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A LOCKING-FREE LAMINATED COMPOSITE PLATE FINITE ELEMENT

Ivan Moura Belo
Programa de Pós-Graduação em Engenharia Mecânica, Pontifícia Universidade Católica do Paraná, Rua Imaculada Conceição, 1155 - Prado Velho, CEP: 80.215-901, Curitiba - Paraná
ivanbelo@yahoo.com.br

João Elias Abdalla Filho
Programa de Pós-Graduação em Engenharia Mecânica, Pontifícia Universidade Católica do Paraná, Rua Imaculada Conceição, 1155 - Prado Velho, CEP: 80.215-901, Curitiba - Paraná
joao.abdalla@pucpr.br

Roberto Dalledone Machado
Programa de Pós-Graduação em Engenharia Mecânica, Pontifícia Universidade Católica do Paraná, Rua Imaculada Conceição, 1155 - Prado Velho, CEP: 80.215-901, Curitiba - Paraná
roberto.machado@pucpr.br

Abstract. *A plate finite element which is locking-free is developed using strain gradient notation for the analysis of laminated composites. The element is based on a first-order shear deformation theory and on the equivalent lamina assumption. Strains and stresses can be calculated at different points through the laminate's thickness. The parabolic nature of the transverse shear strain distribution is included into the model by the use of an adequate weighting function. The physically interpretable strain gradient notation allows for the detailed a-priori analysis of the finite element model. The polynomial expansions are inspected and spurious terms responsible for locking are precisely identified in the shear strainsexpansions. The element is corrected by simply removing the spurious terms from those expansions, which is an equivalent procedure to performing reduced-order integration. The advantage here is that the error elimination is done prior to the element's computational implementation. The element is implemented into a Fortran finite element code in two versions; namely, with and without spurious terms. Results provided by both models are compared to show the effects of the spurious terms on the solutions. Results are compared to analytic solutions to validate the element and the spurious terms removal procedure.*

Keywords: *Laminated Composites, Plates, Finite Elements, Strain Gradient Notation, Locking*

1. Introduction

In this work, a finite element for the analysis of laminated composite plates developed using strain gradient notation is presented. The element is based on a first-order shear deformation theory, which considers the transverse shear strains according to Mindlin's theory. The element is also based on the equivalent lamina assumption, which treats the laminate as one single, orthotropic lamina plate whose constitutive properties are the average of the properties of all laminae.

Strain gradient notation consists in an alternative notation for writing finite element polynomials. Strain gradient notation is a physically interpretable notation which relates displacements to the kinematic quantities of the continuum. The identification of this relationship is possible due to a procedure which identifies the physical contents of the polynomial coefficients (Dow, 1999). The main advantage of the use of such a notation is that the modeling characteristics of the finite element are clear since the early steps of the formulation. Thus, sources of modeling errors can be identified and also removed from the finite element polynomial expansions prior to the formation of its stiffness matrix.

Strain gradient notation has been previously employed to analyze laminated composites (Dow and Abdalla, 1994) when qualitative errors in finite element analysis were identified for the first time. In that work, the authors did not focus in validating the formulation in what regards the accurate computation of displacement and stresses. The focus of the work was to show the deleterious effects of modeling errors in the representation of the behavior of laminated plates. Strain gradient notation (Dow et al., 1985) is a physically interpretable notation which allows for the clear identification of the modeling capabilities and deficiencies of the element. Under this notation, the coefficients of the polynomial expansions appear explicitly as functions of the continuum kinematic quantities. Thus, spurious terms which are present in the shear strain polynomial expansions of the element are identified. They are flexural terms which cause stiffening of the model by increasing its shear strain energy when bending of the plate occurs. Mesh refinement reduces the effects of the spurious terms and, eventually, it might be able to remove them. In order not to rely on mesh refinement for obtaining satisfactory results, one may resort to reduced-order integration techniques (Hughes, 2000). In the case of a four-node element, one-point numerical integration rather than a four-point one (2×2 strategy) is required to remove the effects of locking. However, having identified the spurious terms precisely here, they can be removed from the shear strain expressions, rendering a corrected element for locking without having to employ such a procedure. Therefore, locking is taken care *a-priori* of the computer implementation and no technique to remove it is necessary during analysis. The element is implemented in a FORTRAN finite element code in both versions, that is, with the spurious terms and after their elimination. Comparison of numerical results show how the spurious terms stiffens the model, requiring more refined meshes in order to attain convergence. Also, the corrected element is validated by comparing numerical solutions with results obtained from an analytic solution (Reddy, 1997).

