DISPLACEMENT-BASED FINITE ELEMENT FORMULATION OF ACOUSTIC FLUID-STRUCTURE INTERACTION WITH LOCALIZED LAGRANGE MULTIPLIERS

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Abstract. *In this work, the localized Lagrange multipliers method is applied to the solution of acoustic internal fluid-structure problems. A finite element displacement formulation is applied both to fluid and solid partitions and two independent sets of Lagrange multipliers are assigned to the structural and fluid interfaces. The irrotationality constraint is also enforced by using Lagrange multipliers, ensuring that no spurious non-zero frequencies are encountered. A number of examples using the proposed formulation is presented and the results are compared to the ones obtained from different finite element formulations, showing good agreement. Finally, some suggestions for future developments are discussed.*

Keywords*: Fluid Solid Interaction, Finite Element, Lagrange multipliers.*

1. Introduction

Fluid-structure interaction is an important field in Mechanical Engineering . The capacity of handling the problems arising in this field arises as a consequence of development of multi-physics approaches, large avaiability of computational resources and an increase on improving design in order to achieve robust and reliable systems for new applications. This approach should be considered in the design of dams, elasto-acoustic problems, wind induced vibrations in bridges or buildings, fluid transportation and so on.

Many different formulations have been developed in the past to deal with this kind of problem. The finite element method is a very practical tool to solve the differential equations, which arise in this context. Since this method was sucessfully applied to the solution of problems in elastodynamics, it was naturally considered as a primary tool for fluid structure problems.

There are two important issues that should be treated before the applications of FEM in this context, namely: how to model the interface between the two fields and how to eliminate the spurious non-zero frequencies which arise in displacement based finite element formulations.

The present paper deals with fluid-structure systems involving non viscous flows and small displacements elastic response of the solid. This corresponds typically to vibro-acoustic problems. For those cases, the most widely adopted FEM formulation for fluid-structure problems uses the pressure as primary variable for fluid and the displacement for solid. The interface condition is assured by assuming that the pressure acting in the solid is equal as structural normal stress component in the solid. This type of formulation will be called as (u-p) formulation. In the so called (u-u) formulation the fluid and the solid primary variables are the displacement. The interface condition is simply implemented, as the displacement component normal to the interface should be the same, for fluid and solid domains.

The non-zero spurious frequencies have been widely reported in the context of finite elements fluid-structure interactions (Kiefling and Feng, 1976, Hamdi, Ousset and Verchery, 1978, Olson and Bathe, 1983). A number of approaches have been suggested for the elimination of spurious non-zero modes. Penalty approach (Hamdi, Ousset and Verchery, 1978), reduced integration (Chen and Taylor, 1990) and non standard finite elements, like Raviart-Thomas (Bermúdez and Rodríguez, 1994). Later Wang and Bathe (1997) showed that the origin of the spurious non-zero frequencies relies on the use of the pure displacement-based formulation and on the incorrect treatment of the interface conditions.

The mixed formulation proposed by Bathe, Nitikipaiboon and Wang (1995) and Wang and Bathe (1997) achieves success on handling the spurious modes by introducing a new variable in order to take into account the irrotationality constraint. This formulation was presented for displacement/pressure formulation.

A different perspective is envisaged when dealing with multi-physics problems. It is often convenient to treat the fluid domain, the solid domain and the interface phenomena as modular units. That allows the use of existent codes for fluid and solid systems to be used with a small amount of modifications. This strategy was followed by Park, Gusmaste,

and Felippa (2000) using the concept of localized Lagrange multipliers. A specific formulation for fluid and solid interaction and the use of rigidity regularization scheme was further proposed in Park, Felippa, and Ohayon(2001).

In the present work, the partitioned formulation proposed by Park, Felippa, and Ohayon (2001) is applied to the numerical solution of acoustic fluid-structure problems. A displacement-displacement formulation is proposed, which grants a natural implementation of interface constraints. The incorporation of the non rotational condition to the proposed formulation prevents the appearance of spurious modes.

2. The localized Lagrange multipliers method

The localized Lagrange multipliers method was proposed by Park, Gusmaste, and Felippa (2000). This method was originally proposed to circumvent the non-uniqueness arising in some applications of classic Lagrange multipliers method in partitioned analysis of structures. In fluid-structure problems this localized applications might be explored to ensure the interface constraints.

