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THE IMPLICITLY RESTARTED ARNOLDI METHOD APPLIED TO TUNED VIBRATION ABSORBERS PROBLEMS

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Abstract. *Krylov subspace methods are widely used in scientific computing. The Implicitly Restarted Arnoldi Method is one of these methods used for solving eigenvalues problems. This paper presents an implementation of this method applied to a generalized eigenvalue problem derived from Tuned Vibration Absorbers (TVA) connected to the main structure. TVA's systems are used in order to reduce vibration levels in structures. This problem leads to a non-proportional damping matrix and hence the eigenvalues of undamped system differs of the damped case. The resulting generalized eigenvalue problem is constituted by non-positive definite matrices and hence, some usual methods could not be applied. The obtained results for a test platform, used to measure loadings induced by human movements, are presented showing the applicability of the implemented method.*

Keywords: Eigenvalue Problem,Krylov Subspace, Implicitly Restarted Arnoldi Method, Tuned Vibration Absorbers

1. Introduction

The advance of structural analysis has been allowed the design of more economic and consequently more flexible structures. Due to this advance, the engineers began to deal, more frequently, with new kind of problems, such as structures more susceptible to vibration. Several solutions were conceived to reduce vibrations: structural modification, stiffness increasing and the use of tuned vibration absorbers (Magluta 1993 and Magluta et al. 2003).

The resulting generalized eigenvalue problem, showed in Eq. (1), is formed by symmetric non-positive definite matrices which implicates in complex eigenvalues. The Arnoldi Method (Arnoldi 1951) was used with restarting techniques, this method is called Implicitly Restarted Arnoldi Method (Lehoucq 1995, Ainsworth 2003), and it was used to solve the generalized eigenvalue problem. IRAM (Implicitly Restarted Arnoldi Method) is an iterative method based on Krylov subspace. This method applies similarity transformations, the projected matrix has lower order than the original one. The eigenvalues evaluation of this resulting matrix is more simple.

In structural dynamics the lower modes are the most important and hence only them should be evaluated. The IRAM has the advantage of evaluating only the interested eigenvalues instead of other methods, like QZ (Moler and Stewart 1973). The QZ method must evaluate all eigenvalues and so the computational costs rises drastically. In this paper, the results obtained with IRAM and QZ are compared. For the comparison a test platform used to measure human loadings was discretized by the Finite Element Method.

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2. The Implicitly Restarted Arnoldi Method

A generalized eigenvalue problem is defined by:

$$\boldsymbol{Ax} = \lambda \boldsymbol{Bx} \tag{1}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices of order *n*, λ are the eigenvalues and $\boldsymbol{x}$ are the corresponding eigenvectors. When $\boldsymbol{B}=\boldsymbol{I}$, the problem is known as standard problem. Let us initially focus on the standard problem. It will be seen in what follows that this formulation can be easily extended to the generalized eigenvalue problem.

The Arnoldi Method projects the matrix of the problem onto the Krylov Subspace:

$$K_m\left(\boldsymbol{A}, \boldsymbol{v_1}\right) = Span\left\{ \boldsymbol{v_1}, \boldsymbol{Av_1}, \boldsymbol{A^2v_1}, \ldots, \boldsymbol{A^{k-1}v_1} \right\} \tag{2}$$

where $\boldsymbol{v_1} \neq \boldsymbol{0}$ is the starting vector.

The projected matrix is the called upper Hessenberg matrix and its elements are equal to zero when $i \geq j+2$:

$$\begin{bmatrix} \alpha_1 & \cdots & & \\ \beta_2 & \alpha_n & \cdots & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \beta_n & \alpha_n \end{bmatrix} \tag{3}$$

where, α_i is the *i*^th^ diagonal element β_i is *i*^th^ subdiagonal element. The Arnoldi Factorization is one of the several methods used to obtain a Hessenberg matrix.

