{v_0 + \sigma_v \xi(\theta)} \sum_{i=1}^{m} \nabla \bullet (\nabla \phi_i) u_i(\theta) - f \tag{18}$$

Using the orthogonality condition between the residue expressed in Eq. (18) and the *k-th* base function V_m we obtain

$$(\varepsilon_m, \phi_k) = 0, \quad \forall k = 1, \dots, m \quad \Rightarrow \quad e^{\sigma_v \xi(\theta)} \sum_{i=1}^{m} (\nabla \phi_i, \nabla \phi_k) u_i(\theta) = -(f, \phi_k) e^{-v_0} \tag{19}$$

The coordinate function will be approached through the expansion truncation given by Eq. (9), with $\Psi_n = \left\{ \psi(\xi) \in L^2(\Theta, \mathcal{F}, P) \middle| \psi \text{ is polynomials of degree} \le n \right\}$, the space of the problem solution defined by Eq. (16) is given by $V_m \otimes \Psi_n$. Introducing an injective mapping of the scripts, the partial sum of the function coordinate can be represented by

$$u_{im}(\theta) = \sum_{j=1}^{m} u_{ij}\psi_j(\xi(\theta)) \tag{20}$$

where $\{u_{ij}, \psi_j\}_{j=1}^{m}$ are the constants to be determined and polynomials chaos, respectively. Substituting Eq. (20) in Eq. (18) we obtain

$$e^{\sigma_v \xi(\theta)} \sum_{i,j=1}^{m,n} ((\phi_i, \phi_k)) \psi_j(\xi(\theta)) u_{ij} = -(f, \phi_k) e^{-v_0} \tag{21}$$

where $((\phi_i, \phi_k)) = (\nabla\phi_i, \nabla\phi_k)_{L^2(\Omega)}$ is the internal product in $L^2(\Omega)$. The residua ε_{mn} can be obtain From Eq. (12).

$$\varepsilon_{mn} = e^{\sigma_v \xi(\theta)} \sum_{i,j=1}^{m,n} ((\phi_i, \phi_k)) \psi_j(\xi(\theta)) u_{ij} + (f, \phi_k) e^{-v_0} \tag{22}$$

The orthogonality condition between the residua and the *l-th* base function of Ψ_n is imposed generating the following equations system

$$\langle \varepsilon_{mn}, \psi_l \rangle = 0, \quad \forall l = 1,...,n \quad \Rightarrow \quad \sum_{i,j=1}^{m,n} ((\phi_i, \phi_k)) \langle e^{\sigma_v \xi} \psi_j, \psi_l \rangle u_{ij} = -(f, \phi_k) e^{-v_0} \tag{23}$$

Analyzing the term $\langle e^{\sigma_v \xi} \psi_m, \psi_l \rangle$ we obtain

$$\langle e^{\sigma_v \xi} \psi_m, \psi_l \rangle = \int_{-\infty}^{+\infty} \psi_m(\xi) \psi_l(\xi) e^{-\frac{1}{2}\xi^2 + \sigma_v \xi} d\xi = e^{\frac{1}{2}\sigma_v^2} \langle (\psi_m \circ \tau_{\sigma_v}), (\psi_l \circ \tau_{\sigma_v}) \rangle \tag{24}$$

where $\tau_{\sigma_v} : L^2(\Theta, \mathcal{F}, P) \to L^2(\Theta, \mathcal{F}, P)$ is the translation operator which is defined by

$$\tau_{\sigma_v}(\psi(\xi)) = \psi(\xi + \sigma_v) \tag{25}$$

Substituting Eq. (24) in Eq. (23) gives

$$\sum_{i,j=1}^{m,n} ((\phi_i, \phi_k)) e^{\frac{1}{2}\sigma_v^2} \langle (\psi_m \circ \tau_{\sigma_v}), (\psi_l \circ \tau_{\sigma_v}) \rangle u_{ij} = -(f, \phi_k) e^{-v_0} \tag{26}$$

Denoting $d_{ml} = \langle (\psi_m \circ \tau_{\sigma_v}), (\psi_l \circ \tau_{\sigma_v}) \rangle$ and $k_{ik} = ((\phi_i, \phi_k))$, Eq. (26) can be rewritten as

$$\sum_{i,j=1}^{m,n} d_{ml} k_{ik} u_{ij} = -(f, \phi_k) e^{-\left(v_0 + \frac{1}{2}\sigma_v^2\right)} \tag{27}$$