2. Theoretical Background

For completeness, this section presents the macromechanical theory adopted to describe laminated composites as well as the necessary expressions pertaining to Mindlin's theory of plates. Also, strain gradient notation is introduced into the formulation of the laminate prompting it for finite element development.

The macromechanical theory for laminated composite plates adopted here is based on the following assumptions: 1) Plane sections normal to the middle surface of the plate remain plane, but not necessarily normal after bending. Thus, the model accounts for transverse shear deformation of the plate; 2) There is a perfect bond between laminae, which prevents them from slipping relatively to each other. This means that the behavior of the laminate may be represented by the behavior of its middle surface, and that compatibility is imposed; and 3) The components of stress and strain which are normal to the middle surface of the plate are negligible. Thus, they are not included in the model.

The laminate has the capability of developing in-plane displacements u and v along the x and y directions, respectively; out-of-plane displacements w, and rotations q and p in the x and y directions, respectively (around the y and x axes, respectively). The displacements w must be independent of the rotations q and p to allow for transverse shear deformation. The kinematic relations of the plate model are the following:

$$u(x,y,z) = u_0(x,y) + z\,q(x,y) \tag{1}$$

$$v(x,y,z) = v_0(x,y) - z\,p(x,y) \tag{2}$$

$$w(x,y,z) = w_0(x,y) \tag{3}$$

$$q(x,y) = \frac{\partial u(x,y,z)}{\partial z} \tag{4}$$

$$p(x,y) = \frac{\partial v(x,y,z)}{\partial z} \tag{5}$$

where u_0 and v_0 are middle surface in-plane displacements, w_0 is the middle surface transverse displacement and z is the coordinate which is associated to the thickness of the plate. The strains are arranged in vector form as shown below:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} u_0,_x \\ v_0,_y \\ u_0,_y + v_0,_x \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} z\,q,_x \\ -z\,p,_y \\ z\,(q,_y - p,_x) \\ w,_y - p \\ w,_x + q \end{Bmatrix} \tag{6}$$

where the first vector contains membrane strains and the second vector contains plate bending strains.

The strain energy of the laminate is the sum of the strain energies of its laminae. Hence,

$$U = \frac{1}{2}\sum_{k=1}^{n} \int_{\Omega} (\varepsilon)_k^T [Q]_k (\varepsilon)_k \, d\Omega_k \tag{7}$$

where k is a typical lamina, n is the total number of laminae of the laminate, ε_k is the vector containing the strains of lamina k, $[Q]_k$ is the constitutive properties matrix of lamina k, and Ω_k is the volume of lamina k. At this point, strain gradient notation is introduced into the formulation. Displacements are related to kinematic quantities of the continuum, which are rigid body modes, strains, and first-order and higher-order derivatives of strains. These kinematic quantities are generally referred to as strain gradients. The relations of displacements and strains to strain gradients are given below:

$$\{d\} = [\phi]\{\varepsilon_{sg}\} \tag{8}$$

$$\{\varepsilon\} = [T_{sg}]\{\varepsilon_{sg}\} \tag{9}$$

where $[\phi]$ and $[T_{sg}]$ are the corresponding transformation matrices, and $\{\varepsilon_{sg}\}$ is the strain gradients vector. Equation (8) and Eq. (9) are combined to eliminate vector $\{\varepsilon_{sg}\}$. The result is substituted into Eq. (7) to yield:

$$U = \frac{1}{2}\{d\}^T[\phi]^{-T}\left(\sum_{k=1}^{n}\int_{\Omega}[T_{sg}]_k^T[Q]_k[T_{sg}]_k \, d\Omega_k\right)[\phi]^{-1}\{d\} \tag{10}$$

which is an expression of the strain energy in strain gradient notation. The quantity between parentheses is called strain energy matrix and it is represented by $[U_M]$. The elements of its principal diagonal contain the quantities of strain energy associated with the pure strain modes of the laminate. The other elements of the matrix contain the quantities of energy associated with the coupling between the various strain modes. Matrix $[U_M]$ may be written as:

$$[U_M] = \int_A \left(\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [T_{sg}]_k^T [Q]_k [T_{sg}]_k \, dz_k \right) dA \tag{11}$$

where the volume integral is broken into an integral over the area of the middle surface of the laminate and an integral over its thickness. This line integral is carried out as the sum of the integrals over the thicknesses of the various laminae having integration limits z_{k-1} e z_k, which represent the bottom and top coordinates of a typical lamina k. The integration over the thickness of the laminate yields its stiffness quantities:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2) dz = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} Q_{ij}^{(k)}(1, z, z^2) dz \tag{12}$$

or

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)}(z_{k-1} - z_k) \tag{13}$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^{(k)}(z_{k-1}^2 - z_k^2) \tag{14}$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{(k)}(z_{k-1}^3 - z_k^3) \tag{15}$$

