ANALYSIS OF LAMINATED COMPOSITE PLATES WITH TRANSVERSE CRACKS BY MLGFM SOLUTION

Marcelo Pinto da Silva
PUC-PR / Programa de Pós-graduação em Engenharia Mecânica
Rua Imaculada Conceição, 1155, CP.: 16.210 – CEP 80.001-970 – Curitiba, PR.
marcelo.p.silva@electrolux.com.br

Roberto Dalledone Machado
PUC-PR / Programa de Pós-graduação em Engenharia Mecânica
Rua Imaculada Conceição, 1155, CP.: 16.210 – CEP 80.001-970 – Curitiba, PR.
roberto.machado@pucpr.br

João Elias Abdalla Filho
PUC-PR / Programa de Pós-graduação em Engenharia Mecânica
Rua Imaculada Conceição, 1155, CP.: 16.210 – CEP 80.001-970 – Curitiba, PR.
joao.abdalla@pucpr.br

Abstract. *The aim of this paper is present and applies a theory for approximate solution of the symmetrical laminated composite plates with transverse cracks in matrix. Strain-displacements relations are investigated for structures without cracks (undamaged) and further including the cracks. Damage is embedded within the constitutive equations based on the principles of Continuum Damage Mechanics and thermodynamics constraints. The damage enclosure is done by a second order tensor containing the internal state variables (that represent the transverse crack geometry and kinematics). Numerical analysis accomplished by Tay and Lim to evaluate the internal state variables under mode I load is used in this paper. The Modified Local Green's Function Method (MLGFM) is used to get the approximate solution. This technique adds the Finite Element Method and Boundary Finite Method but the final equations system are associated with boundary degrees of freedom. The methodology is similar to the Finite Element Method but the final solution is established by the Boundary Element Method. Some results are showed varying stacking sequences, plies number, plies orientation and damage extension. The results are compared with other researches.*

Keywords*: Damage, Transverse Cracks, Laminated Composite Plates, Internal State Variable, Modified Local Green's Function Method.*

1. Introduction

Currently, a great effort in the research of advanced materials is conduced searching characteristics more favorable to the applications for which they destine. Composite materials are examples of structural solutions that confirm this comment. In fact, the use of advanced composite material reinforced by fibres has resulted in many applications where the high strength and low weight is required. But, when cross-ply composite laminated are subject by loads, these materials can produce definitive structural changes such as transverse cracking, fibre-matrix interface debonding, fibre breakage and delamination. So, in order to optimize the design and fabrication of such structures, it is important to understand the failure and fracture modes of these materials subjected to different loading conditions.

The process of damage evolution in composite laminates is generally very complex. The loss of structural integrity in composite structures is related to the types, distribution of damage and the mode of loading. In general, transverse cracks in off-axis plies constitute the first internal damage of unidirectional reinforced fibre composites according to Allen et. al. (1987-a,b), Tay and Lim (1996).

The use of fracture mechanics, especially in the linear elastic fracture analysis, has been very successful for isotropic engineering materials but attempts to apply similar fracture mechanics tools have met some limitations. In the case for generalized distribution of cracks, Continuum Damage Mechanics has demonstrated good approximation and a suitable methodology.

Numerous models for the prediction of stiffness loss in composite laminates are available, for example, Talreja (1984), Allen et. al. (1987-a,b) and Tay et. al. (1997).

In this paper, the Modified Local Green's Function Method is used to investigate the stiffness loss in cross-ply symmetrical laminates with distributed transverse cracks. The transverse matrix crack is modeled within each representative volume of the structure. The Modified Local Green's Function Method is a technique of numerical approach that will be explained in the next section.