Equation (27) has the following matrix representation

$$\sum_{j=1}^{m} K_j U_j = F_0 \tag{28}$$

where

$$K_j = \left[k_{pq}^j\right]_{n \times n}, \; k_{pq}^j = d_{ml} k_{pq} \quad \forall l = 1,...,m \tag{29}$$

Writing the system presented by Eq. (28) in a block form, the above mentioned system can be rewritten as

$$KU = F \qquad \Rightarrow \qquad \begin{pmatrix} K_0 & \cdots & K_{1,m} \\ \vdots & \ddots & \vdots \\ K_{m,1} & \cdots & K_0 \end{pmatrix} \begin{pmatrix} U_0 \\ \vdots \\ U_m \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \\ 0 \end{pmatrix} \tag{30}$$

where K is the stiffness matrix, U is the temperature vector and F *is* the source term. The equations system stiffness matrix, Eq. (30), is symmetric.

In the case where the uncertainty is modeled as a random field, the purpose is to use the KL expansion to represent the thermal conductivity

$$\kappa(x, \theta) = \kappa_0(x) + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i(x) \xi_i(\theta) \tag{31}$$

where ξ are the orthonormal random variables, λ are the eigenvalues, ϕ are the eigenfunctions and $\kappa_0(\cdot)$ is the expected value of the thermal conductivity. The eigenvalues and the eigenfunctions are derived from the proper value problem in Eq. (3). For the studied problem we assume

$$\langle \kappa(x) \rangle = \kappa_0, \qquad \forall x \in \Omega \tag{32}$$

Carrying out the truncation of the KL expansion in the *M-th* term of Eq. (31), the approached problem can be defined as

$$\begin{cases}\nabla\bullet\left(\kappa_0\nabla u\right)+\sum_{i=1}^{M}\sqrt{\lambda_i}\,\xi_i\nabla\bullet\left(\varphi_i\nabla u\right)=f\left(x\right), & \forall\left(x,\theta\right)\in\Omega\times\Theta\\ u\left(x\right)=0, & \forall x\in\partial\Omega\end{cases}\tag{33}$$

Based on the methodology used in the case of the random variable, now the objective is to obtain solutions in the form presented by Eq. (9). Carrying out the truncation of the expansion results in the following residua

$$\varepsilon_{MN}\left(\theta\right)=\kappa_0\sum_{j=1}^{N}\nabla\bullet\left(\nabla\phi_j\right)u_j\left(\theta\right)+\sum_{i,j=1}^{M,N}\sqrt{\lambda_i}\,\xi_i\nabla\bullet\left(\varphi_i\nabla\phi_j\right)u_j\left(\theta\right)-f\tag{34}$$

The residua expressed in Eq. (34) is minimized with respect to the elements of the space base, V_m

$$\left(\varepsilon_{MN},\phi_n\right)=0\quad\Rightarrow\quad\sum_{i,j=1}^{M,N}\left[\kappa_0\left(\left(\phi_j,\phi_n\right)\right)+\sqrt{\lambda_i}\xi_i\left(\varphi_i\nabla\phi_j,\nabla\phi_n\right)\right]u_j\left(\theta\right)=-f\tag{35}$$

Equation (35) represents a system of equations which unknown terms are the coordinate functions of Eq. (9). The coordinate functions are approached using a base of polynomials chaos finite dimensions, defined as $\Psi_n^M=\left\{\psi\left(\left\{\xi_i\right\}_{i=1}^{M}\right)\in L^2\left(\Theta,\mathcal{F},P\right)\middle|\,\psi\text{ is the polinomial of degree}\le n\right\}$. Thus the approximation space of the problem defined by Eq. (33) is given by $V_m\otimes\Psi_n^M$. The residua generated by the coordinate functions approach is

$$\varepsilon_{MNP}=\sum_{m=1}^{P}\sum_{i,j=1}^{M,N}\left[\kappa_0\left(\left(\phi_j,\phi_n\right)\right)+\sqrt{\lambda_i}\xi_i\left(\varphi_i\nabla\phi_j,\nabla\phi_n\right)\right]u_{jm}\psi_m-f\tag{36}$$