$$A_{ij}^* = K \sum_{k=1}^{n} Q_{ij}^{(k)}(z_{k-1} - z_k) \tag{16}$$

where A_{ij} is the membrane stiffness ($i, \ j \ = \ 1, \ 2, \ 6$), B_{ij} is the membrane-bending coupling stiffness ($i, \ j \ = \ 1, \ 2, \ 6$), D_{ij} is the bending stiffness ($i, \ j \ = \ 1, \ 2, \ 6$), and A_{ij}^* is the membrane stiffness associated to the effects of transverse shear ($i, \ j \ = \ 4, \ 5$), h is the thickness of the laminate and K is the shear correction factor ($K = 5/6$).

3. Four-Noded Rectangular Plate Element

This section presents the development of the finite element in strain gradient notation. The element has five degrees of freedom at each node, namely; the in-plane displacements u and v, the out-of-plane displacement w, and the rotations p and q around the x and y axes, respectively, as shown in Fig. 1. To compute numerical results for maximum transverse displacement and stresses, nondimensionalizations are used to present results in graphical form.

The essential field variables of the problem are the in-plane displacements u and v, and the out-of-plane displacement w, and polynomials for these variables must be built to start the finite element formulation. Next, definitions in Eq. (4) and Eq. (5) are employed to define the polynomials for the rotations.

In strain gradient notation these polynomials are:

$$\begin{aligned} u(x, y, z) = {} & [u]_0 + [\varepsilon_x]_0 x + \left[\frac{\gamma_{xy}}{2} - r\right]_0 y + [\varepsilon_{x,y}]_0 xy + \left[\frac{\gamma_{xz}}{2} + q\right]_0 z \\ & + [\varepsilon_{x,z}]_0 xz + [\varepsilon_{x,yz}]_0 xyz + \left[\frac{\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 yz \end{aligned} \tag{17}$$

$$\begin{aligned} v(x, y, z) = {} & [v]_0 + \left[\frac{\gamma_{xy}}{2} + r\right]_0 x + [\varepsilon_y]_0 y + [\varepsilon_{y,x}]_0 xy + \left[\frac{\gamma_{yz}}{2} - p\right]_0 z \\ & + [\varepsilon_{y,z}]_0 yz + [\varepsilon_{y,xz}]_0 xyz + \left[\frac{\gamma_{xy,z} + \gamma_{yz,x} - \gamma_{xz,y}}{2}\right]_0 xz \end{aligned} \tag{18}$$

$$w(x, y) = [w]_0 + \left[\frac{\gamma_{xz}}{2} - q\right]_0 x + \left[\frac{\gamma_{yz}}{2} + p\right]_0 y + \left[\frac{-\gamma_{xy,z} + \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 xy \tag{19}$$

$$q(x, y) = \left[\frac{\gamma_{xz}}{2} + q\right]_0 x + [\varepsilon_{x,z}]_0 x + \left[\frac{\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 y + [\varepsilon_{x,yz}]_0 xy \tag{20}$$