2. The Modified Local Green's FunctionMethod (MMFGL)

The Modified Local Green´s Function Method is an integral approximated method, which joins the Finite Element and the Boundary Element Methods. Fundamental solutions are obtained automatically by the MLGFM, which generates them from the Green's functions projections on the shape functions space. The fundamental solutions are determined by the domain approximation, as in the conventional FE analysis. The aim of the method is to associate the FEM and the BEM, applying transverse integration technique and reciprocity relations to determine, in a local level, the partial differential operator into an ordinary partial operator [Barcellos and Silva (1987)]. The matrices solution system is determined without the explicit knowledge of the Green's Function. By a domain approximation, using finite elements, discrete projections of the Green's Function are generated and they correspond to fundamental solutions to be applied on the boundary equation system.

The MLGM was employed in several branches of Mechanics, some of which are the Elasticity 2-D and 3-D, isotropic plates [Barbieri et al (1993)], shells, Fracture Mechanics, and others. It was also applied to the laminated composite plate problems [Machado and Barcellos (1992)], using first and high orders shear deformation theories. In most applications, the MLGFM showed high precision and superconvergency. Due the special characteristics of the method, the present work adopted it to solve the generalized crack problems in laminated plates.

Two associated problems are automatically treated by the MLFGM, the first one in the domain and the other one on the boundary. The Green's Functions are generated in the local level and they are determined by the FE approximation. Once the Green's projections are developed, they are used to generate the matrices, which are employed in the MLGFM equation system:

$$\mathbf{A\,u_\Omega = B\,f + C\,a} \quad \text{(in the domain)} \qquad \mathbf{D\,u_\Gamma = E\,f + F\,a} \quad \text{(on the boundary)} \tag{1}$$

where $\mathbf{u_\Omega}$ and $\mathbf{u_\Gamma}$ are the displacements in the domain and the boundary and the matrices **A**, **B**, **C**, **D**, **E** and **F** are defined as:

$$\mathbf{A} = \int_\Omega [\psi(Q)]^T [\psi(Q)] d\Omega_Q \qquad \mathbf{B} = \int_\Omega [\psi(Q)]^T G_\Gamma(Q) d\Omega_Q$$
$$\mathbf{C} = \int_\Omega [\psi(Q)]^T G_\Omega(Q) d\Omega_Q \qquad \mathbf{D} = \int_\Gamma [\phi(q)]^T [\phi(q)] d\Gamma_q \tag{2}$$
$$\mathbf{E} = \int_\Gamma [\phi(q)]^T G_\Gamma(q) d\Gamma_q \qquad \mathbf{F} = \int_\Gamma [\phi(q)]^T G_\Omega(q) d\Gamma_q$$

The matrices $[\psi(Q)]$ and $[\phi(q)]$ are the conventional shape functions over the domain and the boundary, while the Local Green's Functions projections are determined by:

$$G_\Gamma(Q) = \int_\Gamma G^T(p,Q)\,[\phi(p)]\,d\Gamma_p \qquad G_\Omega(Q) = \int_\Omega G^T(P,Q)\,[\psi(P)]\,d\Omega_P$$
$$G_\Gamma(q) = \int_\Gamma G^T(p,q)\,[\phi(p)]\,d\Gamma_p \qquad G_\Omega(q) = \int_\Omega G^T(P,q)\,[\psi(P)]\,d\Omega(16) \tag{3}$$

3. Representation of the damage state though stress-strain relations

The continuum damage mechanics model used in this work is based on that proposed originally by Kachanov (1972) and later used for composite laminates by Allen et al. (1987-a,b). It employs a set of second-order tensorial quantities called the internal state variables (ISVs) which are strain-like variables containing kinematics features of the distributed transverse cracks or damage. So, the effects of transverse cracks on the mechanical behavior of laminated composites can be described in terms of the material properties and the damage vector. In this work, the classical laminated plate theory is used to formulate the equations for composite laminates with transverse cracks. Assuming that a laminated can be represented by located plain elements in its average surface, the loads in one determined point in this surface can be evaluated by the following expressions:

$$\{N\} = \int_{-t/2}^{t/2} \{\sigma_x \quad \sigma_y \quad \tau_{xy}\} dz \qquad \{M\} = \int_{-t/2}^{t/2} \{\sigma_x \quad \sigma_y \quad \tau_{xy}\} z dz \tag{4}$$

where $\{N\}$ and $\{M\}$ are the resultants of forces and moments vectors, σ_x, σ_y, τ_{xy} are stresses on the plate and t is the laminated thickness.

