VIBRATION ANALYSIS OF ROTATING SHAFTS COUPLED BY RIGID DISKS

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Abstract: *This work is a preliminary study of dynamical response of a mechanical transmission system including two coupled axis with one pair of gears. The mechanical model is represented by two flexible axis coupled with one pair of rigid disks. The model is simplified and don't consider elasticity of the teeth gear. Numerical analysis was performed utilizing the Finite Element Method. Each axis is modeled with five one-dimensional elements. Five variables are considered at each node – rotation due to torsion, two displacements due to bending in two orthogonal planes and two rotations due to bending in the same planes. All calculus, deduction of the kinetics and potential energy equations, the application of Lagrange's equations for obtaining mass and stiffness matrixes for this model, were performed with 'Mathematica 5'. The Newmark method was used for solving of the differentials equations systems.*

Keywords. Mechanical Vibrations, Rotating Shafts, Finite Element

1. Introduction

To design power transmission systems, it is essential to understand the dynamic behavior of gears. In the literature, many studies of the dynamic behavior of gear systems can be found. The first vibratory models for gear dynamics were given in the 1950th, although the concern with gear loads back to the eighteenth century. Experimental studies and more reliable models on the dynamics models on dynamic behavior of gears have been reported in the past 30 years and heretofore (H.N.Ozguven, 1991, .A.Andersson, L. Vedmar.,2003).

Even though there are several mathematical models developed for gear dynamics.

It has been observed that dynamic models suggested for systems may vary considerably, and yet it may still be possible to obtain similar predictions by using completely different models for certain systems. However, this depends on the relative dynamic properties of the systems.

The present work is divided in two parts. Analysis of dynamic behavior of first shaft with disc was done initially**.** The Finite Element Method is used, adopting beam element with two nodal points and five degree of freedom for each node. The motion equations are obtained from Lagrange's equations, and describe the motion in two transverse planes. The shaft and disc are modeled with equal length finite elements. The mass and stiffness matrixes of shaft and disc are different because of their unequal diameters. At first, mass and stiffness matrixes for the shaft and for the disc, are separately calculates for each case, and adequately assembled into one global mass matrix and one global stiffness matrix. In the following part of this work, matrixes for second shaft and disc were assembled analogously. The final procedure is to couple the matrixes for the first axis and for the second axis into global matrices for the full system. The present work is preliminary study of a more complete problem that may include the stiffness of the gear teeth. The stiffness of the teeth is not considered presently. The main assumption is that the linear velocity of the second disk is equal to the linear velocity of first disk at the contact point.

2. Mathematical development

The model of analyzed system is presented in Fig.1. The model consist of two parts:

1) first shaft with disk;
2) second shaft with disk.

Each part is divided in five one-dimensions beam finite elements of equal length with two nodal points. Every element has five degrees of freedom according to Fig.2, two displacements and three rotations. The finite elements of this system can be examined as beam and torsion elements. The kinetic energy, bending strain energy of element in Fig.2 can be expressed as

$$T_{total}(t) = T_{beam-xy}(t) + T_{beam-xz}(t) + T_{tors}(t) \quad (1)$$

$$V_{total}(t) = V_{beam-xy}(t) + V_{beam-xz}(t) + V_{tors}(t) \quad (2)$$

where $V_{total}(t)$, $T_{total}(t)$ – the total strain and kinetic energies
$T_{beam-xy}$, $T_{beam-xz}$, $V_{beam-xy}$, $V_{beam-xz}$ – the kinetic and strain energies in plains *xy* and *xz*
T_{tors} and V_{tors} – kinetic and strain energies of torsion

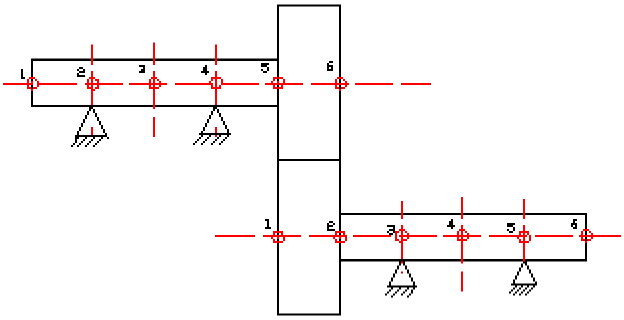


Fig.1. Model of the system.