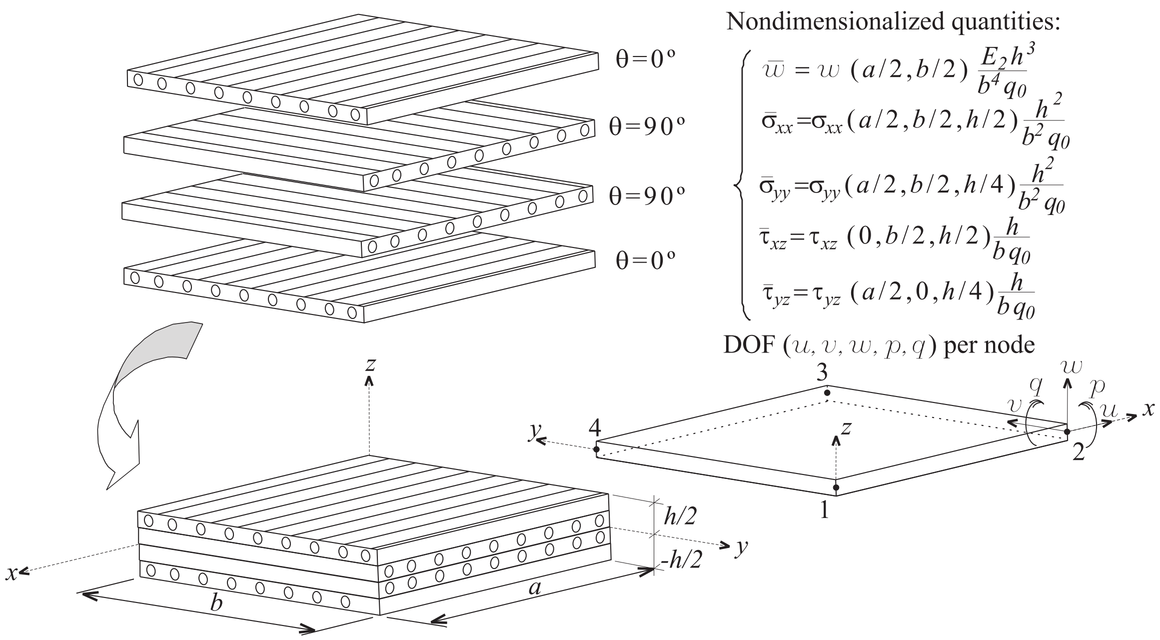

Figure 1. Four-node plate finite element, stacking sequence and nondimensionalized displacement and stresses

$$p(x,y) = \left[p - \frac{\gamma_{yz}}{2}\right]_0 + \left[\frac{-\gamma_{xy,z} - \gamma_{yz,x} + \gamma_{xz,y}}{2}\right]_0 x + [-\varepsilon_{y,z}]_0 y + [-\varepsilon_{y,xz}]_0 xy \tag{21}$$

Inspection of these expressions show that the displacements are comprised of terms which are functions of rigid body modes, constant normal and shear strains, and first and second-order derivatives of normal and shear strains.

Applying the definitions of the theory of elasticity, the strain polynomial expansions result:

$$\varepsilon_x = [\varepsilon_x]_0 + [\varepsilon_{x,y}]_0 x + [\varepsilon_{x,z}]_0 z + [\varepsilon_{x,yz}]_0 yz \tag{22}$$

$$\varepsilon_y = [\varepsilon_y]_0 + [\varepsilon_{y,x}]_0 y + [\varepsilon_{y,z}]_0 z + [\varepsilon_{y,xz}]_0 xz \tag{23}$$

$$\gamma_{xy} = [\gamma_{xy}]_0 + [\varepsilon_{x,y}]_0 x + [\varepsilon_{y,x}]_0 y + [\gamma_{xy,z}]_0 z + [\varepsilon_{y,xz}]_0 yz + [\varepsilon_{x,yz}]_0 xz \tag{24}$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x + [\varepsilon_{y,z}]_0 y + [\varepsilon_{y,xz}]_0 xy \tag{25}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\gamma_{xz,y}]_0 y + [\varepsilon_{x,z}]_0 x + [\varepsilon_{x,yz}]_0 xy \tag{26}$$

Strain gradient notation, which is physically interpretable, allows for an *a-priori* evaluation of the modeling capabilities of the finite element. Equation (22) and Eq. (23) show that the normal strain expansions contain only strain states which are associated to the corresponding normal strains. All the coefficients are terms of the corresponding Taylor series expansions. However, the expansions for shear strains, Eq. (24), Eq. (25) and Eq. (26), contain terms which do not belong to their expansions in Taylor series; namely, the first and second-order derivatives of normal strains. These spurious terms are called parasitic shear because they increase the shear strain energy of the element unduly when they are activated during the elementt's deformation. They are the cause of locking. The reason for the presence of spurious terms is the use of inconsistent polynomials for displacement w and rotations p and q. For consistency, the polynomial for w must be one order higher than the polynomials for p and q because the transverse shear strains are defined as the sum of rotations and first derivatives of the out-of-plane displacement. In the present case, all three polynomials are of the same order, which then causes the presence of erroneous terms. In order to correct the element, these spurious terms are removed from the shear strain polynomial expansions, resulting in the following expressions:

$$\gamma_{xy} = [\gamma_{xy}]_0 + [\gamma_{xy,z}]_0 z \tag{27}$$

$$\gamma_{yz} = [\gamma_{yz}]_0 + [\gamma_{yz,x}]_0 x \tag{28}$$

$$\gamma_{xz} = [\gamma_{xz}]_0 + [\gamma_{xz,y}]_0 y \tag{29}$$

Computationally, the presence or the elimination of the spurious terms is accounted for in matrix $[T_{sg}]$, which in turn affects the values of stiffness components as shown in the expression of the stiffness matrix in strain gradient notation:

$$[K] = [\phi^{-T}] \int_A \left(\sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} [T_{sg}]_k^T [Q]_k [T_{sg}]_k \, dz_k \right) dA[\phi^{-1}] = [\phi^{-T}][U_M][\phi^{-1}] \tag{30}$$

4. Application

In this section, one laminated composite plate problem is solved using the model described above. The problem is solved first with meshes containing the spurious terms, and then after their elimination. Solutions are compared to show the effects of the spurious terms. Further, solutions are compared with the analytic solutions to show the effectiveness of the proposed model. Analytic solutions are constructed using solutions derived by (Reddy, 1997).