If the $\{\varepsilon_0\}$ e $\{\kappa_0\}$ are strain and mid-plane curvature vectors, $[A]$, $[B]$ e $[D]$ are the laminate extensional stiffness matrix, coupling stiffness matrix and bending stiffness matrix, respectively; $\{D_N\}$e $\{D_M\}$ are damage vectors related to the force and moment resultants, the expressions (4) can be written as:

$$\{N\} = [A]\{\varepsilon_0\} + [B]\{\kappa_0\} + \{D^N\} \qquad \{M\} = [B]\{\varepsilon_0\} + [D]\{\kappa_0\} + \{D^M\} \tag{5}$$

In the expressions above, the matrices are defined as:

$$[A] = \sum_{k=1}^{N} (z_k - z_{k-1}) \left[\bar{Q}\right]_k \qquad [B] = \frac{1}{2}\sum_{k=1}^{N} (z_k^2 - z_{k-1}^2) \left[\bar{Q}\right]_k$$

$$[D] = \frac{1}{3}\sum_{k=1}^{N} (z_k^3 - z_{k-1}^3) \left[\bar{Q}\right]_k \qquad \{\varepsilon^0\} = \int_{-t/2}^{t/2} \{\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}\} dz \tag{6}$$

$$\{\kappa^0\} = \int_{-t/2}^{t/2} \{\kappa_x \quad \kappa_y \quad \kappa_{xy}\} dz \qquad \{D^N\} = \sum_{k=1}^{N} (z_k - z_{k-1}) \left[\bar{Q}\right]_k \left\{\bar{\alpha}\right\}_k$$

$$\{D^M\} = \frac{1}{2}\sum_{k=1}^{N} (z_k^2 - z_{k-1}^2) \left[\bar{Q}\right]_k \left\{\bar{\alpha}\right\}_k$$

where z_{k-1} e z_k, respectively, the mid-plane distances of the laminate until the bottom and the top surfaces of a generic plate k, $[\overline{Q}]_k$ are the transformed reduced stiffness matrix and $\{\overline{\alpha}\}_k$ is the vector of the internal state variable (ISVs), in global coordinates.

The usual rules of matrix transformation (from local to global), though the transformation matrix $[T]_k$ can be represented by (for $[\overline{Q}]_k$ and $\{\overline{\alpha}\}_k$):

$$\left[\bar{Q}\right]_k = [T]_k^{-1} [Q]_k [T]_k \qquad \left\{\bar{\alpha}\right\}_k = [T]_k \{\alpha\}_k \tag{7}$$

The vector of ISVs in material coordinates is given by:

$$\left\{\alpha\right\}_k = \sum_{\eta=1}^{P} \{0 \quad \alpha_{22}^{\eta} \quad \alpha_{12}^{\eta}\}_k \tag{8}$$

where P is the total number of damage types being considered.

4. Implementation of the damage model by the Modified Local Green's Function Method (MLGFM)

Considering a conventional approach of the structure by finite elements, the weak formulation of the plates equilibrium equations results in the following system of algebraic equations for a typical finite elements:

$$\begin{bmatrix} K_{11} & K_{12} & K_{16} \\ K_{21} & K_{22} & K_{26} \\ K_{61} & K_{62} & K_{66} \end{bmatrix} \begin{Bmatrix} d_x \\ d_y \\ d_z \end{Bmatrix} = \begin{Bmatrix} F_1^a \\ F_2^a \\ F_6^a \end{Bmatrix} + \begin{Bmatrix} F_1^d \\ F_2^d \\ F_6^d \end{Bmatrix} \tag{9}$$