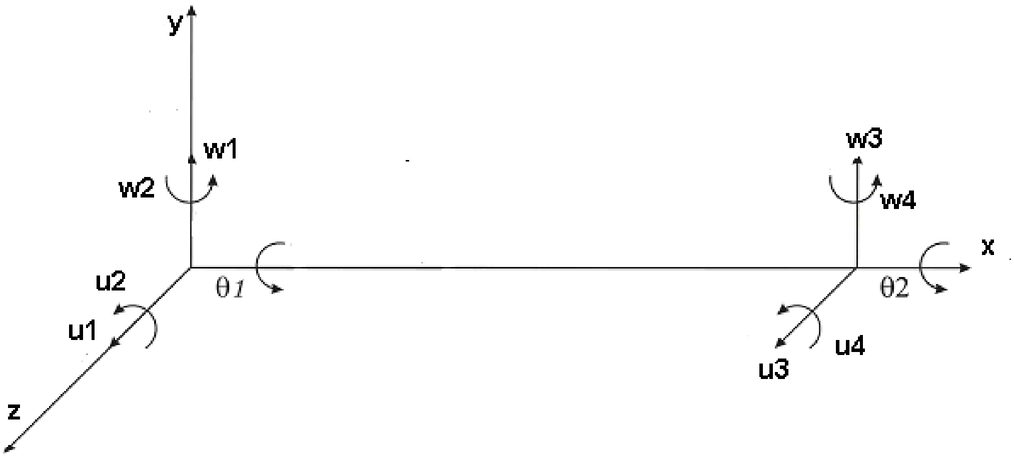


Figure 2 – Nodal displacements for a two noded beam finite element

We now consider a beam element according to the Euler-Bernoulli theory. Figure 3 shows a uniform beam element which may be subjected to transversal forces in both *xy* and *xz* planes, and torsional and flexural moments. In this case, translational and rotational nodal displacements due to bending are labeled as $w_1(t)$, $w_2(t)$, $w_3(t)$, $w_4(t)$. The transverse displacement within the element can be expressed as

$$w(x,t) = \sum_{i=1}^{4} N_i(x) w_i(t) \quad (3)$$

where *Ni(t)* are shape functions. Since

$$w(0,t) = w_1(t) \quad , \quad \frac{\partial w(0,t)}{\partial x} = w_2(t) \quad , \quad w(l,t) = w_3(t) \quad , \quad \frac{\partial w(l,t)}{\partial x} = w_4(t) \quad (4)$$

Ni(t) must satisfy the following boundary conditions:

$$N_1(0) = 1 \quad , \quad N_1(l) = 0 \quad , \quad \frac{dN_1}{dx}(0) = 0 \quad , \quad \frac{dN_1}{dx}(l) = 0 \quad (5)$$

$$N_2(0) = 0 \quad , \quad N_2(l) = 0 \quad , \quad \frac{dN_2}{dx}(0) = 1 \quad , \quad \frac{dN_2}{dx}(l) = 0 \quad (6)$$

$$N_3(0)=0 \quad , \quad N_1(l)=1 \quad , \quad \frac{dN_3}{dx}(0)=0 \quad , \quad \frac{dN_3}{dx}(l)=0 \tag{7}$$

$$N_4(0)=1 \quad , \quad N_4(l)=0 \quad , \quad \frac{dN_4}{dx}(0)=0 \quad , \quad \frac{dN_4}{dx}(l)=1 \tag{8}$$

where l – length of the element

Since four conditions are known for each *Ni(t)*, we assume a polynomial involving four constants (that is, a cubic equation) for *Ni(t)*:

$$N_i(x)=a_i+b_ix+c_ix^2+d_ix^3 \qquad i=1, 2, 3, 4 \tag{9}$$

Substituting the four sets of boundary conditions from Eqs.(5) to (8) into Eq.(9) one obtains the following shape functions:

$$N_1(x)=1-3\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)^3 \tag{10}$$

$$N_2(x)=x-2l\left(\frac{x}{l}\right)^2+l\left(\frac{x}{l}\right)^3 \tag{11}$$

$$N_3(x)=3\left(\frac{x}{l}\right)^2-2\left(\frac{x}{l}\right)^3 \tag{12}$$

$$N_4(x)=-l\left(\frac{x}{l}\right)^2+l\left(\frac{x}{l}\right)^3 \tag{13}$$

Thus the transverse displacement within the element becomes

$$w(x,t)=\left(1-3\frac{x^2}{l^2}+2\frac{x^3}{l^3}\right)w_1(t)+\left(\frac{x}{l}-2\frac{x^2}{l^2}+\frac{x^3}{l^3}\right)lw_2(t)+\left(3\frac{x^2}{l^2}-2\frac{x^3}{l^3}\right)w_3(t)+\left(-\frac{x^2}{l^2}+\frac{x^3}{l^3}\right)lw_4(t) \tag{14}$$

The kinetic energy can be expressed as (Meirovith, 1997):

$$T_{beam-xy}(t)=\frac{1}{2}\int_0^l \rho A\left\{\frac{\partial w(x,t)}{\partial t}\right\}^2 dx \tag{15}$$

where ρ -is density of material

A – is transfers area of section

Replacing Eq (14) into (15) and integrating one obtains

$$T_{beam-xy}(t)=\frac{1}{420}Al\rho(78w_1'^2(t)+2l^2w_2'^2(t)+78w_3'^2(t)-22lw_3'(t)w_4'(t)+2l^2w_4'^4(t)+$$
$$w_1'(t)(22lw_2'(t)+54w_3'(t)-13lw_4'(t))+lw_2'(t)(13w_3'(t)-3lw_4'(t))) \tag{16}$$

The strain energy .(Meirovitch 1977) is

$$V_{beam-xy}(t)=\frac{1}{2}\int_0^l EI\left\{\frac{\partial^2 w(x,t)}{\partial x^2}\right\}^2 dx \tag{17}$$

where E – is Young's modulus

I – is area moment of inertia

Replacing Eq (14) into (17) and integrating one may obtain

$$V_{beam-xy}(t)=\frac{1}{l^3}(2EI(3w_1{}^2(t)+l^2w_2{}^2(t)+3w_3{}^2(t)-3lw_3(t)w_4(t)+$$

(18)

$$l^2w_4{}^2(t)+lw_2(t)(-3w_3(t)+lw_4(t))+3w_1(t)(lw_2(t)-2w_3(t)+lw_4(t))))$$

$T_{beam-xz}$ and $V_{beam-xz}$ can be obtained analogously and are given by

$$T_{beam-xz}(t)=\frac{1}{420}Al\rho(78u_1'^2(t)+2l^2u_2'^2(t)+78u_3'^2(t)+22lu_3'(t)u_4'(t)+2l^2u_4'^2(t)-$$

(19)

$$lu_2'(t)(13u_3'(t)+3lu_4'(t))+u_1'(t)(-22lu_2'(t)+54u_3'(t)+13u_4'(t)))$$

$$V_{beam-xz}(t)=\frac{1}{l^3}(2EI(3u_1{}^2(t)+l^2u_2{}^2(t)+3u_3{}^2(t)+3lu_3(t)u_4(t)+$$