The problem is a square simply supported plate subjected to a uniform load ($q_o = 10\ N/m^2$). It is composed of four laminae with lamination scheme ($0^o/90^o/90^o/0^o$). The mechanical properties and dimensions of the laminae are the following: $E_1 = 175\ GPa$, $E_2 = 7\ GPa$, $G_{12} = 3.5\ GPa$, $G_{13} = 3.5\ GPa$, $G_{23} = 1.4\ GPa$, $\nu_{12} = 0.25$, $h_1 = h_2 = h_3 = h_4 = 2.5\ mm$, $a = 1\ m$, $b = 1\ m$. To show the effects of locking, a thin plate with side-to-thickness ratio $a/h = 100$ was chosen. Normal stresses $\overline{\sigma}_{xx}$ and $\overline{\sigma}_{yy}$ and transverse shear stresses $\overline{\tau}_{xz}$ and $\overline{\tau}_{yz}$ are calculated at given points of the plate. Normal stresses are calculated at the center point, while transverse shear stresses are calculated at middle points of the borders.

The problem is solved using five uniform meshes; namely, 2×2, 4×4, 8×8, 16×16, and 32×32. The meshes are run first with the elements containing parasitic shear terms, and then with the elements corrected for parasitic shear. Figure 2(a) to Fig. 3(d) show the results of these analyses. In the plots, FSDT is the analytic solution for the first-order shear deformation theory. The solutions containing parasitic shear are referred to as with/PS while those not containing parasitic shear are referred to as wout/PS. In general, the results show poor convergence rate of the model containing parasitic shear whereas the model without spurious terms converges rather quickly. Figures 2(a) and Fig. 2(c) show that the spurious terms delay convergence of the normal stresses and that even the finer meshes (32×32) do not present results that match the analytic solutions. On the other hand, Fig. 2(b) and Fig. 2(d) show that when the spurious terms have been removed thicker meshes already present acceptable results and that the finer meshes (32×32) match the analytic solutions.