where K_{ij} are the usual elements of the stiffness matrix, (d_x, d_y, d_z) is the element displacement vector, $\{F^a\}$ and $\{F^d\}$ are the applied force vector and the element damage force vector, respectively, which can be expressed as:

$$F_1^a = \int_{\Omega^e} p_x^a \psi_j dxdy \qquad F_2^a = \int_{\Omega^e} p_y^a \psi_j dxdy$$
$$F_6^a = \int_{\Omega^e} p_z^a \phi_j dxdy \qquad F_1^d = \int_{\Omega^e} p_x^d \psi_j dxdy \tag{10}$$
$$F_2^d = \int_{\Omega^e} p_y^d \psi_j dxdy \qquad F_6^d = \int_{\Omega^e} p_z^d \phi_j dxdy$$

In the above $\left\{p_x^d, p_y^d, p_z^d\right\}$ is defined as:

$$p_x^d = \begin{bmatrix} \frac{\partial}{\partial x}\sum_{k=1}^{N}\left\{\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy}\right\}_k + \\ + \frac{\partial}{\partial y}\sum_{k=1}^{N}\left\{\overline{Q}_{16}\alpha_{xx} + \overline{Q}_{26}\alpha_{yy} + \overline{Q}_{66}\alpha_{xy}\right\}_k \end{bmatrix}\left(z_k - z_{k-1}\right) \tag{11}$$

$$p_y^d = \begin{bmatrix} \frac{\partial}{\partial x}\sum_{k=1}^{N}\left\{\overline{Q}_{16}\alpha_{xx} + \overline{Q}_{26}\alpha_{yy} + \overline{Q}_{66}\alpha_{xy}\right\}_k + \\ + \frac{\partial}{\partial y}\sum_{k=1}^{N}\left\{\overline{Q}_{12}\alpha_{xx} + \overline{Q}_{22}\alpha_{yy} + \overline{Q}_{26}\alpha_{xy}\right\}_k \end{bmatrix}\left(z_k - z_{k-1}\right) \tag{12}$$

$$p_z^d = \begin{bmatrix} \frac{1}{2}\frac{\partial^2}{\partial x^2}\sum_{k=1}^{N}\left\{\overline{Q}_{11}\alpha_{xx} + \overline{Q}_{12}\alpha_{yy} + \overline{Q}_{16}\alpha_{xy}\right\}_k + \\ + \frac{\partial^2}{\partial x \partial y}\sum_{k=1}^{N}\left\{\overline{Q}_{16}\alpha_{xx} + \overline{Q}_{26}\alpha_{yy} + \overline{Q}_{66}\alpha_{xy}\right\}_k + \\ + \frac{1}{2}\frac{\partial^2}{\partial y^2}\sum_{k=1}^{N}\left\{\overline{Q}_{12}\alpha_{xx} + \overline{Q}_{22}\alpha_{yy} + \overline{Q}_{26}\alpha_{xy}\right\}_k \end{bmatrix}\left({z_k}^2 - {z_{k-1}}^2\right) \tag{13}$$

So, the damage vector related to force $\left\{F_k^d\right\}$ is solved if the ISVs are known and replacing the expressions (11) to (13) in expressions (10). This vector is added to the nodal forces of the structure completing the system of equations by introduction of the boundary conditions.

The actual paper uses a different proceeding to expose above. It utilized another computational method describe in section 2 by Modified Local Green's Function Method. In this method, the system expressed in (9) is not applied directly like a conventional finite element method (FEM). The MLGFM is a solution integral method and the problem variables are determined on boundary like the boundary element method (BEM). The fundamentals solutions, normally solved by BEM are automatically created by domain matrix associated with FEM. So, the solutions achieved by BEM are of boundary, but the determination of the Green's function matrix is based on the FEM, described by expression (9).

