A PLATE FINITE ELEMENT COUPLING FIRST ORDER MODEL AND A POSTERIORI ERROR ESTIMATOR

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Abstract. *The main objective of this work is to propose a finite element model for vibration and static analysis of laminated composite plates based on the First Order Shear Deformation Theory (FSDT). The paper presents a triangular finite element with two different interpolations, linear for rotations and quadratic for displacements, and it is coupled with a posteriori error estimator based on the local directional interpolation error. Numerical studies are conducted on a number of laminated arrangements and boundary conditions and the present results are in agreement with 3-D and higher order theories solutions.*

Keywords: *triangular elements, composite laminated, finite element, error estimator, mesh refinement*

1. Introduction

Composite materials are identified when two or more different phases are recognized in their macroscopical structures. Each phase has specific properties that contribute to produce more efficient material, which have better physical properties than the components alone.

These materials have been used in different engineering application like: automobile and aerospace industry, medical and sports equipments. The interest is in developing a new material which might hold better properties for engineering applications. Such tendency can be verified in work of Bogdanovich and Sierakowski (1999).

Composite laminated is the most composite material used in structural applications (Herakovich,1998). There are composed by a lot of orthotropic laminas glued by an adhesive layer.

Few works in literature propose to apply finite elements for composite laminated analysis using triangular elements. This type of element has better performances for the mesh automatic generation, and for plane surface modeling with irregular geometry. When an adaptative approach is considered, this is a useful tool, .

This work purposes a triangular finite element for the static stresses and modal analysis to laminated composite. This element relies on the FSDT and purposes a different interpolation for displacement fields components. In order to compatibilize the sectional rotation associated with the curvature variation, the transverse medium plane displacement is interpolated with one superior order for the rotations interpolations due to the shear.

More over this, the element is coupled with a posteriori error estimator based on the local directional interpolation error. This strategy is presented by Borges, Feijóo and Zouain (1999) and Borges *et. al.* (2001). This form of adaptation shows improved results in localizing regions of abrupt variation of the variables, as vibration mode or vertical displacements, whose location is not a priori know .

There for, this work presents an triangular finite element for composite laminated coupled with an error estimator for adaptive mesh refinement.

2. FSDT for composite laminated

In the present work, an approach based on the principles of virtual potencies is used and this is the most natural way to described the laws that govern the continuum media. Beside this, the model is generated naturally for application in the finite element method.

The internal potencies for a plate is described as in Eq. (1):

$$< \tilde{\mathbf{T}}, \nabla\tilde{\mathbf{v}} > = \overbrace{\int_{\Sigma_0}\int_z \Big[\mathbf{T_0}\cdot(\nabla\mathbf{v})_\mathbf{0}\Big] dz\, d\Sigma_0}^{\text{medium plane stress terms}} + \overbrace{\int_{\Sigma_0}\int_z \Big[\mathbf{t_{0n}}\cdot\Big(\frac{\partial \mathbf{v}_0}{\partial z} + \nabla v_z\Big)\Big] dz\, d\Sigma_0}^{\text{shear stress terms}} + \overbrace{\int_{\Sigma_0}\int_z \Big[\sigma_\mathbf{n}\frac{\partial v_z}{\partial z}\Big] dz\, d\Sigma_0}^{\text{normal stress terms}} \quad (1)$$

where $\mathbf{v_0}$ is the displacement in the medium plane and v_z is the normal displacements to medium plane (Σ_0). $\mathbf{T_0}$ represents the stress in the medium plane. σ_n is the normal stress to the medium plane. $\mathbf{t_{0n}}$ and $\mathbf{t_{n0}}$ are the components of the stress tensor associated with the transverse shear loads.