The interaction among substructures of partitioned structures enforced by Lagrange multipliers is very well established. In displacement formulations, the kinematics' constraints corresponding to a single node shared by a number of substructures, as depicted in fig. 1a, are expressed as, for substructure 1 (figure 1b).

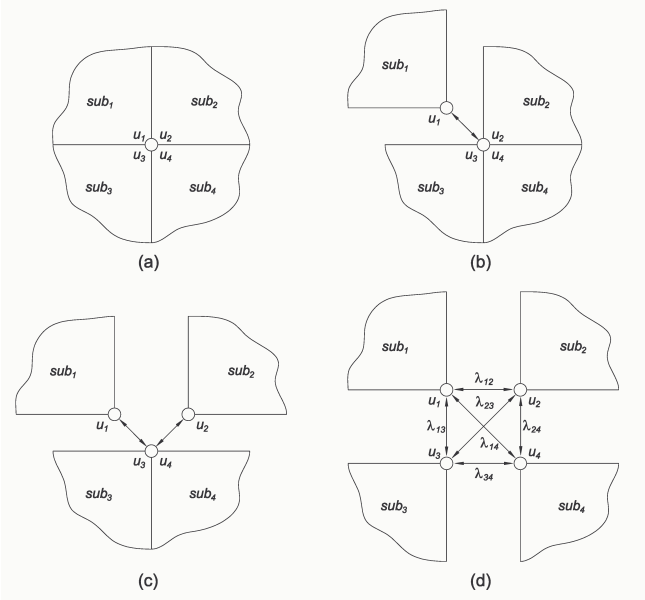

Figure 1 - A node shared by 4 structures: the classical treatment by Lagrange multipliers method leads to 6 Lagrange multipliers

$$c_{12} = \mathbf{u}_1 - \mathbf{u}_2 = 0 \qquad c_{13} = \mathbf{u}_1 - \mathbf{u}_3 = 0 \qquad c_{14} = \mathbf{u}_1 - \mathbf{u}_4 = 0$$

for substructure 2 (fig. 1c)

$$c_{23} = \mathbf{u}_2 - \mathbf{u}_3 = 0 \qquad c_{24} = \mathbf{u}_2 - \mathbf{u}_4 = 0$$

and, finally, to substructure 3 (fig. 1d)

$$c_{34} = \mathbf{u}_3 - \mathbf{u}_4 = 0$$

Those constraints introduced above are incorporated to a variational formulation by adding to it the following functional :

$$\pi(\mathbf{u}, \lambda) = \lambda_{12} c_{12} + \lambda_{13} c_{13} + \lambda_{14} c_{14} + \lambda_{23} c_{23} + \lambda_{24} c_{24} + \lambda_{34} c_{34} \tag{1}$$

which is recast in a matrix form as

$$\pi(\mathbf{u},\lambda) = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ \lambda_{34} \end{bmatrix}^T \begin{bmatrix} c_{12} \\ c_{13} \\ c_{14} \\ c_{23} \\ c_{24} \\ c_{34} \end{bmatrix} = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \\ \lambda_{23} \\ \lambda_{24} \\ \lambda_{34} \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix} \tag{2}$$

Note that the solution of this linear system is not unique. One can eliminate the linear dependent Lagrange multipliers leading to

$$\pi(u,\lambda) = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix}^T \begin{bmatrix} c_{12} \\ c_{13} \\ c_{14} \end{bmatrix} = \begin{bmatrix} \lambda_{12} \\ \lambda_{13} \\ \lambda_{14} \end{bmatrix}^T \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix} \tag{3}$$

So, retaining, for instance, λ_{12}, λ_{13} *e* λ_{14} as the Lagrange multipliers insures the uniqueness of solution and, therefore, the partitioning process can be done in many different ways. The choice a unique set of Lagrange multipliers has been a major issue in the classical force method. If we have *n* structures sharing the same node, the number of combination of Lagrange multipliers will be *n(n-1)/2*, while the number of linearly independents restrictions should be just *n-1*.

The localized version of Lagrange multipliers could be used to circumvent this drawback. In this method, a reference point is created. All the constraints should be referred to this point. This process is showed in fig. 2.