5. Internal state variables for composite laminated plates with generalized damage

Damage is an irreversible process and it must be understood by thermodynamics constraints. This analysis lead damage is approach by internal state variables (ISVs) that represents the micromechanics of internal damage caused by the presence of transverse cracks in individual off-axis plies of composite laminates. In this case, Lim and Tay (1996) propose the damage can be related only for two internal variables associated with strains that occur in laminate plan (plane stress state): α_{12} and α_{22}. The proposed internal state variables (ISVs) are related with shear strains in the ply plane and normal strains (perpendicular direction of fibres), respectively, according to applied loads and plies directions. For the case of symmetrical laminated, this corresponds to Mode I crack opening and coupled Mode I and III.

Thus, ISVs can be expressed by:

$$\alpha_{22} = \frac{1}{V}\int_{S_c} u_2 n_2 dS \qquad \alpha_{12} = \frac{1}{V}\int_{S_c} u_1 n_2 dS \tag{14}$$

where u_2 is the crack opening displacement, n_2 unit normal to the crack surface, u_1 is the crack sliding displacement, V is the representative element volume and S_c is the crack surface area.

The ISVs can be very conveniently obtained through finite element parametric studies. The idea is to model the repetitive representative element volume containing a transverse crack, as Fig. 1. As shown in Fig. 1, the 90° ply is sandwiched between two 0° plies, and all damage in the form of transverse matrix cracking is assumed to be contained within the 90° ply only, or else a symmetrical laminate [0°/90°/0°]. A uniform displacement in the positive *"x"* direction is applied in one of side. We assume that transverse cracks are uniformly distributed throughout the laminate. The size and shape of the representative volume depends on the relative thicknesses of the different plies and the crack density (number of cracks per unit volume). A series of finite element parametric studies has been performed in order to establish the effects of crack opening profiles, relative ply thicknesses and crack density on the stiffness of the laminate.