FSDT will be used aiming at including the shear effects. In this theory, the adopted hypothesis is that the displacement v_z is not a function of z, as in Equation (2)

$$\frac{\partial v_z}{\partial z} = 0 \quad \rightarrow \quad v_z = v_z(\mathbf{\Sigma_0}) \tag{2}$$

Besides that the transverse section remains plane, but no longer normal to the medium plane for the plate. Hence:

$$\frac{\partial \mathbf{v_0}}{\partial z} + \nabla v_z = \phi(x,y) \quad \rightarrow \quad \gamma(x,y) = \phi(x,y) - \nabla v_z \tag{3}$$

where is $\gamma(x,y)$ the transversal rotational section due to the transverse shear loads and ϕ is the total rotation for the section, as showed in the Fig.1. This equation proves that the displacements can be interpolated with functions one degree above than rotation terms.

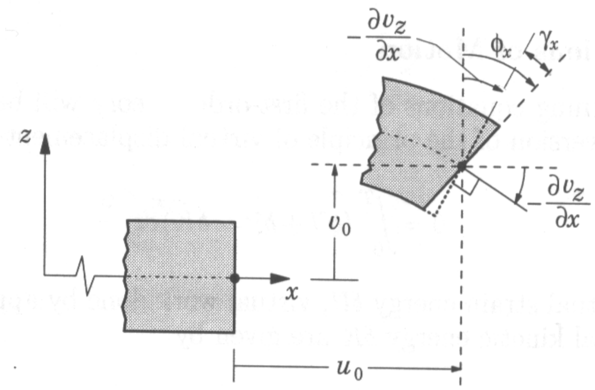


Figure 1. Relation between variables on the FSDT

Integrating along z, one has:

$$\mathbf{v_0} = \mathbf{u_0} - z\nabla v_z + z \cdot \phi(x,y) \tag{4}$$

where $\mathbf{u_0}$ represents the membrane displacement for the plate medium plane, as showed in the Fig.1.

Eq.(1) can be recasted applying Eq.(2) and Eq.(3), such as:

$$< \tilde{\mathbf{T}}, \nabla\tilde{\mathbf{v}} > = \int_{\Sigma_0} \mathbf{N_0} \cdot \nabla\mathbf{u_0}\, d\Sigma_0 + \int_{\Sigma_0} \mathbf{M_0} \cdot \nabla\gamma\, d\Sigma_0 + \int_{\Sigma_0} \mathbf{V_0} \cdot (\gamma + \nabla v_z)\, d\Sigma_0 \tag{5}$$

where:

$$\mathbf{N_0} = \int_z \mathbf{T_0}\, dz \quad ; \quad \mathbf{M_0} = \int_z \mathbf{T_0} z\, dz \quad ; \quad \mathbf{V_0} = \int_z \mathbf{t_{0n}}\, dz \tag{6}$$

When applying the divergence theorem, the compatible loads are identified. Therefore, the compatible loads with FSDT are:

- Uniformly loads distributed in medium plane, such that:
  - $\mathbf{f_0}$ Parallel when medium plane, such as:

    $div(\mathbf{N_0}) = -\mathbf{f_0}$
  - f_z Perpendicular to medium plane, such as:

    $div(\mathbf{V_0}) = -f_z$

- $\mathbf{f_c}$ Perpendicular shear to medium plane, such as:

  $div(\mathbf{M_0}) + \mathbf{V_0} = -\mathbf{f_c}$

- Loads along medium plane boundary:
  - $\bar{\mathbf{f}}_\mathbf{0}$ Paralell to medium plane, such as:

    $\mathbf{N_0 m}\Big|_{\partial\Sigma_0} = \bar{\mathbf{f}}_\mathbf{0}$

  - $\bar{f}_z$ Shear load perpendicular to medium plane, such as:

    $V_0 m\Big|_{\partial\Sigma_0} = \bar{f}_z$

  - $\bar{M}_c$ Distributed moment, to:

    $M_0 m\Big|_{\partial\Sigma_0} = \bar{M}_c$

2.1 Constitutive relation

For an orthotropic lamina, the constitutive relations can be written as:

$$\mathbf{T_0} = \mathbf{Q}(\nabla\mathbf{u_0} + z\nabla\gamma) \quad ; \quad \mathbf{t_{0n}} = \mathbf{L}(\gamma + \nabla v_z) \tag{7}$$

where Q is the membrane terms of the stiffness matrix and L is the shear terms of the stiffness matrix. By introducing Eq.(7) in Eq.(6), the generalized loads are represented by:

$$\mathbf{N_0} = \mathbf{D_m}\nabla^s\mathbf{u_0} + \mathbf{D_{mf}}\nabla\gamma \quad ; \quad \mathbf{M_0} = \mathbf{D_{mf}}\nabla^s\mathbf{u_0} + \mathbf{D_f}\nabla\gamma \quad ; \quad \mathbf{V_0} = \mathbf{D_c}\gamma + \mathbf{D_{mf}}\nabla v_z \tag{8}$$

being $\mathbf{D_m}$ called elasticity extensional matrix or membrane matrix, $\mathbf{D_f}$ being the bending elasticity matrix, $\mathbf{D_{mf}}$ being flexion-shear couple and $\mathbf{D_c}$ being the shear elasticity matrix. The matrices described before are expressed by:

$$[\mathbf{D_m}, \mathbf{D_{mf}}, \mathbf{D_f}] = \sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} \mathbf{Q}^{(k)}[1, z, z^2]dz \quad ; \quad \mathbf{D_c} = k_c\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}} \mathbf{L}^{(k)}dz \tag{9}$$

The factor k_c is a shear correction factor. Such factor was introduced by Reissner (1945) and Mindlin (1951) for isotropic plates in order to adjust the transversal shear modeling and the relation between mean shear tension and the shear tension in medium plane. This factor incorporates the effects of non-linear transversal shear behavior, that are not admitted by FSDT theory. For laminated composites, the same values are adopted to validate numerical codes as showed by Carrera (2002).

2.2 Cinematic formulation

Regarding the virtual displacements, the formulation for the FSDT can be written for internal potencies and compatible loads as:

$$< \tilde{\mathbf{T}}, \nabla\tilde{\mathbf{v}} > = \int_{\Sigma_0}[\mathbf{D_m}\nabla^s\mathbf{u_0} \ + \ \mathbf{D_{mf}}\nabla\gamma]\cdot\nabla^\mathbf{s}\mathbf{u_0^*}\,d\Sigma_0 \ + \int_{\Sigma_0}[\mathbf{D_{mf}}\nabla^s\mathbf{u_0} \ + \ \mathbf{D_f}\nabla\gamma]\cdot\nabla\gamma^*)]\,d\mathbf{\Sigma_0} + \int_{\Sigma_0}[\mathbf{D_c}\gamma + \mathbf{D_c}\nabla v_z]\cdot\gamma^* + \int_{\Sigma_0}[\mathbf{D_c}\gamma + \mathbf{D_c}\nabla v_z]\cdot\nabla v_z^*\,d\mathbf{\Sigma_0} \ = \ \int_{\Sigma_0}[\mathbf{f_0}\cdot\mathbf{u_0^*} + f_z v_z^* + \mathbf{f_c}\cdot\gamma^*]\,\mathbf{d\Sigma_0} + \int_{\partial\Sigma_0}[\bar{\mathbf{f}}_\mathbf{0}\cdot\mathbf{u_0^*} + \bar{f}_z v_z^* + \bar{M}_c\cdot\gamma^*]\,d\partial\mathbf{\Sigma_0} \tag{10}$$

In the same order, the external potencies for a plate are described like the Eq. (11):

$$< \tilde{\rho}\ddot{\tilde{\mathbf{v}}}, \tilde{\mathbf{v}}^* > = \int_{\Sigma_0}\int_z \tilde{\rho}\,\ddot{\tilde{\mathbf{v}}}\cdot\tilde{\mathbf{v}}^*\,dz\,d\Sigma_0 \tag{11}$$