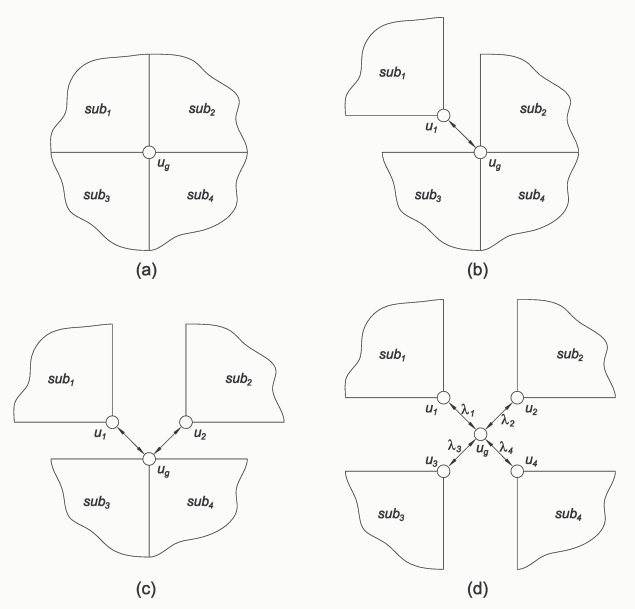

Figure 2 - A node shared by 4 structures: the localized treatment by Lagrange multipliers method leads to 4 Lagrange multipliers

The constraint for node 1 will be

$$c_1 = \mathbf{u}_1 - \mathbf{u}_g$$

for the second, third and fourth node

$$c_2 = \mathbf{u}_2 - \mathbf{u}_g \qquad c_3 = \mathbf{u}_3 - \mathbf{u}_g \qquad c_4 = \mathbf{u}_4 - \mathbf{u}_g$$

the constraints are the same, despite the order. The equation set is unique too. The new functional introduces the independent displacement $\mathbf{u}_g$ and reads as

$$\pi(\mathbf{u}, \lambda_l, \mathbf{u}_g) = \sum_{i=1}^{N_{partições}} \lambda_i c_i$$

3. Variational formulation of fluid structure problems

A variational formulation corresponding to vibro-acoustic problems (Park, Felippa, and Ohayon, 2001) is introduced bellow:

$$\frac{1}{2}\int_{\Omega_s} \sigma^T \varepsilon d\Omega_s + \frac{1}{2}\rho_f c_f^2 \int_{\Omega_f} \left(\nabla^2 \mathbf{u}_f\right)^2 d\Omega_f + \int_{\Omega_s} \mathbf{u}_s^T (\mathbf{f}_s - \rho_s \ddot{\mathbf{u}}_s) d\Omega_s + \int_{\Omega_s} \mathbf{u}_f^T (\mathbf{f}_f - \rho_s \ddot{\mathbf{u}}_f) d\Omega_f = \int_{\Gamma_s} \mathbf{u}_s^T \mathbf{T}_s d\Gamma + \int_{\Gamma_f} \mathbf{u}_f^T \mathbf{T}_f d\Gamma \tag{4}$$

where ρs e ρf are the densities of solid and fluid, c_f is the sound velocity in fluid, $\mathbf{u}_s$ and $\mathbf{u}_f$ are the solid and fluid displacement vectors, $\ddot{\mathbf{u}}_s$ and $\ddot{\mathbf{u}}_f$ are the solid and fluid acceleration vectors, σ e ε are the structural strain and stress vectors, $\mathbf{f}_s$ and $\mathbf{f}_f$ are the solid and fluid body, $\mathbf{T}_s$ and $\mathbf{T}_f$ are the force vectors acting in the boundary Γ_s and Γ_f of fluid and solid masses. Ω_s and Ω_f indicates the solid and a fluid domains. The pressure can be obtained by $p = \rho_f c_f^2 \nabla \mathbf{u}_f$. The irrotationality condition in the fluid ($\nabla \times \mathbf{u}_f = 0$ in Ω_f), is introduced as a constraint in the variational formulation.