2.1. Arnoldi Factorization

Definition: If $\boldsymbol{A} \in \Re^{n \times n}$, then the following relation:

$$\boldsymbol{AV_k} = \boldsymbol{V_k H_k} + \boldsymbol{f_k e_k^T}, \tag{4}$$

is an Arnoldi Factorization in *k* steps. When *k* < *n*, the factorization is truncated. $\boldsymbol{V_k} \in \Re^{n \times k}$ is the matrix whose columns are the Arnoldi vectors and form an orthogonal basis for the Krylov subspace. $\boldsymbol{H_k}$ is the projection of $\boldsymbol{A}$ onto this subspace, $\boldsymbol{e_k}$ is the k^{th} vector of the canonical basis. For *k* = *n* and for exact arithmetic, the residual vector $\boldsymbol{f_k}$ must vanish and therefore the eigenvalues of the Hessenberg matrix are exactly the eigenvalues of $\boldsymbol{A}$. Figure 1 shows the algorithm, written in pseudo code, which performs this factorization.

Input ($\underline{A}$, $\underline{v}_1$)

$\underline{v}_1 = \frac{\underline{v}_1}{\|\underline{v}_1\|}$; $\underline{w} = \underline{A}\underline{v}_1$; $\alpha = \underline{v}_1^H \underline{w}$;

$\underline{f}_1 = \underline{w} - \alpha \underline{v}_1$;

$\underline{V}_1 = (\underline{v}_1)$;

$\underline{H}_1 = (\alpha)$;

For j = 1, 2, 3, .., k-1

(i) $\beta_j = \|\underline{f}_j\|$; $\underline{v}_{j+1} = \frac{\underline{f}_j}{\beta_j}$;

(ii) $\underline{V}_{j+1} = (\underline{V}_j; \underline{v}_{j+1})$; $\hat{\underline{H}}_j = \begin{pmatrix} \underline{H}_j \\ \beta_j \underline{e}_j^T \end{pmatrix}$;

(iii) $\underline{w} = \underline{A}\underline{v}_{j+1}$;

(iv) $\underline{h} = \underline{V}_{j+1}^H \underline{w}$; $\underline{f}_j = \underline{w} - \underline{V}_{j+1}\underline{h}$;

(v) $\underline{H}_{j+1} = (\hat{\underline{H}}_j; \underline{h})$;

End

Figure 1. Arnoldi Factorization in *k* steps

In practical situations, the approximations of the desired eigenvalues of $\boldsymbol{A}$ should appear when *k* rises, although some problems with loss of orthogonality of the Arnoldi Vectors begins to occur. Sorensen (1992) proposed a restarting technique, applying polynomial filters to improve the convergence of the desired eigenvalues. This technique is known as Implicitly Restarted Arnoldi Method.

2.2. Restarting Technique

The IRAM combines the Arnoldi Factorization in *k* steps with the Implicitly Shifted QR algorithm (Golub and Van Loan 1996). The resulting algorithm is showed in Fig. (2).

Input ($\underline{A}$, $\underline{V}_m$, $\underline{H}_m$, $\underline{f}_m$) after *m* steps from algorithm 1

For $i = 1, 2, \ldots,$ until convergence

(i) Compute $\underline{H}_m$ spectrum and select *p* "shifts" $\mu_1, \mu_2, \ldots, \mu_p$

(ii) $\underline{q}^H = \underline{e}_m^T$;

(iii) For j = 1, 2, ..., *p*

(iv) Factor$[\underline{Q}, \underline{R}] = qr(\underline{H}_m - \mu_j)$;

(v) $\underline{H}_m = \underline{Q}^H \underline{H}_m \underline{Q}$; $\underline{V}_m = \underline{V}_m \underline{Q}$;

(vi) $\underline{q} = \underline{q}^H \underline{Q}$;

End

(vii) $\underline{f}_k = \underline{v}_{k+1}\beta_k + \underline{f}_m \sigma_k$;

(viii) $\underline{V}_k = \underline{V}_m(1:n, 1:k)$; $\underline{H}_k = \underline{H}_m(1:k, 1:k)$;

(ix) Perform a new Arnoldi factorization starting from the k^{th} step, applying *p* steps to obtain a new m-step factorization.

