Displacement of Semi-infinite Beams on an Elastic Foundation under Bending-Compression Loading

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Abstract.*. Abstract The behavior of semi-infinite uniform beams on an elastic foundation is formulated for equilibrium. A close form solution for the displacement field, using Laplace transforms, is presented. A linear elastic brittle material, loaded in a quasi-static manner is considered. Once interface loading is deformation dependent, with dissipation occurring, determination of the deformation at different load levels is pursued. Deflections and bending moments are determined along the beam, taking into consideration the loading variables. Results show predominance of bending for contact with inclined walls.*

Keywords : *Beam deflections; Elastic foundation; Contact loading; Deformation dependence*

1. Introduction

In this work the problem of beams of constant cross section, resting on a linear elastic foundation, loaded by in-plane as well as out-of-plane resultants, Fig. 1, is addressed. The uniform beam problem is a well studied problem in ice mechanics. Its study started with (Hetenyi, M.,1946) in the area of rail road engineering. Later on (Croasdale, K.R., 1980) obtained the solution corresponding to the failure loads in ice beams, under particular loadings. Here the solution is presented in a more elegant way, also covering other forms of boundary prescription.

2. Formulation

2.1 Equilibrium

Equilibrium of an elastic beam element, Fig. 1, constituted of a homogeneous material of rectangular uniform cross section, supported on a linear elastic foundation, being κ the foundation constant, under a bending compression loading , satisfies the pair of differential equations:

$$D(N) \cong 0$$

$$D^2(E'ID^2(w)) - D[ND(w)] + b\kappa w \cong 0; \qquad D^i = \frac{d^i}{dy^i}, i \in Z^+ \tag{1}$$

The first equation above refers to in-plane force equilibrium, y direction, whereas the second corresponds to the lateral, out-of-plane, moment equilibrium. Combination of both equations, while disregarding the displacement of the neutral line with respect to the equal area axis, caused by the cross coupling, results in:

$$D^4(w) + 4\gamma_0^4 D^2(w) + 4\delta_0^4 w = 0 \tag{2}$$

being w the lateral displacement. The coefficients are defined by:

$$\gamma_0 = \left(\frac{n_0}{4E'i_0}\right)^{\frac{1}{4}}; \qquad n_0 \geq 0 ; \qquad \delta_0 = \left(\frac{\kappa}{4E'i_0}\right)^{\frac{1}{4}}; \tag{3}$$

representing n_0 the value of the normal force per unit width b_0, supposed constant. The bending proportional stiffness term is i_0 and E' is the equivalent beam Young's modulus. Therefore:

$$n_0 = \frac{N_0}{b_0}; \qquad i_0 = \frac{h_0^{\ 3}}{12}; \qquad E' = \frac{E}{(1-\nu^2)} \tag{4}$$

where N_0 is the normal force at the origin, supposed positive if compressive. E is the elastic modulus of the material and ν its Poisson's ratio. This beam model is referred to as Kirchhoff model (Mellor, M., 1983).

2.2 Interface Loading

Contact at the left interface supposes a loading that may be prescribed by a couple m_0 and a shear force v_0, both per unit width. Relationship between these quantities and the kinematics at the interface establishes that

$$y = 0; \qquad \begin{aligned} m_0 &= -E' i_0 D^2(w)_0 \\ v_0 &= -E' i_0 D^3(w)_0 - n_0 D(w)_0 \end{aligned} \tag{5}$$

what evidences dependence upon the deformation of the beam at the origin. Interface contact problems suppose the existence of local conditions that differ from a beam model. Disregarded disturbances caused by this close field, the beam model may be applied.

2.3 Boundary Conditions

At the other end, the far end, the boundary conditions reflect regularity, so that;

$$y = L; \qquad \begin{aligned} w_L &= 0 \\ D(w)_L &= 0 \end{aligned} \tag{6}$$

where the subscript is employed to describe the position where the variable is to be computed. For the semi-infinite body, we need to set $L \to \infty$. Lateral loading will add an additional term to the right hand side of Eq. (2).