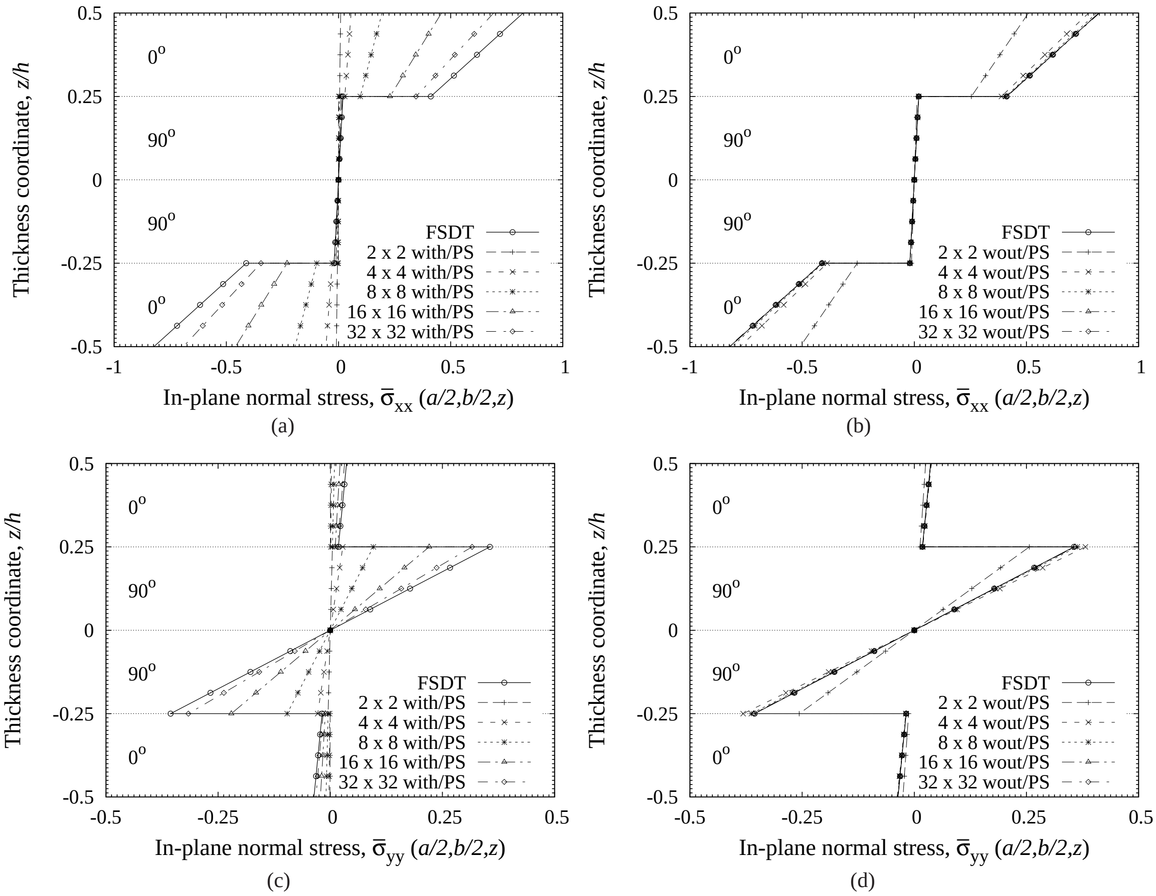

Figure 2. Nondimensionalized in-plane normal stresses computed through the thickness of the laminate (a) $\overline{\sigma}_{xx}$ with/PS, (b) $\overline{\sigma}_{xx}$ wout/PS, (c) $\overline{\sigma}_{yy}$ with/PS and (d) $\overline{\sigma}_{yy}$ wout/PS

As shown in Fig. 3(a) and Fig. 3(c), the spurious terms cause the transverse shear stresses solutions to diverge from the FSDT solutions as mesh refinement is performed leading to completely erroneous results. The enlarged details of the plots show this erroneous behavior. After removal of spurious terms, solutions converge well within the analytic solutions as shown in Fig. 3(b) and Fig. 3(d). The enlarged details of the plots show that there is very good agreement between the

32×32 meshes solutions and the analytic solutions.