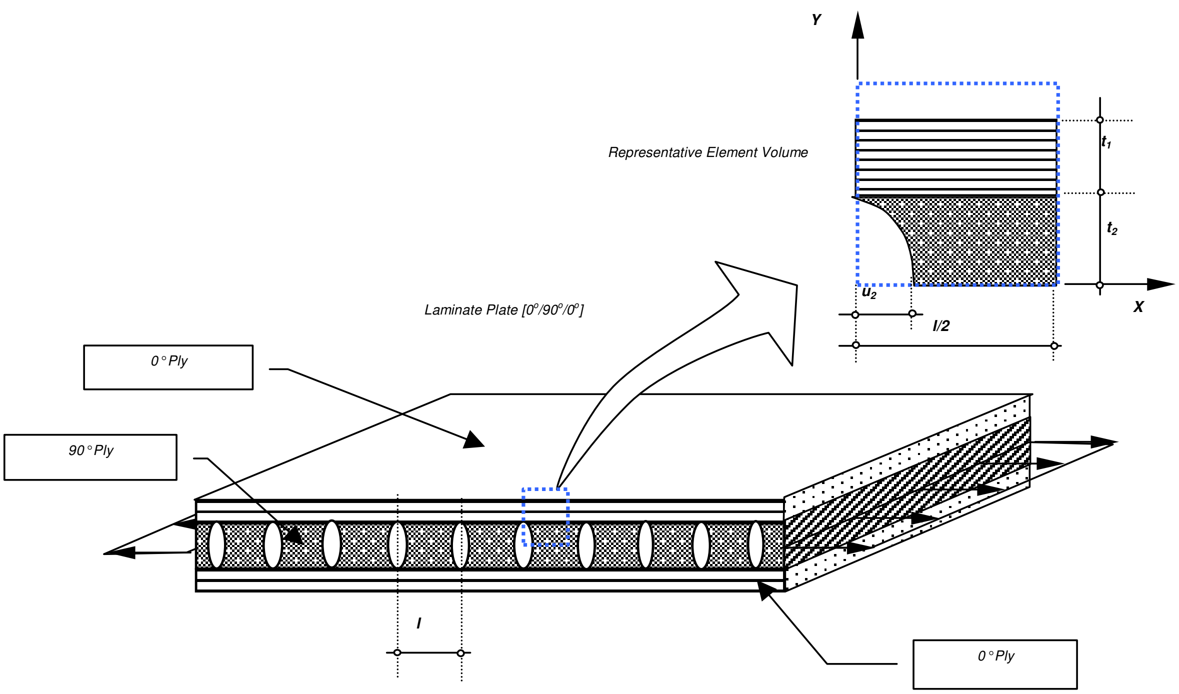

Figure 1. Laminate Plate [0°/90°/0°] with generalized cracks: parameters and representative element volume definition

A set of parametric variables are defined:

$$t = t_1 + t_2 \qquad \rho = \frac{2t}{l} \qquad \theta = \frac{t_1}{t_2} \qquad \delta = \frac{u_2}{t_2} \qquad \psi = \frac{u(\xi)}{u_2} \tag{15}$$

where t_1 and t_2 are the thicknesses of the 0° and 90° plies, respectively, t is the entire laminate thickness, l is the crack spacing between two adjacent cracks, ρ is the normalized crack density, θ is the adjacent ply thickness constraint ratio, δ

are the normalized crack opening displacement, u_2 is the maximum crack opening displacement, ψ is the normalized crack opening profile function and ξ is the normalized distance in the thickness direction from the center of the crack.

The maximum crack opening displacement u_2 can be solved through a simple finite element analysis. The δ value is determined through expression (15) and the displacement u_2 found by finite element study. Two types of functions are proposed relating δ with ρ and also δ with [Tay and Lim (1996)]:

$$\delta = c_1\left(e^{-a_1\rho}\right) + c_2\left(e^{-b_1\rho}\right) + c_3 \qquad \delta = c_4\left(e^{a_2\theta}\right) + c_5\left(e^{b_2\theta}\right) + c_6 \tag{16}$$

The constants a_1, a_2, b_1, b_2, c_1, c_2, c_3, c_4, c_5, and c_6 introduced in expressions (16) depend on the material. The constants values for Graphite-Epoxy (Gr/Ep) and Glass-Epoxy (Gl/Ep) are shown in Table 1.

Table 1. Material Constants for (54) and (55) expressions.

Material	**Constants for ρ**					**Constants for θ**				
	c_1	c_2	c_3	a_1	b_1	c_4	c_5	c_6	a_2	b_2
Graphite Epoxy	-8.86E-2	0.21	2.20E-2	1.95	0.74	-4.51E-2	-0.17	0.30	-1.49	6.95E-4
Glass Epoxy	1.03	-0.81	2.28E-2	0.94	1.00	-0.14	0.00	0.20	-0.91	-0.91

Therefore, the δ values can given by expressions (16). The ISVs are functions of the maximum crack opening displacement u_2 - according to expressions (14). If the crack density is defined by $\zeta = 1/l$, the ISVs for Mode I and coupled Mode I and III can be simplified to:

Mode I:
$$\alpha_{22} = \frac{8}{5} u_2 \zeta \tag{17}$$

Coupled Mode I and III:
$$\alpha_{22} = \frac{8}{5}\frac{u_2\zeta}{\operatorname{sen}\phi} \qquad \alpha_{12} = \frac{8}{5}\frac{u_2\zeta}{\tan\phi \operatorname{sen}\phi} \tag{18}$$

Thus, the Mode I ISV for a cross ply laminate with transverse cracks is fully determined once the maximum crack opening displacement u_2 and crack density ζ are known. For coupled Mode I and III is necessary know the angle ϕ. Hence, the damage forces can be determined through expressions (10) and added to the system of algebraic equations (9) verifying the stiffness loss of the structure. This simplified assumption obtains good results and allow include a computational model in element finite program avoiding great changes.