(20)

$$l^2u_4{}^2(t)-3u_1(t)(lu_2(t)+2u_3(t)+lu_4(t))+lu_2(t)(3u_3(t)+lu_4(t))))$$

Now, consider a uniform element with the x axis, taken along the centroidal axis. Let I_p denote the polar moment of inertia about the centroidal axis and GJ represent the torsion stiffness ($J=I_p$ for a circular cross section). The torsional displacement within the element can be expressed in terms of the rotations $\theta_1(t)$ and $\theta_2(t)$ as

$$\theta(x,t)=N_1(x)\theta_1(t)+N_2(x)\theta_2(t) \tag{21}$$

where $N_1(t)$ and $N_2(t)$ are the shape functions . Since $\theta(0,t)=\theta_1(t)$ and $\theta(l,t)=\theta_2(t)$, the shape functions must satisfy the boundary conditions

$$N_1(0)=1, N_1(l)=0 \tag{22}$$

$$N_2(0)=0, N_2(l)=1 \tag{23}$$

$N_i(t)$ is assumed to be a linear function as

$$N_i(x)=a_i+b_ix, i=1,2 \tag{24}$$

By using boundary conditions (22) and (23) the constant a_i and b_i in Eq.(24) can be determined to obtain

$$N_1(x)=1-\frac{x}{l},$$

(25)

$$N_2(x)=\frac{x}{l}$$

and hence, from Eq.(21),

$$\theta(x,t)=\left(1-\frac{x}{l}\right)\theta_1(t)+\frac{x}{l}\theta_2(t) \tag{26}$$

The kinetic energy and strain energy for torsion are given by

$$T_{tors}(t)=\frac{1}{2}\int_0^l \rho I_p\left\{\frac{\partial\theta(x,t)}{\partial t}\right\}^2dx \tag{27}$$

Substituting Eq (26) into (27) and integrating one may obtain

$$T_{tors}(t) = \frac{1}{6} I_p l \rho (\theta_1'^2(t) + \theta_1'(t)\theta_2'(t) + \theta_2'^2(t)) \tag{28}$$

$$V_{tors}(t) = \frac{1}{2} \int_0^l GJ \left\{ \frac{\partial \theta(x,t)}{\partial x} \right\}^2 dx \tag{29}$$

where G – is shear modulus

J – is polar moment of inercia

Substituting Eq (26) into (29) and integrating one may obtain

$$V_{tors}(t) = \frac{GJ(\theta_1(t) - \theta_2(t))^2}{2l} \tag{30}$$

Replacing Eqs (16),(19) and (28) into (1) and Eqs.(18),(20) and (29) into (2), equations of total kinetic energy and strain energy for element are obtained. Equations (1) and (2) can, finally, be expressed as a function of the nodal displacements θ_1, w_1, w_2, u_1, u_2, θ_2, w_3, w_4, u_3, u_4. To accomplish the integrations involved in the equations of kinetic and potential energies dependent on the variable x, software Mathematica® was used. In order to obtain the equations of motion for the finite element, Lagrange's equations were adopted (Meirovitch, 1986 and 1997):

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}i} \right) - \frac{\partial T}{\partial qi} + \frac{\partial V}{\partial qi} = Qi \tag{31}$$

To accomplish the differentiations involved in Lagrange's equations, the software Mathematica® was utilized. Taking the nodal displacements θ_1, w_1, w_2, u_1, u_2, θ_2, w_3, w_4, u_3, u_4 as generalized coordinates in Lagrange's equations, the equation of motion can be derived and expressed in the matrix form:

$$[m] \cdot \{\ddot{q}\} + [c] \cdot \{\dot{q}\} + [k] \cdot \{q\} = \{Q\} \tag{32}$$

where [m], [c], [k] - are mass, dumping, stiffness matrixes

$\{\ddot{q}\}$, $\{\dot{q}\}$, $\{q\}$, $\{Q\}$ - are acceleration, velocity, displacemant, force vectors