Again the orthogonality condition of the residua is imposed

$$\left\langle\varepsilon_{MNP},\psi_q\right\rangle=0\quad\Rightarrow\quad\sum_{m=1}^{P}\sum_{i,j=1}^{M,N}\left[\kappa_0\left(\left(\phi_j,\phi_n\right)\right)\delta_{mq}+\sqrt{\lambda_i}\left\langle\xi_i\psi_m,\psi_q\right\rangle\left(\varphi_i\nabla\phi_j,\nabla\phi_n\right)\right]u_{jm}=-f\tag{37}$$

Denoting $k_{jn}=\left(\left(\phi_j,\phi_n\right)\right)$, $c_{imq}=\left\langle\xi_i\psi_m,\psi_q\right\rangle$ and $k_{jn}^{i}=\left(\varphi_i\nabla\phi_j,\nabla\phi_n\right)$, Eq. (37) can be rewritten as

$$\sum_{m=1}^{P}\sum_{i,j=1}^{M,N}\left(\kappa_0k_{jn}\delta_{mq}+\sqrt{\lambda_i}c_{imq}k_{jn}^{i}\right)u_{jm}=-f\tag{38}$$

Equation (38) has the following matrix representation

$$\sum_{i,j=1}^{M,P}\left(K_0+K_{ij}\right)U_j=F_0\tag{39}$$

where

$$K_0=\left[k_{ij}\right]_{n\times n},k_{ij}=\kappa_0\left(\left(\phi_i,\phi_j\right)\right)\quad K_{ij}=\left[k_{pq}^{ij}\right]_{n\times n},\ k_{pq}^{ij}=\sqrt{\lambda_i}c_{ijn}\left(\varphi_i\nabla\phi_p,\nabla\phi_q\right)\quad i=1,...,M\ \ j,n=1,...,P\tag{40}$$

Writing the system presented by Eq. (40) in a block form, the above mentioned system can be rewritten as

$$KU=F\qquad\Rightarrow\qquad\begin{pmatrix}K_0&\cdots&K_{1,P}\\ \vdots&\ddots&\vdots\\ K_{P,1}&\cdots&K_0\end{pmatrix}\begin{pmatrix}U_1\\ \vdots\\ U_P\end{pmatrix}=\begin{pmatrix}F_0\\ 0\\ 0\end{pmatrix}\tag{41}$$

where $K=\left[k_{ij}\right]_{N.P\times N.P}$, with $K\in M_{N.P}\left(\mathbb{R}\right)$ and $U,F\in M_{N.P\times1}\left(\mathbb{R}\right)$, are the system stiffness matrix, the vectors temperature and the source, respectively.

6. Numerical results

In this section we present the numerical results of the one-dimensional thermal conduction problem subject to the Dirichlet boundary conditions. The problem domain is $\Omega=\left(0,1\right)$ and the source term is defined as

$$f\left(x\right)=4000,\qquad\forall x\in\left(0,1\right)\tag{42}$$

and the boundary conditions are given by

$$\begin{cases}u\left(0\right)=0\\ u\left(1\right)=0\end{cases}\tag{43}$$

Two different cases of uncertainty modeling of the thermal conductivity will be considered. In the first case, the uncertainty of the thermal conductivity is modeled as a lognormal random variable. In the second case, the uncertainty of the thermal conductivity is modeled as Gaussian random field. To make sure of the obtained results quality, the obtained numerical solutions through the Galerkin method are analyzed by comparing the moments in terms of the first- and second-moment of the random field and the temperature. For this reason the variance error function is defined as

$$e(x) = 100\% \times \frac{\left|\left(\sigma_u^2 - \sigma_{uMC}^2\right)(x)\right|}{\sigma_{uMC}^2(x)}, \qquad \forall x \in (0,1) \tag{44}$$

where σ_u^2 is the variance function of the temperature random field and σ_{uMC}^2 are the results obtained using the Galerkin method. The statistical moments obtained using the Monte Carlo simulation were done through 5,000 structural realizations. All the variables are in the International System of Units (SI).

6.1. Case 1 – Random variable

In this problem the thermal conductivity is modeled as a lognormal random variable, where the expected value and standard deviation are.

$$\mu_\kappa = 200 \qquad \sigma_\kappa = 20 \tag{45}$$

where v_0 and σ_v are the expected value and standard deviation from the Gaussian lognormal variable ν. The different numerical solutions result from the base elements number generated by the polynomials chaos. Figures 1a and 1b show the variance function and the variance relative error, respectively. It can be concluded from Figure 1b that the error function has practically a constant value.