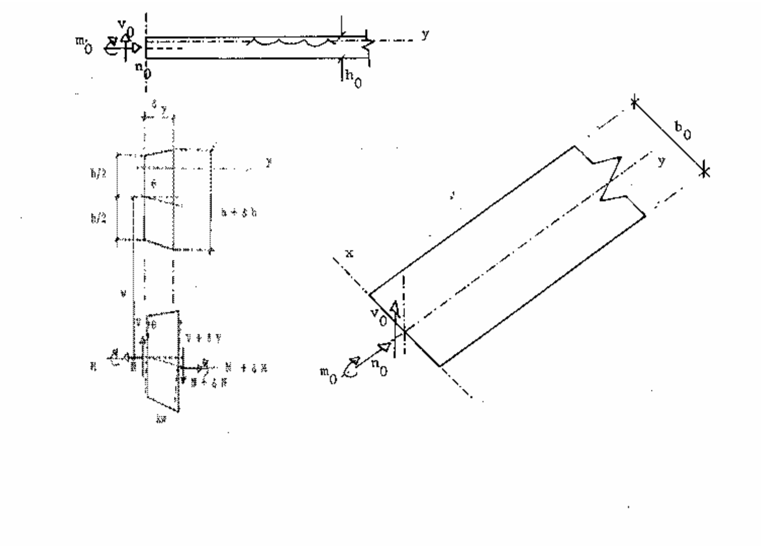

Figure 1. Uniform beam model

3. D.E. Solution

3.1 Forward Laplace Transform Mapping

A characteristic of the differential equation of equilibrium, Eq. (2), is that, aside the central in-plane load dependent coefficient γ_0, all other coefficients are constant. Furthermore it is valid on a semi-infinite interval, what makes it ideal for a Laplace transform type of solution:

$$\mathcal{L}[\mathrm{w}]=\int_0^\infty w(y)\exp(-sy)dy \qquad \mathcal{L}[\mathrm{w(y)}]=\mathrm{l(s)} \tag{7}$$

Applying the transform onto Eq. (2) leads to the expression

$$l(s)=\frac{s^3 w_0+s^2 D(w_0)+s[D^2(w_0)+4\gamma_0{}^4 w_0]+[D^3(w_0)+4\gamma_0{}^4 D(w_0)]}{s^4+4\gamma_0{}^4 s^2+4\delta_0{}^4} \tag{8}$$

whose pair of complex conjugated roots, $<\lambda;-\lambda;\bar{\lambda};-\bar{\lambda}>$ is:

$$\lambda=\alpha+i\beta; \qquad \bar{\lambda}=\alpha-i\beta \tag{9}$$

being:

$$\alpha=\sqrt{\delta_0{}^2-\gamma_0{}^4}; \qquad \beta=\sqrt{\delta_0{}^2+\gamma_0{}^4} \tag{10}$$

Partial fraction expansion of Eq. (9) may be used as a step towards the inversion mapping. Therefore if the expansion uses the roots above as a basis,

$$l(s)=\frac{\mathbf{A}}{(s+\lambda)}+\frac{\mathbf{B}}{(s-\lambda)}+\frac{\mathbf{C}}{(s+\bar{\lambda})}+\frac{\mathbf{D}}{(s-\bar{\lambda})} \tag{11}$$

with complex coefficients in the numerator

$$< \mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} > = < A_r + iA_i, B_r + iB_i, C_r + iC_i, D_r + iD_i > \tag{12}$$

then eight unknowns result. On equating the left and right sides of this equation, a set of relations is obtained.
For the real components the system obtained is:

$$[M]\{\chi_r\} = \{\iota_r\}; \qquad [M] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}; \quad \{\iota_r\} = \begin{Bmatrix} w_0 \\ \dfrac{(4\gamma_0{}^4 - 1)D(w)_0 + D^3(w)_0}{2\alpha} \\ 0 \\ 0 \end{Bmatrix} \tag{13}$$