Organazing the Eq.(11), as functions of the inertia terms results in:

$$< \tilde{\rho}\ddot{\tilde{\mathbf{v}}}, \tilde{\mathbf{v}}^* > = \int_{\Sigma_0} I_0\left[\ddot{\mathbf{u}}_\mathbf{0}\cdot\mathbf{u_0^*} + \ddot{v}_z v_z^*\right]d\Sigma_0 + \int_{\Sigma_0} I_1\left[\ddot{\gamma}\cdot\mathbf{u_0^*} + \ddot{\mathbf{u}}_\mathbf{0}\cdot\gamma^*\right]d\Sigma_0 + \int_{\Sigma_0} I_2\left[\ddot{\gamma}\cdot\gamma^*\right]d\Sigma_0 \tag{12}$$

where $\tilde{\rho}$ represents the specific mass of the plate and:

$$I_0 = \int_z \tilde{\rho}(\mathbf{X_0}, z)\,dz \quad ; \quad I_1 = \int_z \tilde{\rho}(\mathbf{X_0}, z)\,z\,dz \quad ; \quad I_2 = \int_z \tilde{\rho}(\mathbf{X_0}, z)\,z^2\,dz \tag{13}$$

3. Finite element model

Due to its importance and complexity, the plates study was one of the first FEM applications, and still demands great research efforts. The developed elements which are based on the Classical and Higher Order Theories display dificulties once deal with the continuity C^1. Carrera (2002) certifies that FSDT is the theory that combines both low computational costs and higher efficiency for global variables as natural frequencies and central deflection. In FSDT, the rotations and displacements are analyzed separately, considering the transversal shear. This element has the deal only continuity C^0. However, dealing with thin plates, the effect called shear locking occurs in this model. This problem occurs because the same interpolation function is adopted for all degrees of freedom. For solving this problem, the interpolation used in this work is sufficient and do not uses integration complex mechanisms.

The equation obtained for virtual potencies, in this paper holds first order derivatives. Thus, the vertical displacement degrees of freedom and plane medium membrane degrees of freedom are interpolated with Lagrange quadratic functions and the rotations degrees of freedom are interpolated with Lagrange linear functions. Thus, the element propose in this work is quadratic triangular with 24 degrees of freedom.

For any element e, the variables of this problem are interpolated as:

$$\mathbf{u_0}^e(x,y) = \begin{bmatrix} u_x(x,y) \\ u_y(x,y) \end{bmatrix}^e = \sum_{i=1}^{n_0} \psi_i(x,y)\mathbf{u_0}_i^e \quad ; \quad v_z^e(x,y) = \sum_{i=1}^{n_z} \psi_i(x,y)\Delta_i^e$$

$$\gamma^e(x,y) = \begin{bmatrix} \gamma_x \\ \gamma_y \end{bmatrix}^e = \sum_{i=1}^{n_c} \phi_i(x,y)\gamma_i^e \tag{14}$$

where n_0, n_z and n_c are nodes numbers used to interpolate the displacements. The vector $\mathbf{u_0}_i^e$ has the displacements (u_x, u_y) in node i of the element e. The vector Δ_i^e has the displacement v_z in node i of the element e. The vector γ_i^e has the angles (γ_x, γ_y), as normal rotation to planes x and y. The functions ψ_i and ϕ_i are Lagrange interpolation functions. In this work, ψ_i has one superior order to ϕ_i.

For this model, the deformation operators are:

$$\nabla^s = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} \end{bmatrix} \quad ; \quad \nabla = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

Applying the deformation operators in the associated terms defined in equations (ee), the terms associated of each members are defined as:

- Terms to the stiffness matrix

$$[\mathbf{K_0}]_{ij}^e = \int_{\Sigma_0^e} \mathbf{B_0^T}_i(x^e)\mathbf{D_m}\mathbf{B_0}_j(x^e)\, d\Sigma_0^e \quad ; \quad [\mathbf{K_{0f}}]_{ij}^e = \int_{\Sigma_0^e} \mathbf{B_0^T}_i(x^e)\mathbf{D_{mf}}\mathbf{B_p}_j(x^e)\, d\Sigma_0^e \tag{15}$$

$$[\mathbf{K_f}]_{ij}^e = \int_{\Sigma_0^e} \mathbf{B_p^T}_i(x^e)\mathbf{D_f}\mathbf{B_p}_j(x^e)\, d\Sigma_0^e \quad ; \quad [\mathbf{K_{c1}}]_{ij}^e = \int_{\Sigma_0^e} \gamma^{\mathbf{T}}_i(x^e)\mathbf{D_c}\gamma_j(x^e)\, d\Sigma_0^e \tag{16}$$

$$[\mathbf{K_{c2}}]_{ij}^e = \int_{\Sigma_0^e} \mathbf{B_r^T}_i(x^e)\mathbf{D_c}\gamma_j(x^e)\, d\Sigma_0^e \quad ; \quad [\mathbf{K_{c3}}]_{ij}^e = \int_{\Sigma_0^e} \mathbf{B_r^T}_i(x^e)\mathbf{D_c}\mathbf{B_r}_j(x^e)\, d\Sigma_0^e \tag{17}$$

- Terms to the load vector

$$\mathbf{F_0}_j = \int_{\Sigma_0} \mathbf{f_0}\psi_j(x^e)\, \mathbf{d\Sigma_0} + \int_{\partial\Sigma_0} \bar{\mathbf{f}}_\mathbf{0}\psi_j(x^e)\, d\partial\mathbf{\Sigma_0} \quad ; \quad \mathbf{F_f}_j = \int_{\Sigma_0} f_z\psi_j(x^e)\mathbf{d\Sigma_0} + \int_{\partial\Sigma_0} \bar{f}_z\psi_j(x^e)\, d\partial\mathbf{\Sigma_0} \tag{18}$$

$$\mathbf{F_c}_j = \int_{\Sigma_0} f_c\phi_j(x^e)\mathbf{d\Sigma_0} + \int_{\partial\Sigma_0} \bar{M}_c\phi_j(x^e)\, d\partial\mathbf{\Sigma_0} \tag{19}$$

- Terms to the mass matrix

$$[\mathbf{M_0}]_{ij}^e = \int_{\Sigma_0^e} \psi_{\mathbf{0}}^{\mathbf{T}}{}_i(x^e)I_0\psi_{\mathbf{0}j}(x^e)\, d\Sigma_0^e \quad ; \quad [\mathbf{M_{0p}}]_{ij}^e = \int_{\Sigma_0^e} \psi_{\mathbf{0}}^{\mathbf{T}}{}_i(x^e)I_1\psi_{0j}(x^e)\, d\Sigma_0^e \tag{20}$$

$$[\mathbf{M_p}]^e_{ij} = \int_{\Sigma_0^e} \psi^{\mathbf{T}}{}_i(x^e) I_0 \psi_j(x^e)\, d\Sigma_0^e \quad ; \quad [\mathbf{M_{pp}}]^e_{ij} = \int_{\Sigma_0^e} \phi^{\mathbf{T}}{}_i(x^e) I_2 \phi_j(x^e)\, d\Sigma_0^e \tag{21}$$

Thus, the elementar stiffness and mass matrix ans load vector are defined as:

$$[\mathbf{K}^e] = \begin{bmatrix} [\mathbf{K_0}{}^e] & | & 0 & | & [\mathbf{K_{0f}}{}^e] \\ ---- & ---- & ---- & ---- & ---- \\ 0 & | & [\mathbf{K_{c3}}{}^e] & | & [\mathbf{K_{c2}}{}^e] \\ ---- & ---- & ---- & ---- & ---- \\ [\mathbf{K_{0f}^T}{}^e] & | & [\mathbf{K_{c2}^T}{}^e] & | & [\mathbf{K_f}{}^e] + [\mathbf{K_{c1}}{}^e] \end{bmatrix} \quad ; \quad [\mathcal{L}^e] = \begin{bmatrix} [\mathbf{F_0} + \mathbf{F_0^{\theta}}^e] \\ ----- \\ [\mathbf{F_f} + \mathbf{F_f^{\theta}}^e] \\ ----- \\ [\mathbf{F_c} + \mathbf{F_c^{\theta}}^e] \end{bmatrix}$$

