A Matrix Method for Torsional Vibration of Multi-step Non-uniform Rods

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***Abstract**. A matrix method usually applied to study the wave propagation in layered media is used to model the torsional vibration of non-uniform rods subjected to external excitation. The problem is represented through a system of differential equations written in matrix form. The state matrix varies in a piece-wise constant fashion in the axial direction of the rod. The solution is based on the Riccati transformation, also called invariant imbedding approach. The main advantage of the method is that non-uniform rods with an arbitrary number of elements can be easily analyzed through the proposed algorithm. Also, the stiffness and inertia of all the elements of the shaft are considered, providing an analytical solution for the case of rods with uniform sections carrying concentrated elements. Non-uniform rods can be analyzed by approximating the parts with variable cross section by a large number of small uniform sections.*

***Keywords**: torsional vibration, non-uniform rods, impedance*

1. Introduction

The study of torsional vibration of non-uniform rods with circular cross section is important to understand the dynamic response of many structural members and machine parts used in engineering practices. In fact, most of machine components as shafts with variable cross section, are frequently used to transmit power. Although various methods for determining natural frequencies and mode shapes of uniform and non-uniform rods with concentrated elements have already been presented (Li, 2001; Wu and Yang, 1995), the literature review indicates that the analytical solutions are limited to some kinds of non-uniform rods or are difficult to implement. Besides the applications for the study of torsional vibration of mechanical components with variable cross section, the authors were very encouraged for writing the present paper for realizing that the a theory usually applied to investigate the wave propagation in layered media could be used to study problems of torsional vibration of non-uniform rods, providing advantages over other methods presented in the literature.

The method proposed in this paper was already used by the first author for solving the problem of vibration of piezoelectric composite beams that are heterogeneous also in the span direction (Braga *et al.*, 2000). For solving this problem, the governing equations are written in matrix form and the state matrix, that vary in a piece-wise constant fashion in the span direction, is constructed according to the properties of each cross section of the beam. The solution is based on the concept of impedance that is formulated considering the partial waves that propagate in opposite directions of the beam. A similar approach is proposed for the solution of torsional vibration of non-uniform rods. In this case, the non-uniform rod with general geometry is treated as a rod formed by several parts or layers of uniform cross section in the span direction. The state matrix will therefore, vary in a piece-wise constant manner. The proposed solution is based on the propagation of harmonic torsional waves along the rod. It may be shown also that the method is based on a discrete form of the Riccati transformation (Dieci, 1988; Keller, 1982; Chin, 1984).

2. The model for uniform rods

A non-uniform rod having an arbitrary geometry is shown in Fig.1 for illustration of the problem. It is assumed that the motion can be regarded as rotation of the cross-sectional area as a whole and without warping. First, the proposed model will be introduced considering the case of a uniform rod.

The relationship between the angular displacement $\theta(x,t)$ and the twisting moment $M(x,t)$ is given by:

$$\frac{\partial\theta(x,t)}{\partial x} = \frac{M(x,t)}{GJ} \tag{1}$$

and the equation of motion for the free vibration of a shaft

$$\frac{\partial M(x,t)}{\partial x} = I(x)\frac{\partial^2\theta(x,t)}{\partial t^2} \tag{2}$$

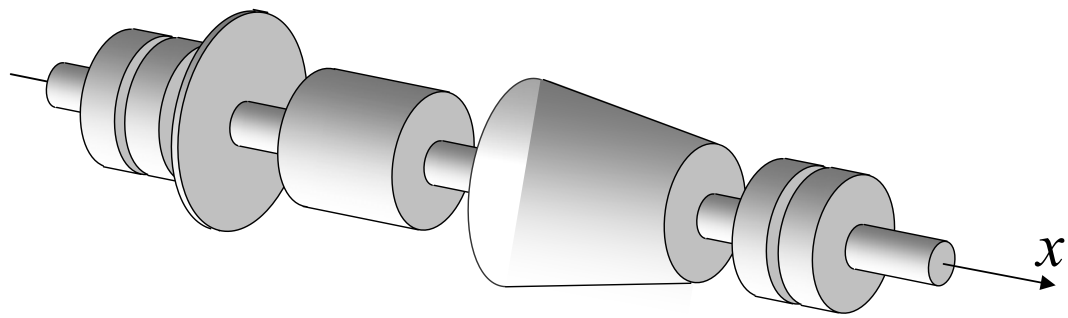

Figure 1. Non-uniform rod with arbitrary geometry.