with a solution vector $\lfloor \chi_r \rfloor = \lfloor A_r \quad B_r \quad C_r \quad D_r \rfloor$ where:

$$\begin{aligned} A_r &= \frac{1}{4} w_0 + \frac{(4\gamma_0{}^4 - 1)}{8\alpha} D(w)_0 + \frac{1}{8\alpha} D^3(w)_0; \qquad C_r = A_r \\ B_r &= \frac{1}{4} w_0 - \frac{(4\gamma_0{}^4 - 1)}{8\alpha} D(w)_0 - \frac{1}{8\alpha} D^3(w)_0; \qquad D_r = B_r \end{aligned} \tag{14}$$

For the imaginary part, on the other hand, the same procedure will render the coefficients:

$$\begin{aligned} A_i &= \frac{1}{8\beta} D^3(w)_0 + \frac{(4\gamma_0{}^4 + 1)}{8\beta} D(w)_0 - \frac{1}{8\alpha\beta} D^3(w)_0 - \frac{2\gamma_0{}^4}{8\alpha\beta} w_0; \qquad C_i = -A_i \\ B_i &= -\frac{1}{8\beta} D^3(w)_0 + \frac{(4\gamma_0{}^4 + 1)}{8\beta} D(w)_0 - \frac{1}{8\alpha\beta} D^2(w)_0 - \frac{2\gamma_0{}^2}{8\alpha\beta} w_0; \quad D_i = -B_i \end{aligned} \tag{15}$$

completing then the solution.

3.2 Inverse Laplace Transform Mapping

The s-space solution can be mapped back into a y-space solution, which is the solution sought, if it is observed that:

$$\mathcal{L}^{-1}\left[\frac{1}{s - < \lambda, -\lambda, \bar{\lambda}, -\bar{\lambda} >}\right] = \exp(< \lambda, -\lambda, \bar{\lambda}, -\bar{\lambda} > y) \tag{16}$$

Or, on substituting Eq. (17) into Eq. (9),

$$w(y) = \exp(-\alpha y)[\mathbf{A}\exp(-i\beta y) + \mathbf{C}\exp(i\beta y)] + \exp(\alpha y)[\mathbf{D}\exp(-i\beta y) + \mathbf{B}\exp(i\beta y)] \tag{17}$$

which is a general form of lateral displacement field. Imposing the regularity conditions, Eqs. (7), and noting the relationship among real and imaginary parts of the coefficients, the positive exponential terms are dropped and the remaining term becomes

$$w(y) = \exp(-\alpha y)[C\cos(\beta y) + S\sin(\beta y)]; \quad C = 2A_r \quad S = 2A_i \tag{18}$$

Coefficients on this expression may be obtained considering the left interface loading, Eqs. (6). Resulting values are

$$A_r = \frac{\tau_1 m_0 + \tau_2 v_0}{oE' i_0}; \qquad \begin{array}{l} \tau_1 = -\beta(\alpha^2 + \beta^2) \\ \tau_2 = -2\alpha\beta \end{array} \tag{19}$$

for the first one, and :

$$A_i = \frac{\eta_1 m_0 + \eta_2 v_0}{oE' i_0}; \qquad \begin{array}{l} \eta_1 = \alpha(\alpha^2 + \beta^2) \\ \eta_2 = \beta^2 - \alpha^2 \end{array} \tag{20}$$

for the other one. The denominator term is

$$o = \beta(\alpha^2 + \beta^2)(3\alpha^2 - \beta^2) \tag{21}$$

4. Application Problem

4.1 Quasi-static Loading

When an ice floe driven by currents and wind is loaded at contact with an inclined wall of an offshore platform, loading is essentially quasi-static, but coming from the impulse generated by the inertia change of the ice. It produces flexure, shear and compression. The rate of change of the linear momentum $\mathbf{l}$ equals the sum of the external forces, normal and shear, per unit width at the interface, whereas the change of the angular momentum $\mathbf{a}$ equals the rate of change of the interface bending moment. Values of normal n_0, bending moment m_0 and shear force v_0 at the interface depend on the contact between platform and beam.