The localized version of the Lagrange multiplier method is adopted to enforce the interface constraint between the fluid ad solid domain and the classical Lagrange multipliers are used to handle the irrotationality. Arranging the fluid and solid displacements as $\mathbf{u} = \{\mathbf{u}_s \quad \mathbf{u}_f\}^T$ the following functional can be obtained

$$\Pi(\mathbf{u}, \lambda_l, \mathbf{u}_b, \mu) = \mathbf{u}^T \left(\frac{1}{2} \begin{bmatrix} \mathbf{K}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_f \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{M}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_f \end{bmatrix} \ddot{\mathbf{u}} - \begin{bmatrix} \mathbf{F}_s \\ \mathbf{F}_f \end{bmatrix} \right) + \mu^T \Phi_f^T \mathbf{u}_f + \lambda_f^T \Psi^T \left(\mathbf{B}^T \mathbf{u} - \mathbf{L}_b \mathbf{u}_g \right) \tag{5}$$

where

$$\mathbf{K}_s = \int_{\Omega_s} \mathbf{H}^T \mathbf{D}^T \mathbf{C} \mathbf{D} \mathbf{H} d\Omega_s \qquad \mathbf{K}_f = \int_{\Omega_f} \mathbf{N}^T \rho_f c_f^2 \mathbf{N} d\Omega_f$$

$$\mathbf{M}_s = \int_{\Omega_s} \mathbf{H}^T \rho_s \mathbf{H} d\Omega_s \qquad \mathbf{M}_f = \int_{\Omega_f} \mathbf{H}^T \rho_f \mathbf{H} d\Omega_f$$

$$\mathbf{F}_s = \int_{\Omega_s} \mathbf{H}^T \mathbf{f}_s d\Omega_s + \int_{\Gamma_s} \mathbf{H}_\Gamma^T \mathbf{T}_s d\Gamma_s \qquad \mathbf{F}_f = \int_{\Omega_f} \mathbf{H}^T \mathbf{f}_f d\Omega_f + \int_{\Gamma_f} \mathbf{H}_\Gamma^T \mathbf{T}_f d\Gamma_f$$

$$\varepsilon = \mathbf{D}\mathbf{H}\mathbf{u}$$

$$\sigma = \mathbf{C}\varepsilon$$

$$\nabla \times u = (\nabla \times \mathbf{H})\mathbf{u} = \Phi_f^T \mathbf{u} \qquad \nabla \cdot u = (\nabla \cdot \mathbf{H})\mathbf{u} = \mathbf{N}\mathbf{u}$$

and $\mathbf{H}$ is the displacement interpolation matrix, $\mathbf{C}$ is the elasticity matrix and $\mathbf{D}$ is the deformation matrix. Ψ is the rigidity regularization operator, as described by Park, Felippa, and Ohayon(2001). The variation of $\Pi(\mathbf{u}, \lambda_l, \mathbf{u}_b, \mu)$ in relation of primary variables results

$$
\begin{aligned}
&\delta\Pi\left(\mathbf{u}_s,\mathbf{u}_f,\lambda_s,\lambda_f,\mathbf{u}_b,\mu\right)= \\
&\quad \delta\mathbf{u}_s^T\left(\mathbf{K}_s\mathbf{u}_s+\mathbf{M}_s\ddot{\mathbf{u}}_s-\mathbf{f}_s+B_s\lambda_s\right)+ \\
&\quad \delta\mathbf{u}_f^T\left(\mathbf{K}_f\mathbf{u}_f+\mathbf{M}_f\ddot{\mathbf{u}}_f-\mathbf{f}_f+B_f\lambda_f+C_f\mu\right)+ \\
&\quad \delta\lambda_s^T\left(B_s^T\mathbf{u}_s-\Psi_{bs}\mathbf{u}_b\right)+\delta\lambda_f^T\left(B_f^T\mathbf{u}_f-\Psi_{bf}\mathbf{u}_b\right)+ \\
&\quad \delta\mathbf{u}_b^T\left(-\Psi_{bs}^T\lambda_s-\Psi_{bf}^T\lambda_f\right)+\delta\mu^T C_f^T\mathbf{u}_f
\end{aligned}
\tag{6}
$$

Finally, making $\delta\Pi\left(\mathbf{u}_s,\mathbf{u}_f,\lambda_s,\lambda_f,\mathbf{u}_b,\mu\right)=0$ we obtain the following system of equations

$$
\begin{bmatrix}
\mathbf{K}_s+\frac{d^2}{dt^2}\mathbf{M}_s & 0 & B_s & 0 & 0 & 0 \\
0 & \mathbf{K}_f+\frac{d^2}{dt^2}\mathbf{M}_f & 0 & B_f & \Phi_f & 0 \\
B_s^T & 0 & 0 & 0 & 0 & -\Psi_{bs} \\
0 & B_f^T & 0 & 0 & 0 & -\Psi_{bf}^T \\
0 & \Phi_f^T & 0 & 0 & 0 & 0 \\
0 & 0 & -\Psi_{bs}^T & -\Psi_{bf}^T & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\mathbf{u}_s \\ \mathbf{u}_f \\ \lambda_s \\ \lambda_f \\ \mu \\ \mathbf{u}_b
\end{Bmatrix}
=
\begin{Bmatrix}
\mathbf{F}_s \\ \mathbf{F}_f \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}
\tag{7}
$$

4. Numerical results

The proposed numerical formulation is applied to the problem of an non viscous fluid contained firstly in a rigid cavity and after in linear elastic cavity. The geometry of these problems is showed in fig. 3.