6. Applications

The methodology presented in this work was verified in some examples. Two types of graphite-epoxi symmetrical laminated plates subjected to middle plane parallel loads are considered: a Gr/Ep $[0^o/90^o]$s with 0,508 mm of thickness, and a Gr/Ep $[0^o/90^o_3]$s with 1,016 mm of thickness. The plates are quadrangular and due to symmetrical conditions, just a quarter of the plates are modeled. Lagrangean quadratic quadrangular finite elements are employed in the domain while quadratic boundary elements are applied on the boundary. Two different conditions to determine the δ parameter is also considered, using formulation by θ and ρ.

The Fig. 2 to 4 show the stiffness loss of the laminates for different density cracks. Figures present the behaviour of the Gr/Ep $[0^o/90^o]$s and Gr/Ep $[0^o/90^o_3]$s analyzed by the two formulations, by θ and ρ, for different meshes. Both formulations are similar and the MLGFM converges to the LIM and TAY (1997) results and also the experimental values.

7. Conclusions

A simplified model based on the ISV formulation solves the generalized transverse cracks problem. This methodology is restricting to the case of symmetrical composite laminated plates. The approximated solution is obtained by the MLGFM, an integral method which joins the FE and the BE methods. Good results were reached when compared to experimental values and conventional FE models as in the literature. This method was applied to other laminated cases and also shown good results and convergence.

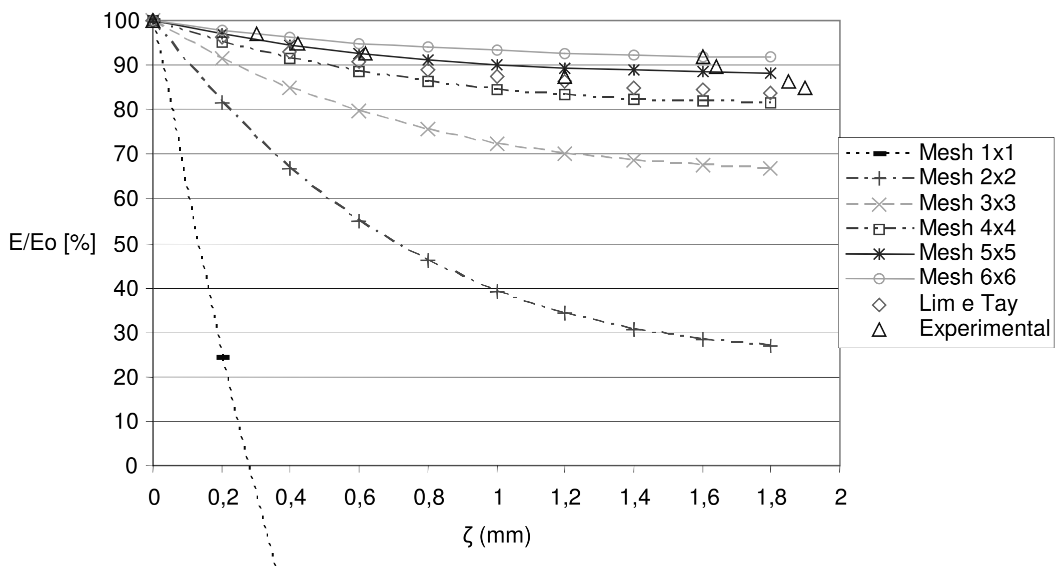

Figure 2 – Stiffness loss E/Eo versus crack density ζ for the Gl/Ep [0°./90°]s laminated, using ρ formulation.