They are resultants that may described by the coefficient of friction μ between ice and the rigid wall of the platform, its slope angle ϕ and the coefficient of eccentricity ζ. Solving the normal $r_0\mathbf{e}_n$ and tangential $\mu r_0\mathbf{e}_t$ to the inclined wall in terms of its horizontal and vertical components (Croasdale, K.R., 1980)

$$y_0 = r_0(\cos\phi + \mu\sin\phi); \qquad z_0 = r_0(\sin\phi - \mu\cos\phi) \tag{22}$$

and denoting the rotation at the origin by θ_0, we may write that :

$$n_0 = -z_0 \sin\theta_0 + y_0 \cos\theta_0; \quad v_0 = z_0 \cos\theta_0 + y_0 \sin\theta_0; \quad m_0 = -n_0 e \tag{23}$$

being $e = h_0\zeta; 0.5 \le \zeta \le 0.5$ the eccentricity. The coefficient μ depends on the existence of sliding or sticking contact conditions, for every value of r_0. A additional consideration has to be introduced here, as the shear force at the origin v_0 will also depend on the direction of the motion of the beam. For the riding-up condition, or up-slope case, under slipping and sticking stages,

$$\begin{array}{ll} v_0 \le n_0 \tan(\theta_t); & \theta_t < \rho \\ v_0 = n_0 \tan(\theta_t - \rho); & \theta_t \ge \rho \end{array} \tag{24}$$

where ρ is a material parameter, related to the way the ice and wall interact, and dependent on surface roughness and temperature among other factors. θ_0 represents the rotations at the origin and $\theta_t = \phi + \theta_0$.

Notice that deformation of the beam acts to create an effective value of friction angle. Table 1 presents the values considered in the present analysis.

Loading and Geometric Parameters	
Slope angle set, degrees	$\phi = \{15,30,45\}$
Eccentricity set	$\zeta = \{-0.5, 0., 0.5\}$
Friction coefficient set	$\mu = \{0.05,0.25\}$
Thickness-width ratio	$h_0/b_0=\{1., 2., 4.\}$

Table 1. Set of parameters used in the analysis

4.2 Material Parameters

For the problem at hand we have to deal with a complex material, whose constitutive equation depends on the type of microstructure considered, time of the year, form of response sought, etc. For ice features in a brittle state, in salt water, Table 2 presents some average values of the properties of this material.

Properties of Ice	
Elastic modulus	$E = 0.50e+10$ Pa
Poisson's ratio	$\nu = 0.30$
Flexural strength	$S_f = 0.70$ MPa
Compressive strength	$S_c = 5.0$ MPa
Foundation constant	$\gamma_{sw} = 1.0045e+4$ Pa/m

Table 2. Some properties of the beam material

4.3 Interface Rotation
4.4

Therefore, because the displacements do depend on the interface conditions, loading at origin depends on the rotation $\theta_0 = D(w)_0$, Eq. (20)-(24),

$$\theta_0 = -2\alpha A_r + 2\beta A_i \tag{25}$$

or, upon substitution of the interface values, Eqs. (27),

$$a \sin\theta_0 + b\cos\theta_0 - \theta_0 = 0 \tag{26}$$

where,

$$a = \frac{-2\alpha e z_0 + y_0}{E' i_0 (3\alpha^2 - \beta^2)}; \qquad b = \frac{2\alpha e y_0 + z_0}{E' i_0 (3\alpha^2 - \beta^2)} \tag{27}$$

are coefficients that depend on θ_0 also. Solution of Eq. (30) requires application of a Newton procedure, with expansion around the no-rotation at the zero stage to produce :

$$\theta_0 \cong -\frac{b_0}{a_0 + \partial_{,\theta_0}(b)_0 - 1} \tag{28}$$

where $a_0 = a(0)$ and $b_0 = b(0)$ depend upon

$$\alpha_0 = \sqrt{\delta_0^{\,2} - \gamma_0^{\,4}(0)}; \qquad \beta_0 = \sqrt{\delta_0^{\,2} + \gamma_0^{\,4}(0)} \tag{29}$$

being $\gamma_0(0) = \sqrt[4]{\dfrac{y_0}{4E'i_0}}$.