Convergence test

End

Figure 2. Algorithm of the IRAM

The principle of the method is to fix k and starting from a m-step Arnoldi Factorization, where m is $m = k + p$. The algorithm evaluates the eigenvalues of $\boldsymbol{H}_m$ and chooses the p shifts to be used in a QR decomposition, the choice of the shifts are guided by the portion of the wanted spectrum of $\boldsymbol{A}$. For example, if the wanted portion of spectrum is the k smallest eigenvalues, the shifts are the p greatest eigenvalues. Once computed the QR decomposition, the next step is the updating of $\boldsymbol{H}_m$ and $\boldsymbol{V}_m$. After these computations, a new m-step Arnoldi Factorization must be done starting from the k-step and performing more p steps obtaining a new m-step Arnoldi Factorization. The procedure should be repeated until the convergence is achieved and so, the k eigenvalues of the Hessenberg matrix are good approximations to the ones of the original problem.

2.3. Solving the Generalized Eigenvalue Problem

One should pre-multiply both sides of Eq. (1) by $\boldsymbol{B}^{-1}$ and so, the standard eigenvalue problem would be transformed into a generalized one, but the evaluation of the inverse of a matrix is a well known unstable numerical procedure, moreover the computational costs of the inverse evaluation are prohibitive.

Another way to perform this transformation is to evaluate the inversion implicitly, replacing the matrix-vector product in the algorithm of Fig. 1 at initial calculations and at step (iii) by the following steps:

(1) Perform Matrix-Vector :

$$\boldsymbol{z} = \boldsymbol{A}\boldsymbol{v}_1 \tag{5}$$

(2) Solve the following system of linear equations for the unknown $\boldsymbol{w}$:

$$\boldsymbol{B}\boldsymbol{w} = \boldsymbol{z} \tag{6}$$

The iterative method GMRES (Saad and Schulz 1986) was used for solving the resulting system of linear equations. This procedure is more stable than using inversion procedure.

2.4.Computing the eigenvalues and eigenvectors

Once the algorithm showed in Fig. (2) was performed, the eigenvalues and eigenvectors of the upper Hessenberg matrix are good approximations for the eigenvalues and eigenvectors of the selected subset of the original problem. With $\boldsymbol{H}_m$, the eigenvalues and eigenvectors are evaluated by:

1. Computing a partial Schur Form:

$$\boldsymbol{H}_m\boldsymbol{Q}_k = \boldsymbol{Q}_k\boldsymbol{R}_k \tag{7}$$

where, the k converged eigenvalues, are contained in the diagonal of the upper triangular matrix $\boldsymbol{R}_k$, and $\boldsymbol{Q}_k$ is an orthogonal matrix.

2. Perform the decomposition:

$$\boldsymbol{R}_k\boldsymbol{S}_k = \boldsymbol{S}_k\boldsymbol{D}_k \tag{8}$$

3. Compute the eigenvectors:

$$\boldsymbol{X}_k = \boldsymbol{V}_k \boldsymbol{S}_k \tag{9}$$

3. The Generalized Eigenvalue Problem Derived from TVA's

The methodology proposed by Magluta obtains the appropriate calibrations for the TVA's systems through an optimization algorithm known as "Goal Programming" (Ignizio, 1979). This optimization technique implicates in an interactive process that requests, in each one of the iterations, the solution of a generalized eigenvalue problem with non-proportional damping, which is the bottleneck process. The non-proportionality is just because to the connection of the TVA's to the main structure. The generalized eigenvalue problem assumes the form:

$$\left(\boldsymbol{A} + \lambda\,\boldsymbol{B}\right)\boldsymbol{x} = \boldsymbol{0} \tag{10}$$

being,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} \tag{11}$$

and,

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{C} & \boldsymbol{M} \\ \boldsymbol{M} & \boldsymbol{0} \end{bmatrix} \tag{12}$$

where, $\boldsymbol{K}$ is the stiffness matrix , $\boldsymbol{C}$ is the damping matrix , $\boldsymbol{M}$ is the mass matrix.