where G is the shear modulus of elasticity, J is the area polar moment of inertia of the cross-section and I is the mass polar moment of inertia per unit length.

Assuming that the time dependence of excitation torque and the angular displacement are harmonic functions represented here by the generic function $g(x,t)$ written in the form

$$g(x,t) = g(x)e^{i\omega t} \tag{3}$$

the equation (2) is rewritten as

$$\frac{dM(x)}{dx} = -\omega^2 I(x)\theta(x) \tag{4}$$

Writing the governing Eqs. (1) and (2) in matrix form as

$$\frac{d\boldsymbol{\zeta}}{dx} = \boldsymbol{T\zeta} \tag{5}$$

where $\boldsymbol{T}$ is the state matrix:

$$\boldsymbol{T} = \begin{bmatrix} 0 & -\omega^2 I \\ \dfrac{1}{GJ} & 0 \end{bmatrix} \tag{6}$$

and $\boldsymbol{\zeta}$ is the state vector:

$$\boldsymbol{\zeta} = \begin{Bmatrix} \theta \\ M \end{Bmatrix} \tag{7}$$

the solution for Eq. (5) may be written in the form

$$\boldsymbol{\zeta}(x) = \boldsymbol{N}(x)\boldsymbol{\zeta}(0) \tag{8}$$

where $\boldsymbol{N}$ is the propagator matrix that relates the state vector in a position x to its value in a initial position $x = 0$, and has the form

$$\boldsymbol{N}(x) = e^{\boldsymbol{T}x} \tag{9}$$

Writing the matrix $\boldsymbol{T}$ in function of their eigenvalues and eigenvectors we can rewrite the equation (9) as

$$\boldsymbol{N}(x) = \boldsymbol{V}\left\{diag\left[e^{k_1 x}, e^{k_2 x}\right]\right\}\boldsymbol{V}^{-1} \tag{10}$$

Where $\boldsymbol{V}$ is the matrix whose columns are the eigenvectors of $\boldsymbol{T}$ and k_α, $\alpha = 1,2$, are the eigenvalues. The eigenvalues of $\boldsymbol{T}$ have opposite signs and are associated with waves that propagate in the positive or negative direction of the x-axis. The eigenvalues of $\boldsymbol{T}$ are the wave numbers of these partial waves. Separating the eigenvalues and eigenvectors according to waves that propagate in the positive and negative directions of the x-axis, the matrix $\boldsymbol{V}$ is decomposed as

$$\boldsymbol{V} = \begin{bmatrix} A_1 & A_2 \\ L_1 & L_2 \end{bmatrix} \tag{11}$$

where A_α and L_α, α= 1,2, are components of the eigenvectors of $\boldsymbol{T}$ and the subscripts 1 and 2 are associated respectively to the waves that propagates in the positive and negative direction of the x-axis.

The state vector in the initial position $x = 0$, may be expressed as a linear combination of the eigenvectors of matrix $\boldsymbol{T}$:

$$\boldsymbol{\zeta}(0) = \boldsymbol{Tc} = \begin{bmatrix} A_1 & A_2 \\ L_1 & L_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \tag{12}$$

where $\boldsymbol{c}$ is a constant vector. Using Eq. (8), the Eq. (12) may be written as

$$\boldsymbol{\zeta}(x) = \begin{bmatrix} A_1 & A_2 \\ L_1 & L_2 \end{bmatrix} \begin{bmatrix} W_1 & \\ & W_2 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} \tag{13}$$

where

$$\mathrm{W}_1 = \mathrm{W}_2^{-1} = \mathrm{e}^{\mathrm{k}_1 \mathrm{x}} \tag{14}$$