5. Results

5.1 Displacement Field

Due to the nature of the problem, determination of the deformed configuration of the beam under compressive-flexural conditions has to be obtained at particular values of the in-plane load n_0. This deformed configuration will always depend upon the interface variables considered. From Eqs.(19)-(22), the lateral displacement field is :

$$w = \frac{e^{-\alpha y}}{oE'i_0}\{[\tau_1 \cos(\beta y) + \eta_1 \sin(\beta y)]m_0 + [\tau_2 \cos(\beta y) + \eta_2 \sin(\beta y)]v_0\} \tag{30}$$

As the interface rotations define the loading variables, once they are determined with the procedure outlined above, for every value of in-plane normal, a deflection profile of the beam may be plotted. In particular effects of the interface variables may be considered and analyzed. Results are presented in Figs. 2, 3 and 4.

In Fig. 2 the effect of the eccentricity of loading $e = -\xi h_0$ on the displacements w a long the beam is described by :

$$w = \hat{w}_e(\frac{y}{\lambda_0}; \frac{h_0}{b_0}; E, \nu, \xi, \phi, \mu; \theta_0; \frac{n_0}{n_c}) \tag{31}$$

for constant values of all variables but the eccentricity itself. In this plot ordinates were made non-dimensional by considering the thickness of the beam as a factor, whereas abscissas were normalized with respect to the wave-like number $\lambda_0 = \dfrac{2\pi}{\delta_0}$. The crushing resistance of the beam $n_c = h_0 S_c$ is used as a factor as well. Observing this figure it is noticed that eccentric edge loading does not produce substantial changes in the lateral displacement profile of the beam. Only at the region closest to the loading edge, where the displacements are largest, appreciable changes occur, with an increase in the amplitude of the displacements corresponding to decreases of the eccentricity of loading.

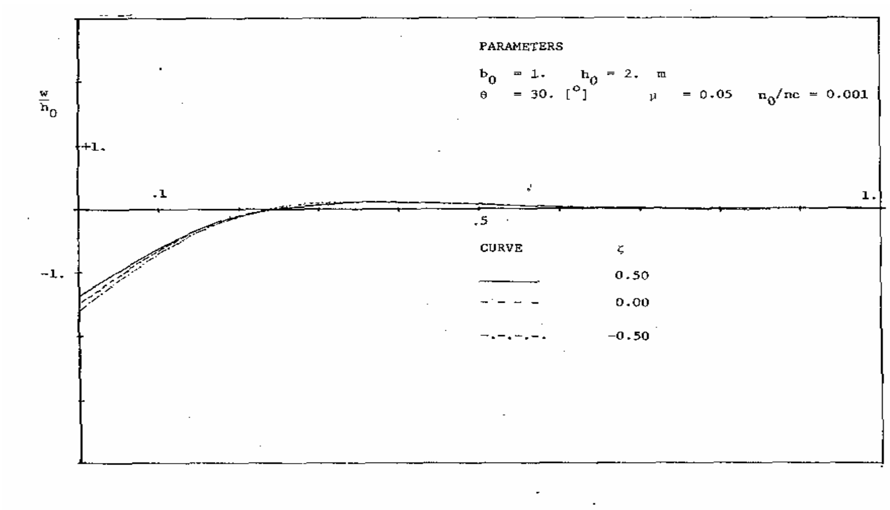

Figure 2. Eccentricity of loading effect on displacements

The effect of the slope angle of the inclined wall produces a much more pronounced effect, as can be observed from the plot presented in Fig. 3:

$$w = \hat{w}_{\phi}\left(\frac{y}{\lambda_0};\frac{h_0}{b_0};E,\nu,\xi,\phi,\mu;\theta_0;\frac{n_0}{n_c}\right) \tag{32}$$

Close to the loading edge, increases in the angle θ result in large increases in the amplitude of displacements. In fact this also implies that the effect of the edge force v_0 is much more important that the bending moment.