The eigenvalues of this proposed problem are complex and complex conjugate and assume the form:

$$\lambda = -\xi \cdot \omega + \omega\sqrt{1-\xi^2}\,i \tag{13}$$

where ω is the natural frequency, ξ is the damping ratio and i is the imaginary number.

4. Numerical Examples

The chosen model used for analysis was a steel-concrete composite simple supported platform. This platform was designed for experimental tests to describe dynamics loads induced by human activities (Faisca, 2003). Due to its dimensions, 2m x 12 m (Fig. 1), the platform was discretized by linear spatial frame elements and the absorbers were represented by discrete spring and damping elements.

The experimental frequencies of the structure were experimentally computed. The structure was excited by consecutive impacts generated by a person jumping. The response was acquired by two accelerometers located at the middle and at a quarter of the span as can be seen in Fig. 2. Once the response spectrum was obtained, the experimental data was analyzed and the obtained results were compared with the numerical ones evaluated by the Implicitly Restarted Arnoldi Method. Table (1) compares the results.

Figure 3 – The simple supported platform

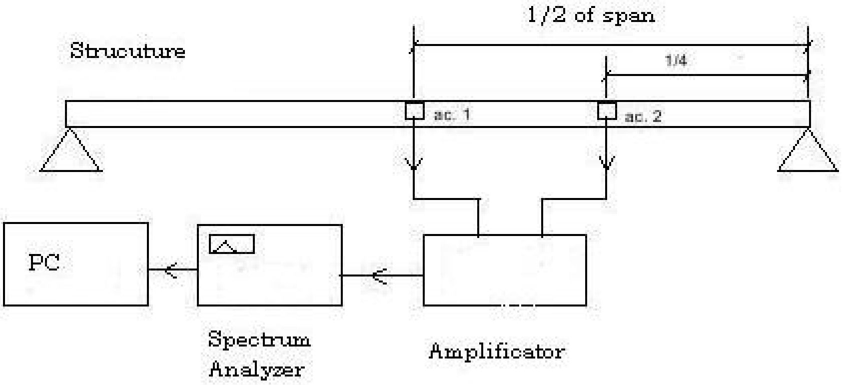

Figure 4– Experimental data acquisition scheme

Table 1. Comparison between numerical and experimental results

Natural Modes	Experimental Frequencies	Numerical Frequencies	Difference (%)
1st Flexural Mode	3.20 Hz	3.17 Hz	0.94
2nd Flexural Mode	12.80 Hz	12.70 Hz	0.78
1st Torsional Mode	28.70 Hz	28.94Hz	0.84

Once the numerical results showed that the numerical model is well adjusted to the experimental data, the next step was to simulate the same structure with TVA's connected to it. The adoption of TVA’s results in complex eigenvalues problem and it must be solved many times during the TVA’s calibration process. Methods able to evaluate complex eigenvalues must be used, as example the QZ method used by Magluta in his work.

Considering the frequencies of human jumping loading (less than 10 Hz), this analysis could be made with only the ten first eigenvalues. In the calibration phase, an optimization technique, known as, Goal Programming (Ignizio, 1979) was used and in each step of the optimization the generalized eigenvalue problem must be solved. The use of QZ method implies in the necessity of evaluation of all eigenvalues while the IRAM cans compute the requested eigenvalues needed for the analysis without loss of accuracy.

In order to compare both methods, it was simulated an example with three TVA's systems. One of the TVA's was located at ¼ of the span, another one situated at the middle and a last one at ¾ of the span. The real and imaginary parts of the computed eigenvalues, λ_r and λ_i ,are presented in Tab. 2.