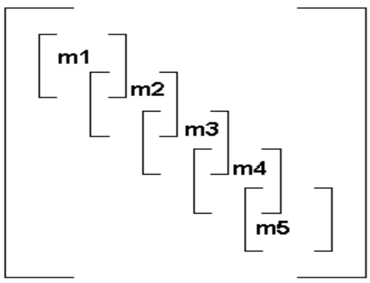
For the matrix equation of motion, the following matrixes of the first shaft elements were obtained:

$$m_1 = \begin{bmatrix}
\frac{J_1\rho l}{3} & 0 & 0 & 0 & 0 & \frac{J_1\rho l}{6} & 0 & 0 & 0 & 0 \\
0 & \frac{13 A_1 l\rho}{35} & \frac{11 A_1 l\rho}{210} & 0 & 0 & 0 & \frac{9 A_1 l\rho}{70} & -\frac{13 A_1 l^2\rho}{420} & 0 & 0 \\
0 & \frac{11 A_1 l^2\rho}{210} & \frac{A_1 l^3\rho}{105} & 0 & 0 & 0 & \frac{13 A_1 l^2\rho}{420} & -\frac{A_1 l^3\rho}{140} & 0 & 0 \\
0 & 0 & 0 & \frac{13 A_1 l\rho}{35} & -\frac{11 A_1 l^2\rho}{210} & 0 & 0 & 0 & \frac{9 A_1 l\rho}{70} & \frac{13 A_1 l^2\rho}{420} \\
0 & 0 & 0 & -\frac{11 A_1 l^2\rho}{210} & \frac{A_1 l^3\rho}{105} & 0 & 0 & 0 & -\frac{13 A_1 l^2\rho}{420} & -\frac{A_1 l^3\rho}{140} \\
\frac{J_1\rho l}{6} & 0 & 0 & 0 & 0 & \frac{J_1\rho l}{3} & 0 & 0 & 0 & 0 \\
0 & \frac{9 A_1 l\rho}{70} & \frac{13 A_1 l^2\rho}{420} & 0 & 0 & 0 & \frac{13 A_1 l\rho}{35} & -\frac{11 A_1 l^2\rho}{210} & 0 & 0 \\
0 & -\frac{13 A_1 l^2\rho}{420} & -\frac{A_1 l^3\rho}{140} & 0 & 0 & 0 & -\frac{11 A_1 l^2\rho}{210} & \frac{A_1 l^3\rho}{105} & 0 & 0 \\
0 & 0 & 0 & \frac{9 A_1 l\rho}{70} & -\frac{13 A_1 l^2\rho}{420} & 0 & 0 & 0 & \frac{13 A_1 l\rho}{35} & \frac{11 A_1 l^2\rho}{210} \\
0 & 0 & 0 & \frac{13 A_1 l^2\rho}{420} & -\frac{A_1 l^3\rho}{140} & 0 & 0 & 0 & \frac{11 A_1 l^2\rho}{70} & \frac{A_1 l^3\rho}{105}
\end{bmatrix} \tag{33}$$

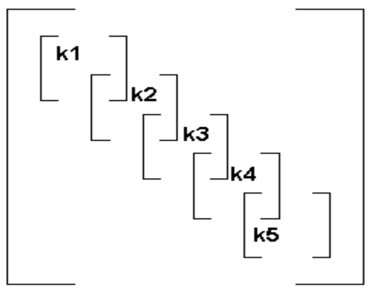
$$k_1 = \begin{bmatrix} \frac{GJ_1}{l} & 0 & 0 & 0 & 0 & -\frac{GJ_1}{l} & 0 & 0 & 0 & 0 \\ 0 & \frac{12EI_1}{l^3} & \frac{6EI_1}{l^2} & 0 & 0 & 0 & -\frac{12EI_1}{l^3} & \frac{6EI_1}{l^2} & 0 & 0 \\ 0 & \frac{6EI_1}{l^2} & \frac{4EI_1}{l} & 0 & 0 & 0 & -\frac{6EI_1}{l^2} & \frac{2EI_1}{l} & 0 & 0 \\ 0 & 0 & 0 & \frac{12EI_1}{l^3} & -\frac{6EI_1}{l^2} & 0 & 0 & 0 & -\frac{12EI_1}{l^3} & -\frac{6EI_1}{l^2} \\ 0 & 0 & 0 & -\frac{6EI_1}{l^2} & \frac{4EI_1}{l} & 0 & 0 & 0 & \frac{6EI_1}{l^2} & \frac{2EI_1}{l} \\ -\frac{GJ_1}{l} & 0 & 0 & 0 & 0 & \frac{GJ_1}{l} & 0 & 0 & 0 & 0 \\ 0 & -\frac{12EI_1}{l^3} & -\frac{6EI_1}{l^2} & 0 & 0 & 0 & \frac{12EI_1}{l^3} & -\frac{6EI}{l^2} & 0 & 0 \\ 0 & \frac{6EI_1}{l^2} & \frac{2EI_1}{l} & 0 & 0 & 0 & -\frac{6EI_1}{l^2} & \frac{4EI_1}{l} & 0 & 0 \\ 0 & 0 & 0 & -\frac{12EI_1}{l^3} & \frac{6EI_1}{l^2} & 0 & 0 & 0 & \frac{12EI_1}{l^3} & \frac{6EI_1}{l^2} \\ 0 & 0 & 0 & -\frac{6EI_1}{l^2} & \frac{2EI_1}{l} & 0 & 0 & 0 & \frac{6EI_1}{l^2} & \frac{4EI_1}{l} \end{bmatrix} \qquad (34)$$