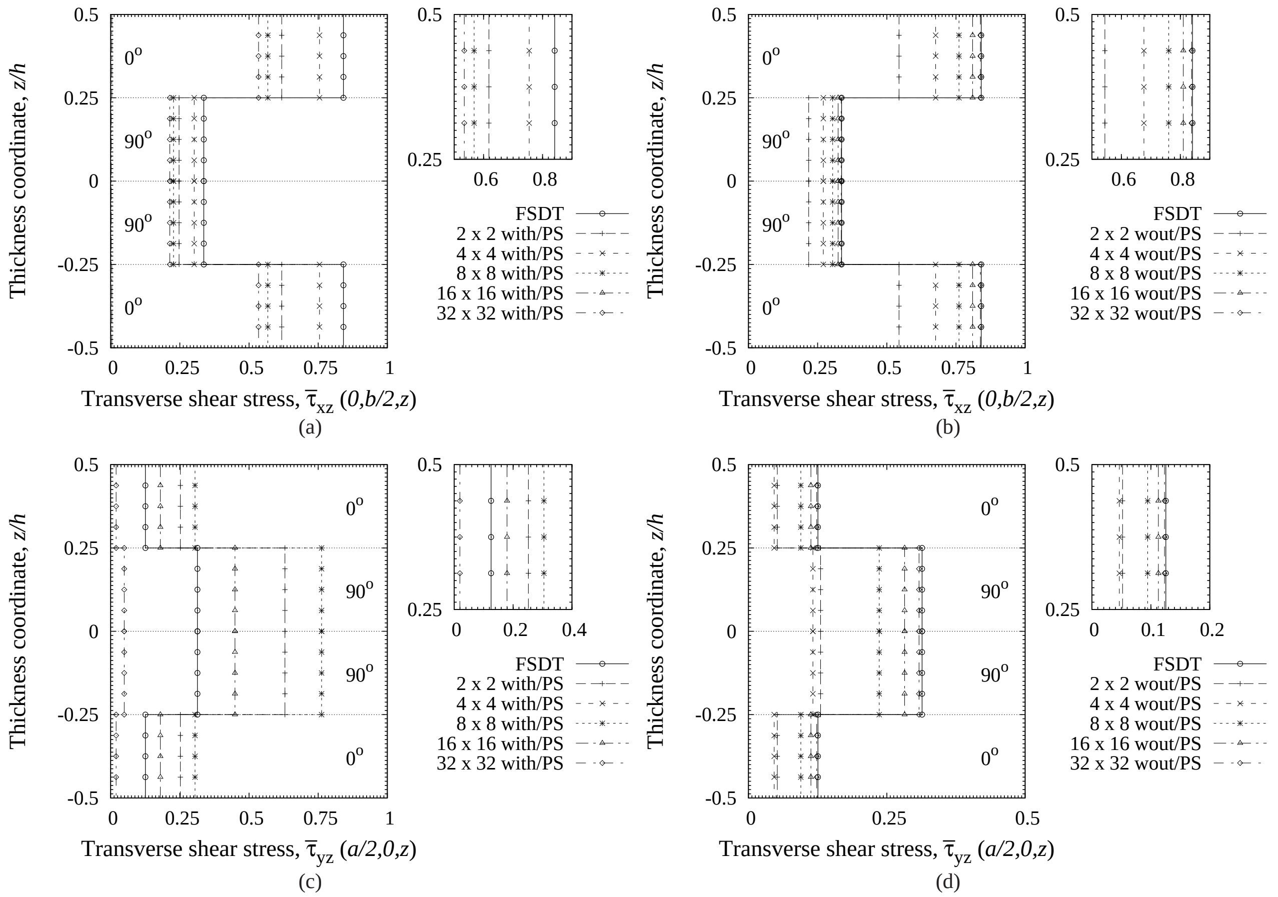

Figure 3. Nondimensionalized transverse shear stresses computed through the thickness of the laminate (a) $\overline{\tau}_{xz}$ with/PS, (b) $\overline{\tau}_{xz}$ wout/PS, (c) $\overline{\tau}_{yz}$ with/PS and (d) $\overline{\tau}_{yz}$ wout/PS

Figures 4(a) and Fig. 4(b) contain plots of nondimensionalized center displacement versus side-to-thickness ratio. Compared to the FSDT solution, the element with parasitc shear underpredicts displacements by 15.7% while the element without parasitic shear underpredicts by about 0.3%. These figures show the diminishing effect of transverse shear deformation on displacements, the effect being negligible for side-to-thickness ratios larger than 20.

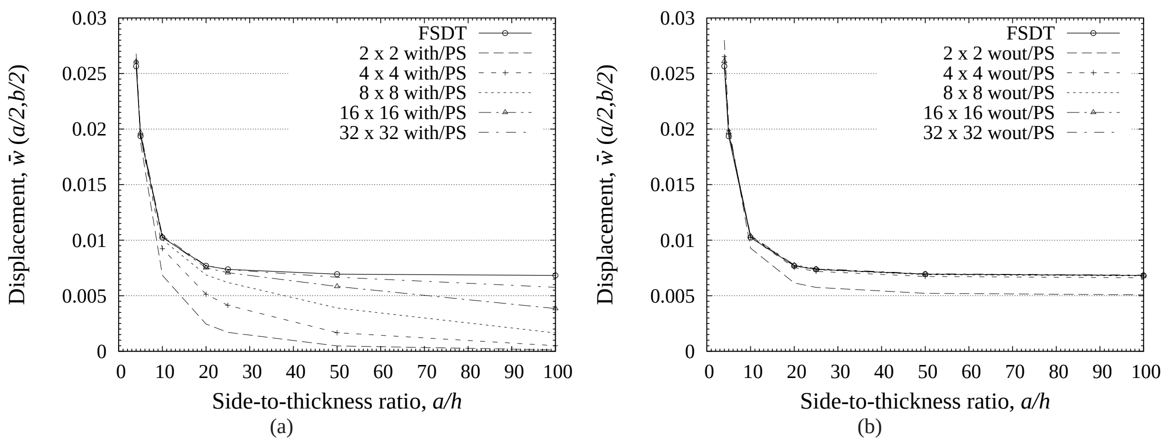

Figure 4. Nondimensionalized center transverse displacement $\overline{w}$ versus side-to-thickness ratio a/h (a) With parasitic shear and (b) Without parasitic shear

The convergence plots in Fig. 5(a) through Fig. 5(d) help to demonstrate the behavior of the solutions. The straight lines in the plots represent the analytic solutions and the other lines represent solutions with parasitic shear and without parasitic shear. The reader may observe how the solutions without parasitic shear approach the analytic solutions asymptoticaly with mesh refinement for both normal and transverse shear stresses whereas the solutions for transverse shear stresses do not present monotonic convergence. In fact, the solutions for transverse shear stresses containing the spurious terms, as they oscillate, do not provide a strong evidence of convergence.