Table 2 – Comparison between computed eigenvalues

Frequencies (main structure + TVA's)	λ_r (QZ)	λ_i (QZ)	λ_r (IRAM)	λ_i (IRAM)
2.676 Hz	1.655	16.721	1.653	16.738
2.908 Hz	3.440	17.935	3.437	17.928
3.450 Hz	2.323	21.539	2.320	21.519
12.69 Hz	0,200	79,744	0,200	79,725
28.95 Hz	0,591	181,804	0,589	182,002

In this analysis the TVA's were calibrated in order to attenuate the amplitudes associated to the first mode. Table 2 shows the effect of the TVA's in this mode. Due to the fact that three TVA's systems were used, 4 modes with frequencies close to the first frequency of the main structure were obtained. It should be pointed out that two of these modes have identical frequencies (2.9 Hz). Comparing Tab. 1 and 2 it can be seen that the second and third modes were not affected. Both real and imaginary parts of the eigenvalues evaluated by the IRAM are close to the eigenvalues computed by the QZ method. The same results were observed for eigenvectors.

Table 3 shows the comparison of the computational time spent to evaluate different number of eigenvalues and eigenvectors for IRAM and QZ methods. It should be pointed out that the QZ method always evaluate all the eigenvalues (40). The better performance of the IRAM method is clearly seen, mainly for smaller quantities of eigenvalues. It should be mention that in the calibration process the generalized eigenvalue problem must be solved in each optimization step and therefore, the difference of computational costs will increase its importance.

Table 3. Comparison between IRAM and QZ routines for distinct number of eigenvalues computed

Number of Eigenvalues\Cpu times	IRAM	QZ
10	0.5 s	3.87 s
20	1.2 s	3.87 s
40	2.97 s	3.87 s

It should be pointed out that the results shown in Table 3 were obtained controlling the residual errors of both methods in very small values, having magnitude of 10^{-8} (QZ) and 10^{-10} for IRAM.

5. Conclusions

The Implicitly Restarted Arnoldi Method is a good methodology to solve generalized eigenvalue problems. This iterative method showed a better performance than QZ, the advantage could be more important when the order of the problem rises or when a process of optimization is used requiring a solution in each step.

6. Acknowledgments

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7. References

Magluta, C., 1993, "Passive Dynamic Absorbers to Attenuate Structural Vibrations", PhD thesis (in Portuguese_, COPPE-UFRJ, Rio de Janeiro, Brazil.

Magluta, C., Ainsworth, G. O. and Roitman, N., 2003, "Comparison Between Multiple Vibration Absorbers and Single Vibration Absorbers Systems", Proceedings of the 17th International Congress of Mechanical Engineering, São Paulo, Brazil.

Arnoldi W. E., 1951, "The principle of minimized iterations in the solution of the matrix eigenvalue problem". Quatr. Journal of Applied Math, No 9, pp. 17-29.

Lehoucq R., 1995, "Analysis and Implementation of Implicitly Restarted Arnoldi Method", PhD thesis, Rice University, Houston.

Ainsworth Jr. G.O., 2003, "Implementation of an Algorithm Based on Arnoldi Method For Solving a Generalized Eigenvalue Problem", Master's Science Thesis (in Portuguese), COPPE-UFRJ,Rio de Janeiro, Brazil.

Moler C. B., Stewart, G. W., 1973, "An algorithm for generalized eigenvalue problems", SIAM Journal of Numerical Analysis, Vol. 10, pp. 241-256.

Sorensen, D. C., 1992, "Implicit Application of Polynomial Filters in a k-Step Arnoldi Method", SIAM Journal of Matrix Analysis and Applications, Vol. 13, No 1, pp. 357-385.

Golun, G. H. and van Loan, C. F., 1996, "Matrix Computations", The Johns Hopkins University Press, Baltimore, MD.

Saad, Y., Schultz, M. H., 1986, "GMRES: A generalized minimal residual algorithm for solving nonsymetric linear systems", SIAM Journal Sci. Stat. Comp., vol. 7, n. 3, pp. 856-869.

Ignizio, J. P.,1979, "Goal Programming and Extensions", Lexington Books, New York., N.Y., U.S.A..

Faisca, R. G., 2003, "Characterization of Dynamic Loads Due to Human Activities", Doctor's Science Thesis (in Portuguese), COPPE-UFRJ, Rio de Janeiro, Brazil.

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