$$[\mathbf{M}^e] = \begin{bmatrix} [\mathbf{M_0}{}^e] & | & 0 & | & [\mathbf{M_{0n}}{}^e] \\ ---- & ---- & ---- & ---- & ---- \\ 0 & | & [\mathbf{M_p}{}^e] & | & 0 \\ ---- & ---- & ---- & ---- & ---- \\ [\mathbf{M_{0n}^T}{}^e] & | & 0 & | & [\mathbf{M_{pp}}{}^e] \end{bmatrix}$$

4. A posteriori error estimator

When a physical problem i analysed using finite element model, there exist some discretization errors caused owing to the use of the finite element model. These errors ate calculated in order to assess the accuracy of the solution obtained (Li and Betess, 1997)

The strategy presented here is refereed by Borges, Feijóo and Zouain (1999) for linear finite elements. Borges *et. al.* (2001) generalize the indicator and the adapting procedure to include quadratic triangles.

The error estimate procedure is adopted for finite element adaptive strategies using *a posteriori* form based on a recovering scheme for the second derivatives of the finite element solution. According to Borges *et. al.* (2001), when using interpolation functions whose degree higher than one, as adopted in this work, the information provided by second derivatives are sufficient to estimate the interpolation error.

The most simple recovery technique is made by weight average method, transforming a discontinue field between elements in a continue field. Marques (2003) shows that this recovery technique is the best for composite laminates because presents good results and is simple for implementation. The field in a node is the weight average with the inverse of the length for the point of superconvergency as in Equation (22) and in Figure 2:

$$T^R(x,y,z) \;=\; \sum_{n=1}^{nelv} \frac{1}{d_n}\, T(x_{sc}, y_{sc}, z_{sc}) \Big/ \sum_{n=1}^{nelv} \frac{1}{d_n} \tag{22}$$

where d_n is the lenght between the superconvergency point of the elements until the node and $nelv$ is the number of elements with the same node.

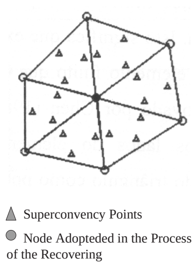


Figure 2. Node and the superconvergency points

An error estimator can be used as a first step for building meshes adopted for domains with localizing regions of rapid or abrupt variations of the variables, whose location are not *a prioiri* know.

The advantages of adapting meshes are well know in this case, particular emphasis is on the anisotropic mesh adaptation process, generated by the directional indicator cited. The goal of that approach is to achieve a mesh adaptive strategy accounting for mesh size refinement, as well modifying of the oriented element stretching. Hence, along the adaptation process, the mesh becomes aligned with the direction of maximum variation of the variable chosen.

In this work, the choice of a variable to be used in order to control the adaptive process regarding desired the analysis. For natural frequencies, the variable is the mode vibrations. For vertical displacements, the variable is the displacements vectors.

Let x denote the vector coordinate of a point in the considered continuum domain and $u(x)$ be the exact solution to the variational formulation of the problem at hand. Then, at any chosen point x_0, Taylor expansion shows $u(x)$ as in Equation (23):

$$u(x) \;=\; \overbrace{u(x_0) \;+\; \nabla u(x_0)\,.\,(x-x_0)}^{\cong u_h} + \frac{1}{2}\,\mathbf{H}(u(x_0))\,.\,(x-x_0)\,.\,(x-x_0) + ... \tag{23}$$

where $\nabla u(x_0)$ and $\mathbf{H}(u(x_0))$ are the gradient and the Hessian operators at x_0. The approximation by finite elements u_h maybe be considered as approximation of the linear and constant parts of the solution. So, the error will be in the quadratic terms in $(x - x_0)$. In the same way that an error estimator based in the gradient for linear interpolation is adopted, an error estimator based in the second order derivatives (also called Hessian matrix) is considered for quadratic interpolation functions.