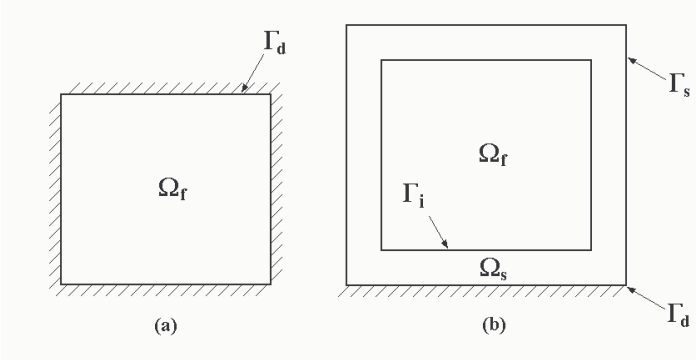

Figure 3 – Model geometry: (a) rigid cavity and (b) elastic linear cavity

Where $\mathbf{\Omega_f}$ and $\mathbf{\Omega_s}$ represents the fluid and solid domain, $\mathbf{\Gamma_i}$ is the interface between fluid and solid, $\mathbf{\Gamma_d}$ is a fixed interface (no movements are allowed for solid domain nodes in this interface for and only movements in a tangential direction are allowed for fluid domain nodes) and $\mathbf{\Gamma_s}$ is a free interface.

For both problems we use a element with 9 degrees of freedom for solid and fluid domains. For fluid mesh we used additional 4 dofs for irrotationality constraint as shown in fig. 4.

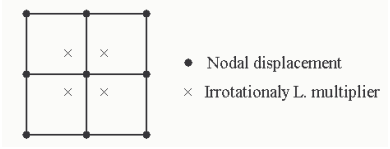

Figure 4 – Element used for fluid domains

4.1. Rigid Cavity

In this case, the analytical solution is availble. For a retangular cavity filled with a fluid and dimensions $H \times W$ the eigenfrequencies are given by

$$\omega_{nm} = c\pi\sqrt{\left(\frac{n}{H}\right)^2 + \left(\frac{m}{W}\right)^2}, \qquad n, m = 0,1,2,3,\ldots \ (n+m \neq 0)$$

Where c is the sound velocity in the fluid. The exact eigenmodes have the displacements given by

$$u^x_{mn} = \frac{n\pi}{H}\sin\frac{n\pi x}{H}\cos\frac{m\pi x}{W}$$
$$u^y_{mn} = \frac{m\pi}{W}\cos\frac{n\pi x}{H}\sin\frac{m\pi x}{W}$$

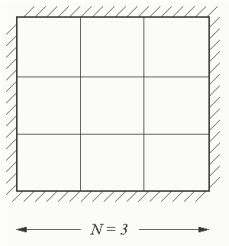

Figure 5 - Finite element mesh used for fluid in rigid cavity

Using the air as fluid we take the following physical parameters:
$\rho_f = 1$ kg/m^3
Sound speed: c = 340 m/s
Figure 5 show the refinement parameter N. In tab. 1 the exact eingenfrequencies and the results obtained by the formulation proposed here are shown.

Table 1 - Comparision between computed and exact frequencies

Mode	N = 2	3	4	5	Exact
(1,0)	1072.15	1068.98	1068.41	1068.25	1068.14
(0,1)	1072.15	1068.98	1068.41	1068.25	1068.14
(1,1)	1514.00	1511.14	1510.77	1510.66	1510.58
(2,0)	2150.34	2159.55	2144.30	2139.70	2136.28
(0,2)	2150.34	2159.55	2144.30	2139.70	2136.28
(1,2)	2396.04	2405.31	2394.22	2390.89	2388.43
(2,1)	2397.56	2405.66	2396.03	2390.90	2388.43
(2,2)	3041.05	3038.44	3027.09	3024.17	3021.16

4.2. Linear elastic cavity

This problem has been solved by (Bermúdez and Rodríguez, 1994) and his numerical results are available for comparisions. Figure 6 show the refinement parameter N. In tab. 2 the eingenfrequencies obtained Bermúdez and Rodríguez (1994) and the results obtained by the formulation proposed here are shown.