The matrixes for the elements for the second axis are very much the same, and the only differences are due to geometric dimensions. The global mass and stiffness matrix of two parts of the model can be obtained assembling the element matrices into global matrixes (D. Norry, J de Freez,1987, Bismarck-Nasr, M. N, 1993), witch consist of the sum of the matrixes of the several elements, and have the following form:

[M₁]=



[K₁]=



(35)

The first part of model has two supporting points. The first support is located at the second node of the shaft and second support is located at the fourth node. Therefore, only rotations are kept free at these nodes. In order to apply these boundary conditions it is necessary to delete lines and columns of global mass and stiffness matrixes for the second and for the fourth nodes, where linear displacements coefficients are situated. Analogously, lines and columns corresponding to the second part of model are deleted. Those are the seventh, ninth, seventeenth and nineteenth columns and lines for first part and twelfth, fourteenth, twenty second and twenty fourth columns and lines for second part of model. At this stage, global mass and stiffness matrixes for the complete system can be assembled.

$$[K]_{Global}=[K_1]+[K_2] \qquad (36)$$

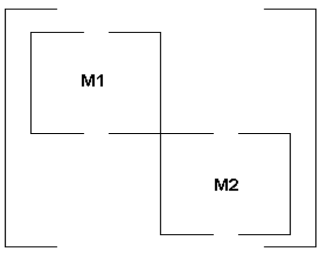
$$[M]_{Global}=[M_1]+[M_2] \qquad (37)$$

where $[K]_{Global}$, $[M]_{Global}$ - are stiffness and mass matrixes of the full system

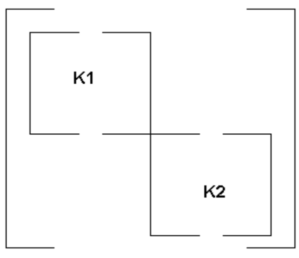
$[K_1]$, $[K_2]$, $[M_1]$, $[M_2]$- are stiffness and mass matrixes of fist and second part of model

Its have form:

$[M]_{Global}$=



$[K]_{Global}$=



(38)

Neglecting the system damping (34) can be written as:

$$[M]_{Global} \cdot \{\ddot{q}\} + [K]_{Global} \cdot \{q\} = \{Q\} \qquad (39)$$

For the sake of simplicity, considering that both gears have the same diameter, the angular velocity of second disk is taken as reverse of the angular velocity of fist disk. To obtain the time domain response, the integration procedure of Newmark was adopted. In order to apply this boundary condition, the system of equations has to be altered. Considering that the angular displacement of the second disc is the opposite of the angular displacement of the first disc, the corresponding line and column of this variable can be eliminated from the matrixes and the column contributions are displaced to the right hand side of the matrix equation obtained from the Newmark procedure.

3. Results

The free vibration solution was obtained in the time domain for the rotating model applying the finite element method. The present model conciders that the absolut values of angular velocities of the disks are equal. It allows the motion prediction of the second shaft.