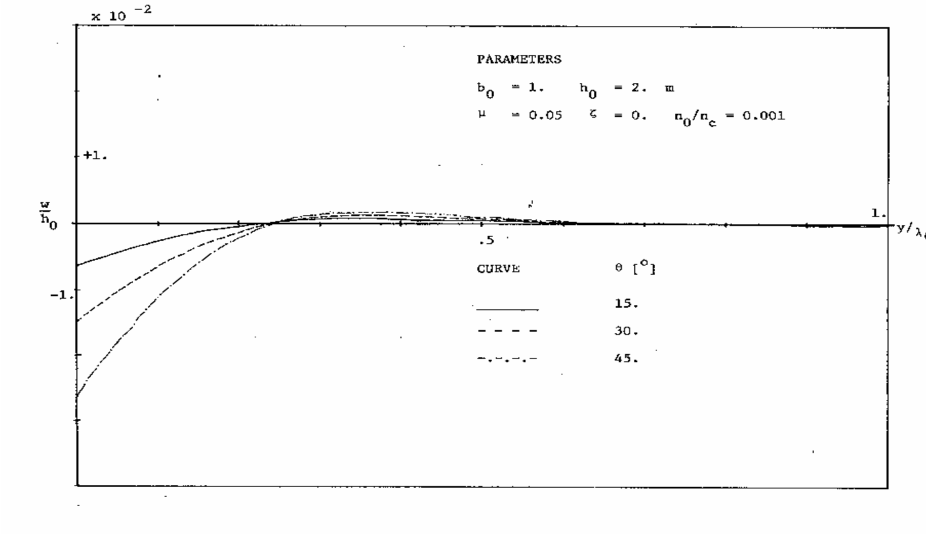

Figure 3. Slope angle effect on displacements

The effect of the coefficient of friction on the displacements is shows in Fig. 4, taking into account two surfaces, steel and fine concrete, and slipping conditions,

$$w = \hat{w}_{\mu}\left(\frac{y}{\lambda_0};\frac{h_0}{b_0};E,\nu,\xi,\phi,\mu;\theta_0;\frac{n_0}{n_c}\right) \tag{33}$$

Overall, it is observed a decrease of the amplitude of lateral displacements, close to the loading edge, when the coefficient of friction is augmented (Mohamed, S. and Tom Carrieres, 1999).

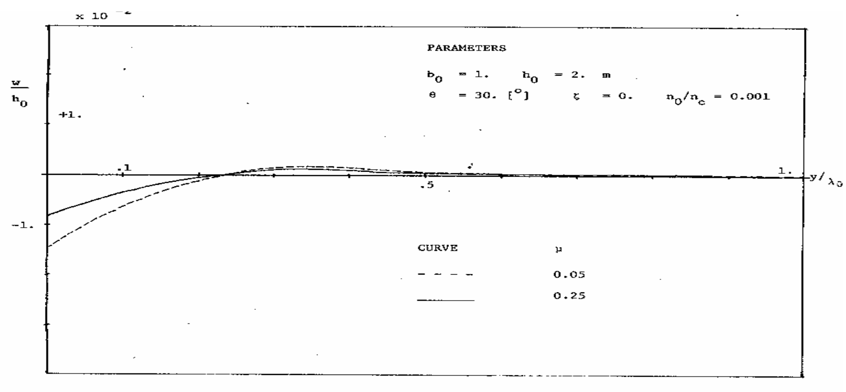

Figure 4. Friction coefficient effect on displacements

6. Conclusions

Solutions obtained here compare quite well with reported results for this kind of problem. In particular the solution may be completed with another form of boundary prescription. Layered forms of prescription of the constitutive equation may well be used with some changes incorporated to this mode (Smirnov, V.N. and Shushlebin, 1990)

7. References

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