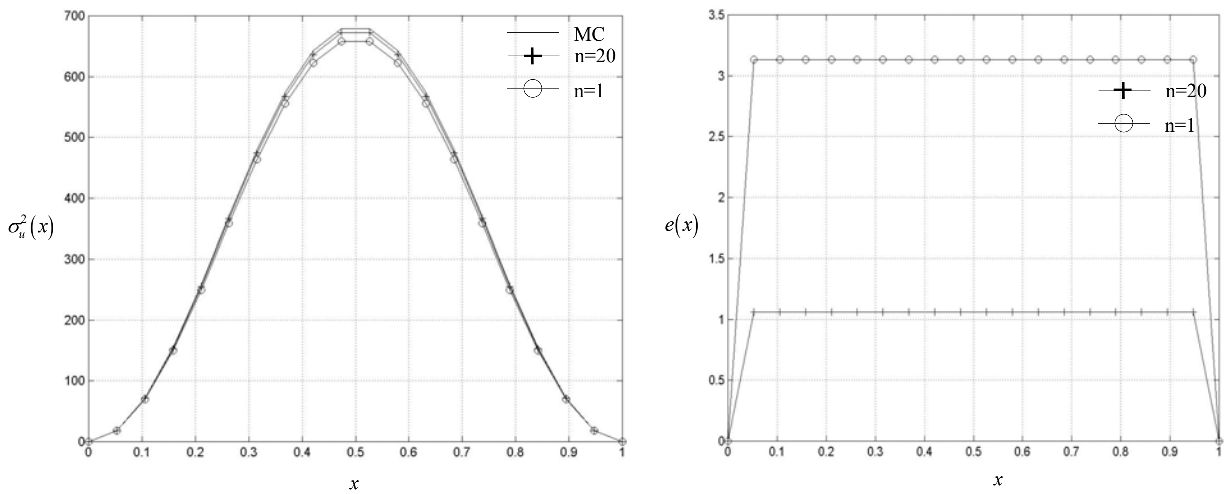

Figure 1: a) Variance function of the random temperature field; b) Relative error function in variance of the random temperature field.

Table 1 presents the expected value function, the temperature field random variance and the relative error function in variance. The functions were evaluated at the position of $x = \frac{1}{2}$.

Table 1: Expected value, variance and relative error function in variance of the temperature field random.

Degree of the polynomials chaos	$\mu_u\left(\frac{1}{2}\right)$	$\sigma_u^2\left(\frac{1}{2}\right)$	$e\left(\frac{1}{2}\right)$
1	307.0580	657.8797	3.1318
10	307.0708	671.9381	1.0618

It can be observed from Table 1 that increasing the polynomials chaos degree, the results become better for the variance. The approached solutions obtained with the polynomial chaos with degree equal to 20 presented the same results than of that with degree equal to 10.

6.2. Case 2 – Random field

In this problem the thermal conductivity is modeled as a Gaussian random field, where the expected value and variance function are given by

$$\kappa_0 = 200, \qquad C_\kappa(x,y) = 400.e^{-\frac{\|x-y\|_2}{l_\kappa}}, \quad \forall (x,y) \in (0,1)^2 \tag{46}$$

where κ_0, C_κ and l_κ are the expected value, the variance function and the correlation length , from de random field of the thermal conductivity. The KL expansion is used to approach the thermal conductivity random field. For the approached solution by means of the Galerkin method were used 5 terms in the KL expansion and polynomial chaos with degree equal to 5.

Figure 2 presents the relative error function in variance of the random temperature field. From the comparison between the figures 1b and 2 it can be observed that the case where the uncertainty is modeled as a random field, the error function assumes higher values compared to the case where the uncertainty is modeled as a random variable, Figures 3a and 3b present the variance functions and the variance relative error of the random temperature field, respectively.