The first and the second shaft are considered to have a constant cross-section. Both shafts are 80 mm long and have 15mm diameter.The disks have equal diametr d=50 mm and wide l=20 mm. The employed material is steel with Young's modulus E= 207 GPa, shear modulus G=79,6 GPa and density ρ=7800 kg/m^3. The shafts were modeled with five elements, considering that one of these elements are reserved for the disc. A tork T=1 $N \cdot m$ is applied at left end of the first shaft. The grafics of rotation and velocities are presented in Figs.(3,4,5,6).

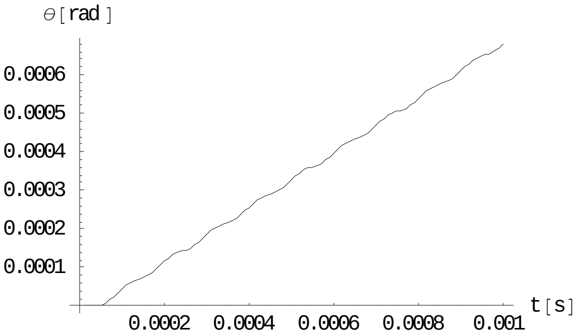


Figure 3. Rotation of first disk.

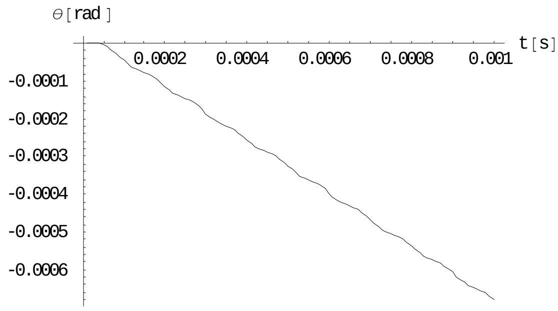


Figure 4.Rotation of second disk.

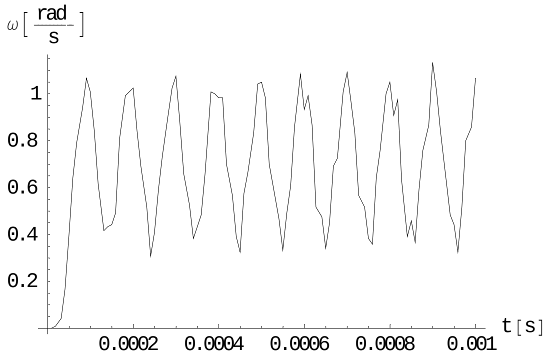


Figure 5. Velocity of first disk.

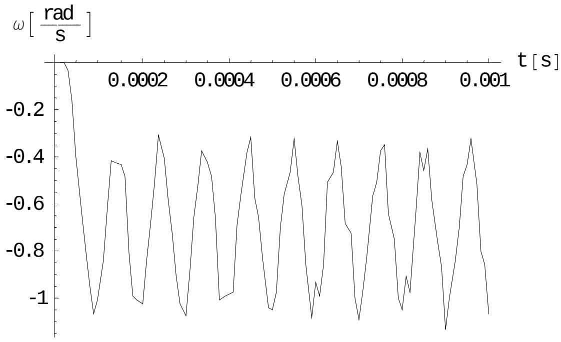


Figure 6. Velocity of second disk.

4. Conclusions

The present work is a preliminary study of a more complete problem that takes into account the influence of the gear teeth stiffness. Presently, two shafts, coupled by two rigid discs are considered. The dynamic response of the second shaft has been investigated. This study permits the determination of the dynamic response of any nodal variable of the model. The main difficulty of the work concerns the consideration of a model which takes into account the shaft bending. Althouth the bending stiffness does not play any role in the response of a pure torsional excitation, as in the problem analyzed here, it will have a important contribution in future works where transversal forces, wich may be caused by mislalinement of the shaft coupling, will be taken into account. A predictible response of the angular displacement of the second shaft was obtained, which validates the applied finete element model.

5. Acknowledgements

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