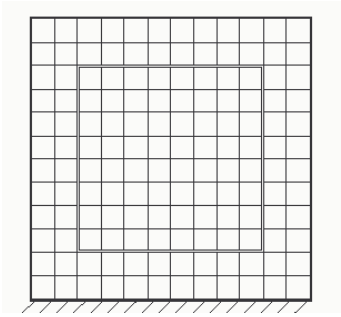

Figure 6 - Finite element mesh used for fluid and solid in linear elastic cavity

In this problem we have used water as fluid and a cavity made of steel, with the physical properties
$\rho_f = 1000$ kg/m^3
Sound speed in water: $c = 1430$ m/s
$\rho_s = 7700$ kg/m^3
Young's modulus: $E = 144\ 10^9$ Pa
Poisson's coeficient: $\nu = 0.35$

Table 2 - Comparison between computed and exact frequencies

Mode	Present Work	Bermúdez and Rodríguez, (1994)
1	634.00	641.84
2	2126.94	2116.40
3	3283.57	3201.48
4	3964.72	3804.12
5	4452.85	4211.62
6	4667.24	4687.93
7	5065.91	5155.25
8	5146.48	5385.81

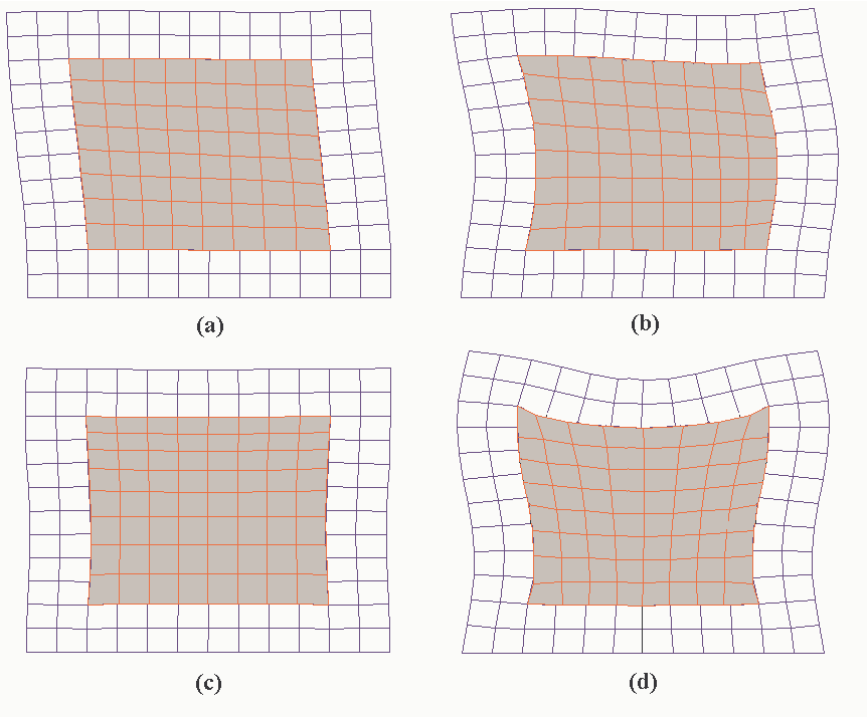

Figure 7 – Vibration modes of coupled system: (a) first vibration mode
(b) second mode (c) sixth mode (d) eighth mode

5. Conclusions

This work presents a methodology for the calculation of vibrations modes in fluid-solid interaction case. The absence of spurious modes is granted enforcing the irrotationality condition in the variational formulation by a set of Lagrange Multipliers and the interface compatibility constraint is enforced with the localized Lagrange multipliers method. The solutions presented here shows good agreement with analytical solutions and the solutions found by other numerical methods available. A drawback of this methodology is the singularity of mass matrix that was noticed by Park, Felippa, and Ohayon(2001). The modularization of the solid, fluid and interface solvers are under development and will be published elsewhere.

6. Acknowledgements

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8. Responsibility notice

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