Through of the metric tensor field, is possible to determinate a quality mesh indicator that establishes both a local and a global error. These factors are used for calculate the number of elements in the new adapted mesh, the decreasing or increasing rate of element size and the stretching factor at each node.

5. Numerical Results

A computer program was developed coupling a posteriori error estimator with a numerical procedure for analysis of laminated composite plates by triangular finite elements. This work presents results about natural frequencies and vertical central displacements for cross-ply $(0^0/90^0/...)$ and angle-ply $(45^0/-45^0/...)$ laminated composite square plates simply supported. A shear correction factor of $5/6$ is used in this examples. The vertical displacement in a plate subjected a normal loads has geometric form similar with first mode of vibration. So, the first mode of vibration was adopted as basis for the refinement process. Tab. 1 shows the material proprieties used in this work and Tab. 2 shows the boundary conditions considered for simply-supported here:

Table 1. Adimensional material proprieties used in the examples.

Material	E_1/E_2	G_{12}/E_2	G_{13}/E_2	G_{23}/E_2	ν_{12}
Composite-1	25	0.6	0.6	0.5	0.25
Composite-2	40	0.6	0.6	0.5	0.25

Table 2. Boundary conditions for simply supported rectangular plates.

Arrangements type	$x = 0, a$	$y = 0, b$
Cross-ply	$u_y = v_z = \gamma_y = 0$	$u_x = v_z = \gamma_x = 0$
Angly-ply	$u_x = v_z = \gamma_y = 0$	$u_y = v_z = \gamma_x = 0$

5.1 Cross-Ply Laminated Composite

The first example in this work is for cross-ply laminates. For fundamental frequencies, Khare *et. al.* (2004) present results for First and Higher Order from an isoparametric finite element based on a shear deformable model. They also show results acquired by for 3-D theory presented by Noor (1973) and by finite elements based on high order theory presented by Putcha and Reddy (1986). Then, this work focuses the process of mesh refinement and compares the solution obtained with and without a posteriori error estimator.

Tab. 3 shows that the results obtained for a present element have good approximation with the results refereed by the authors (obtained by rectangular elements). The side-to-thickness ratio a/h is 5 and the adopted material is identified in Tab. 1 as Composite-2. The numerical results for all cases were obtained through for a mesh $8x8$.

Fig. 3 shows that the results obtained for the triangular element with a posteriori error estimator are better than the ones without error estimator for fundamental frequency and vertical central displacements.

For the vertical central displacement study, Composite-1 as material, stacking sequence $(0/90/0)_2$, side-to-thickness ratio a/h as 100 and uniform load was adopted.

Table 3. Non-dimensionalized fundamental frequency ($\varpi = \omega \sqrt{\rho h^2/E_2}$) of simply supported cross-ply square laminated plates with $a/h = 5$

Number of layers	$(0/90)_2$	$(0/90)_3$	$(0/90)_5$
Noor (1973)	0.42719	0.45091	0.46498
HOST Khare *et. al.* (2004)	0.43225	0.45158	0.46345
FOST Khare *et. al.* (2004)	0.45085	0.46100	0.46579
Putcha and Reddy (1986)	0.44686	0.46005	0.46345
Present FSDT	0.45324	0.46318	0.46808