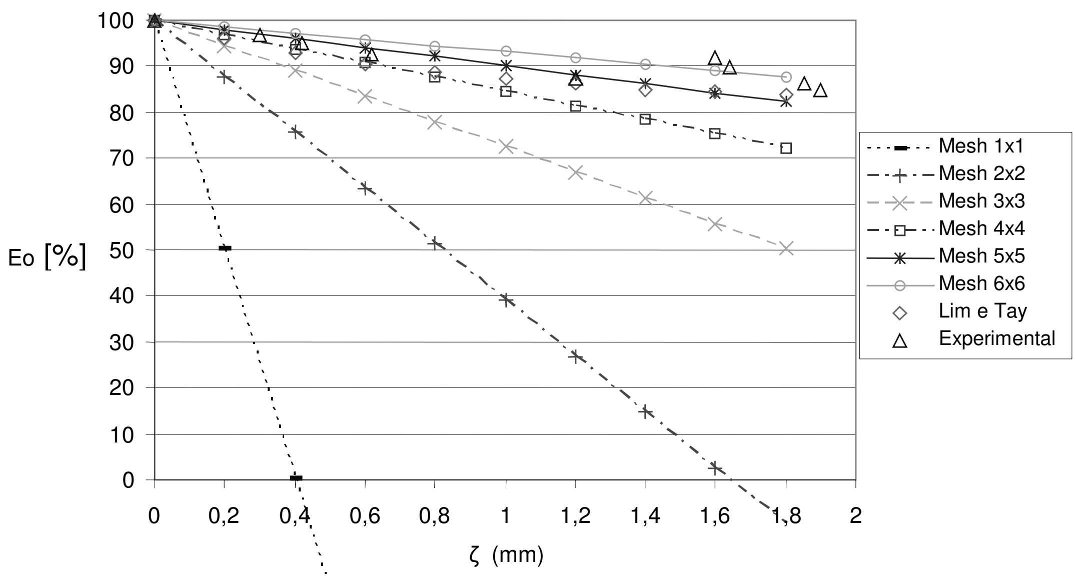

Figure 3 – Stiffness loss E/Eo versus crack density ζ for the Gl/Ep [0°/90°]s laminated, using θ formulation.

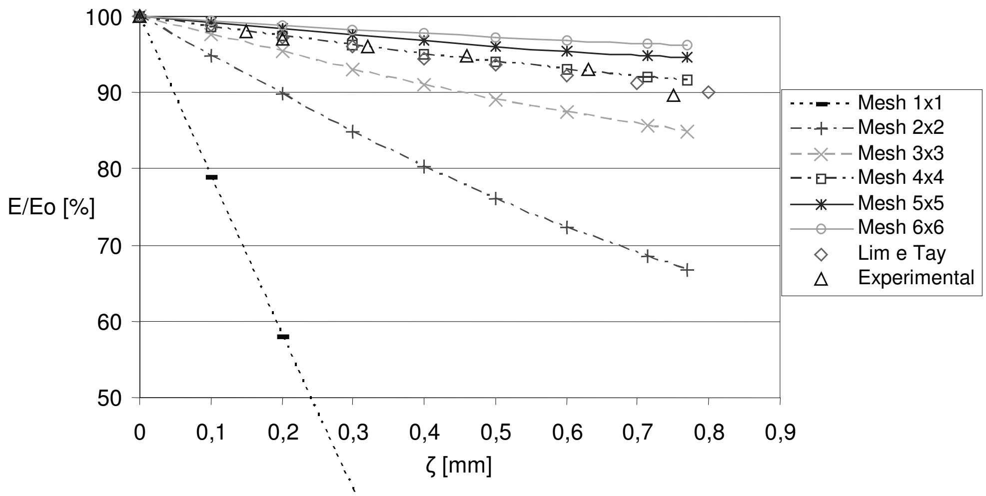

Figure 4 – Stiffness loss E/Eo versus crack density ζ for the Gl/Ep [0^o./$90^o{}_3$]s laminated, using ρ formulation.

8. References

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7. Responsibility notice

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