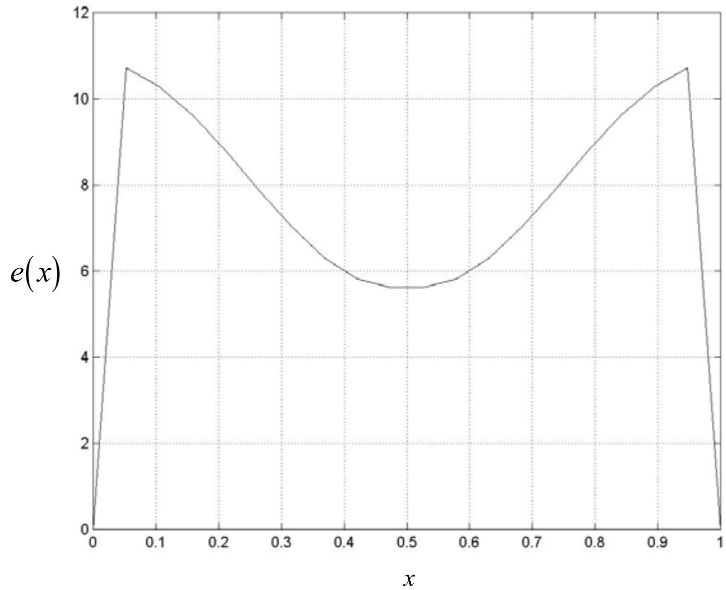

Figure 2: Relative error function in variance of the random temperature field.

It can be noted from figure 3a that the expected values obtained by the Monte Carlo simulation are similar to those obtained by the Galerkin Method. The variance function presents a good approximation as can be observed from figure 3b, even in the case where the thermal conductivity is modeled as a random variable. This is due to the fact that the approximation of the random field was conducted using the KL expansion.

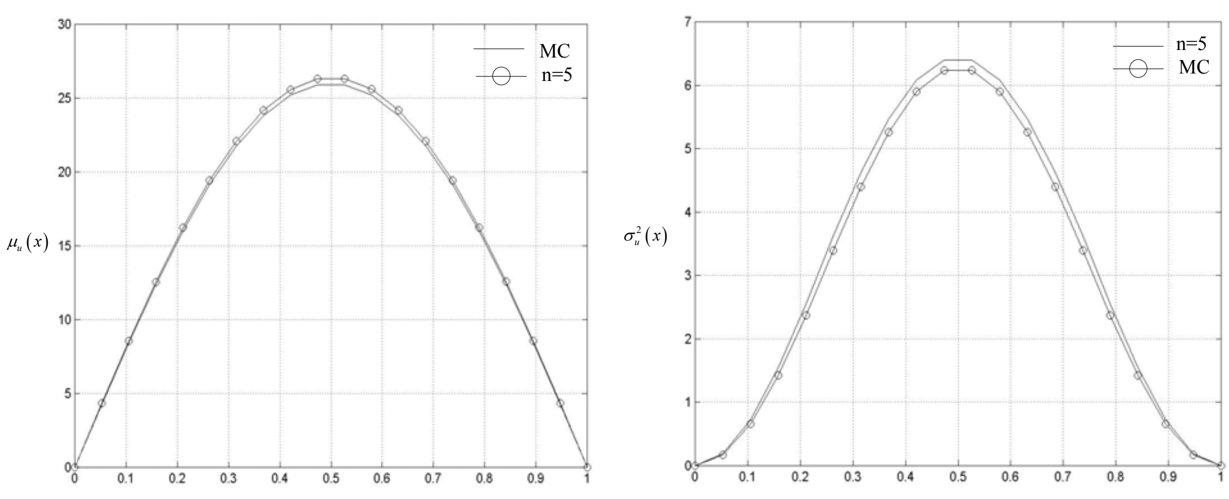

Figure 3: a) Expected value function of the random temperature field; b) Variance function of the random temperature field.

7. Conclusions

The Galerkin Method applied to a heat transfer problem proposed in this work showed satisfactory results. The case where the uncertainty about the thermal conductivity was modeled as a random variable presented better results than the method in which the uncertainty about the thermal conductivity was modeled as a Gaussian random field. But the last mentioned method also presented satisfactory comparative results. The relative error in the variance function presented higher values compared to the case where the uncertainty was modeled as a random variable. It is important to mention that using the formal solution expressed in Eq. (9) one can obtain the probability density function using the Gram-Charlier or Edgeworth expansions. It is also worth mentioning that this technique has a wide application in the areas of engineering, physics and mathematics.

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9. Responsibility notice

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