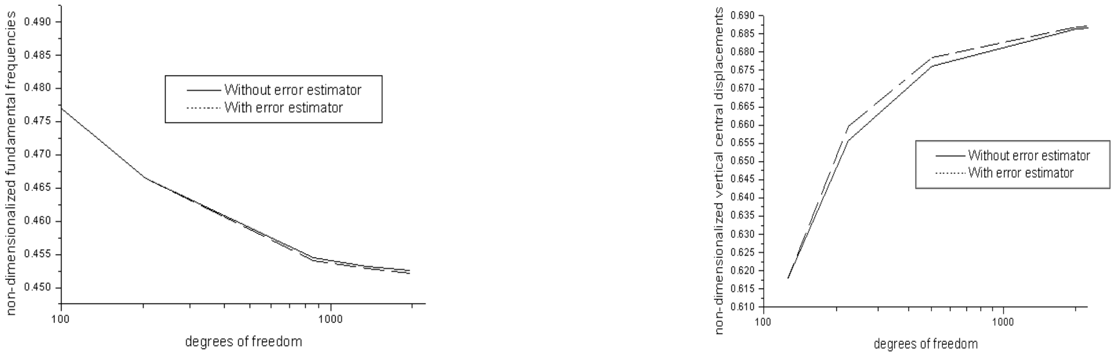


Figure 3. Results of the non-dimensionalized fundamental frequencies (ϖ) and vertical central displacements (v_z) for the refinement process of the simply supported cross-ply square laminated plates

5.2 Angle-Ply Laminated Composite

For natural frequencies, the work presented by Bert and Chen (1978) show analytical results for FSDT and numerical results are found in the work published by Shankara and Iyengar (1996) on rectangular finite elements based on high order theory. This paper will use the results presented by Khare *et. al.* (2004), whose theory was presented in the previous topic.

Tab. 4 shows that the results obtained for a present element has good agreements with the others results. The layers stacking is $(45/-45)_2$ and the material adopted is identified in Tab. 1 as Composite-2. The numerical results for all cases were obtained for a mesh $8x8$.

Fig. 4 shows that results obtained for the triangular element with a posteriori error estimator are better than the ones without error estimator for fundamental frequency and vertical central displacements.

For the vertical central displacement study, Composite-1 as material, stacking sequence $(45/-45)_4$, side-to-thickness ratio a/h as 100 and uniform load was adopted.

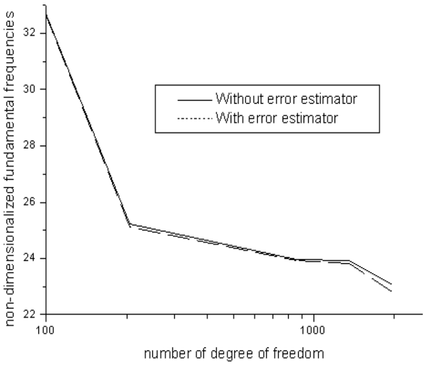
Table 4. Non-dimensionalized fundamental frequency ($\varpi = \omega\ a^2\sqrt{\rho/(h^2 E_2)}$) of simply supported angle-ply square laminated plates with $E_1/E_2 = 40$

Number of layers	$a/h = 10$	$a/h = 30$	$a/h = 50$
Bert and Chen (1978)	18.46	22.74	23.24
HOST Khare *et. al.* (2004)	17.414	23.034	23.900
FOST Khare *et. al.* (2004)	18.563	23.555	24.140
Shankara and Iyengar (1996)	17.974	22.691	23.296
Present FSDT	18.737	23.931	24.510

6. Conclusions

As propose, this work presented a triangular finite element for laminated composite coupled with a posteriori error estimator for improve his results.

The results obtained had good agreements with analytical and numerical solution benchmarks and without shear locking. Good results are obtained without a posteriori error estimator and its are improved with one.



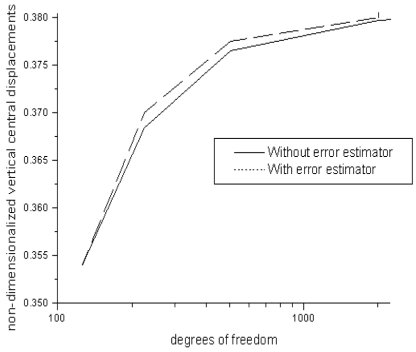


Figure 4. Results of the non-dimensionalized fundamental frequencies (ϖ) and vertical central displacements (v_z) for the refinement process of the simply supported angle-ply square laminated plates

The use of the a posteriori error estimator based on the local directional interpolation error supplies better results than without and is a powerful tool in the mesh refinement.

7. References

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8. Responsibility notice

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