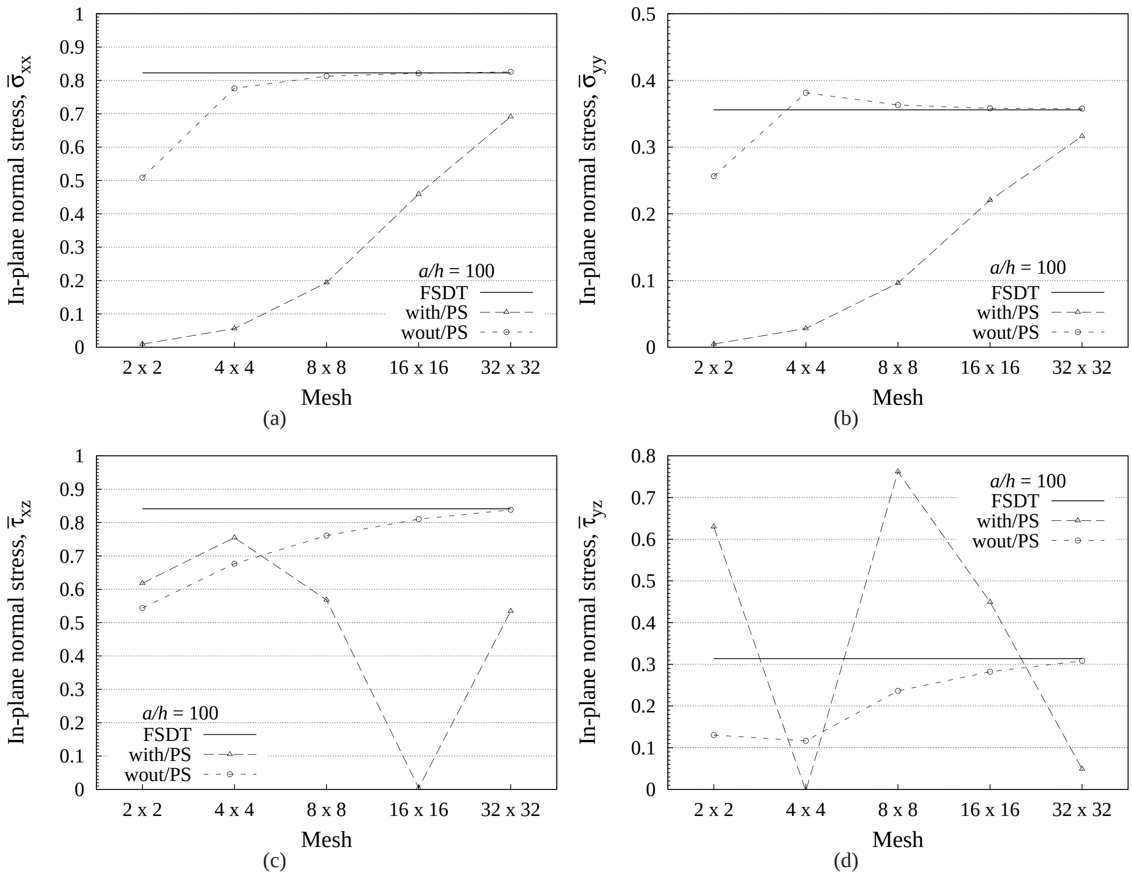

Figure 5. Convergence of the solutions with and without parasitic shear (a) In-plane normal stress $\overline{\sigma}_{xx}$, (b) In-plane normal stress $\overline{\sigma}_{yy}$, (c) Transverse shear stress $\overline{\tau}_{xz}$ and (d) Transverse shear stress $\overline{\tau}_{yz}$

5. Summary and Conclusions

This paper is concerned with the stress analysis of laminated composite plates employing the finite element method. The paper focuses on the locking problem and how it affects the good behavior of a numerical solution. The formulation of a four-node plate element using strain gradient notation is presented. Strain gradient notation is a physically interpretable notation which allows for the *a-priori* determination of the element's modeling capabilities and deficiencies. The shear strain polynomial expansions of the element are inspected and show to possess spurious terms. That is, terms which do not belong to the Taylor series expansions of those kinematic quantities. Numerical analyses demonstrated that those spurious terms are the cause of locking of the model, delaying convergence or not allowing convergence to occur. Using the transparency of the notation, the element is corrected by simply removing the spurious terms from the shear strain expressions. Numerical results comparing solutions of stresses using the model containing the spurious terms and the model corrected for them show the effectiveness of the procedure as the latter converged faster and monotonically. It can be concluded that it is advantageous to use strain gradient notation as spurious terms can be identified precisely and then eliminated definitely from the element's matrices. This is an alternative to employing reduced-order integration techniques to reduce or eliminate locking. Further, it is shown the element provides correct results for stresses as numerical solutions converge nicely to analytic solutions. Finally, it is interesting to note that solutions of transverse shear stresses containing the spurious terms might not converge to the correct solution or even diverge from it. This behavior is shown in the convergence plots. This enforces the need to apply a technique to remove spurious terms from finite elements as completely erroneous results may be provided deceiving the analyst. The simple procedure made available through the use of strain gradient notation is appealing.

6. Acknowledgements

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