At this point, the field variables will be separated in two parts, corresponding to the contributions of the partial waves to the total fields. This is a common practice in the description of wave motion in layered media and is a key point for understanding the theory presented in this section. Using the subscripts 1 and 2 to represent respectively the waves that propagates in the positive and negative direction of the x-axis, the angular displacement $\theta(x)$ and the twisting moment $M(x)$ are now represented as a result of waves that propagates in the positive and negative direction of the x-axis:

$$\theta(x) = \theta_1(x) + \theta_2(x) \tag{15}$$

$$M(x) = M_1(x) + M_2(x) \tag{16}$$

According to Eq. (15) , Eq.(16) and the Eq. (13), the state vector is decomposed as

$$\theta_\alpha(x) = A_\alpha W_\alpha(x) c_\alpha \tag{17}$$

$$M_\alpha(x) = L_\alpha W_\alpha(x) c_\alpha \tag{18}$$

The constants c_α, $\alpha = 1,2$, may be given by

$$c_\alpha = A_\alpha^{-1} \theta_\alpha(0) \tag{19}$$

Using Eq. (19), the Eqs. (17) and (18) are rewritten as

$$\theta_\alpha(x) = W_\alpha(x) \theta_\alpha(0) \tag{20}$$

$$M_\alpha(x) = L_\alpha A_\alpha^{-1} \theta_\alpha(x) \tag{21}$$

Note that W_α(x) is the propagator function that relates the positive (α = 1) and negative (α =2) torsional waves with its value at $x = 0$. Assuming that the time dependence of all fields is harmonic, and using Eq. (21), the twisting moment may be related to the angular velocity $\dot{\theta}_\alpha(x)$ by

$$M_\alpha(x) = Z_\alpha \dot{\theta}_\alpha(x) \tag{22}$$

where

$$Z_\alpha = \left(-\frac{i}{\omega}\right) L_\alpha A_\alpha^{-1} \tag{23}$$

is the *local impedance* of the rod. Note that both W_α(x) and Z_α are functions of the geometry and material properties of each section of the rod and of the pair *(k,ω)*. The Eq. (16) is now rewritten as:

$$M(x) = Z_1 \dot{\theta}_1(x) + Z_2 \dot{\theta}_2(x) \tag{24}$$

Consider now the reflection of a wave from a surface at the beginning of the rod. Suppose that a wave propagating in the negative direction of de x-axis represented by $\theta_2(x)$ and impinging on the surface $x = 0$, is reflected as a wave propagating in the positive direction of the x-axis, represented by $\theta_1(x)$. At $x = 0$, one has:

$$\theta_1(0) = R_1 \theta_2(0) \tag{25}$$

were R_1 is the reflection coefficient at $x = 0$.

Using Eq. (25) and Eq. (20), the Eq. (15) is rewritten as

$$\theta(x) = \left[W_1(x) R_1 W_2(x)^{-1} + 1\right] \theta_2(x)$$

Using this result in Eq. (24), we arrive at an expression relating the twisting moment and the angular velocity:

$$M(x) = G(x)\dot{\theta}(x) \tag{26}$$

where

$$G(x) = \left[Z_1 H(x) + Z_2\right]\left[H(x) + 1\right]^{-1} \tag{27}$$

is the *global impedance* of the rod and

$$H(x) = W_1(x) R W_2(x)^{-1} \tag{28}$$

3. Extending the model to non-uniform rods

When the rod is non-uniform and is excited by external torque, it is necessary to determine the reflection coefficient of each interface and take into account the effect of the external torque. Figure 2 shows a non-uniform rod with N sections and external torques applied along the rod. In order to extend the theory presented in the preceding section for the case of non-uniform rods we now write the following expressions for the interface at $x = L_1$:

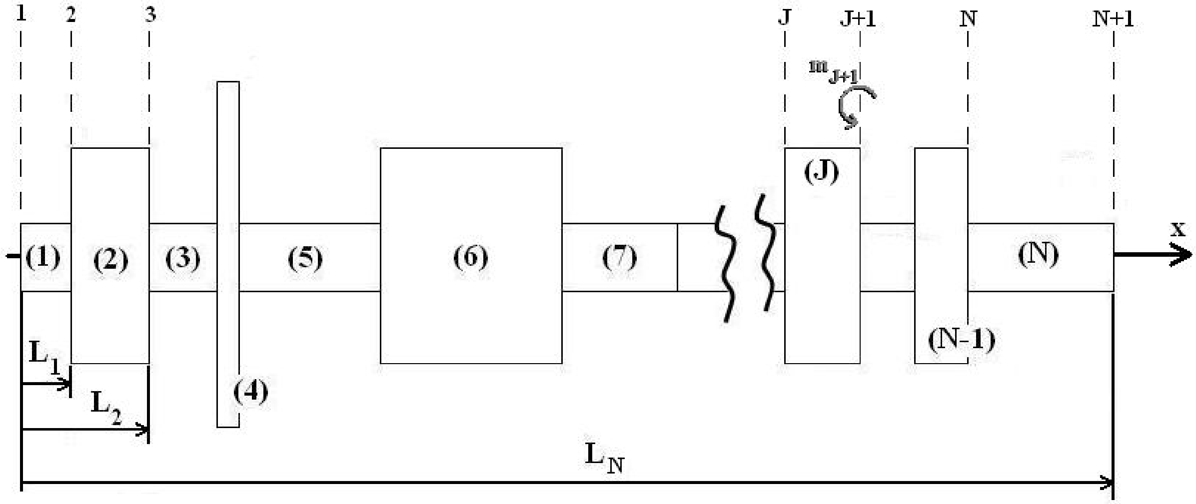

Figura 2. Non-uniform rod elements and interfaces.

for $x = L^-_1$

$$M\left(L_1^-\right) = i\omega G\left(L_1^-\right)\theta\left(L_1^-\right) - m_1 \tag{29}$$

$$\theta\left(L_1^-\right) = \theta_1\left(L_1^-\right) + \theta_2\left(L_1^-\right)$$

for $x = L^+_1$

$$M(L_1^+) = i\omega Z_1\theta_1(L_1^+) + i\omega Z_2\theta_2(L_1^+) \tag{30}$$

$$\theta_1\left(L_1^+\right) = R\theta_2\left(L_1^+\right) + Sm_1$$

and equations

$$\theta\left(L_1^+\right) = \left(1 + R\right)\theta_2\left(L_1^+\right) + Sm_1 \tag{31}$$

$$M\left(L_1^+\right) = i\omega\left(Z_1R + Z_2\right)\theta_2\left(L_1^+\right) + i\omega Z_1Sm_1 \tag{32}$$

where R is the coefficient of reflection and S is the coefficient that allows to compute the effect of the external torque in the other sections. These coefficients are determined by applying the following conditions at the interface:

$$M\left(L_1^+\right) = M\left(L_1^-\right) \tag{33}$$

$$\dot{\theta}\left(L_1^+\right) = \dot{\theta}\left(L_1^-\right) \tag{34}$$

Using Eqs. (29) to (34), we may write the following relation

$$G\left(1 + R\right)\dot{\theta}_2\left(L_1^+\right) + GSm_1 - m_1 = \left(Z_1R + Z_2\right)\dot{\theta}_2\left(L_1^+\right)Z_1Sm_1 \tag{35}$$

From Eq. (35), we arrive at the expressions for R and S:

$$R = \left(Z_1 - G\right)^{-1}\left(G - Z_2\right) \tag{36}$$

$$S = \left(G - Z_1\right)^{-1} \tag{37}$$

Finally for non-uniform rods with external torques, Eq. (26) is rewritten as

$$M(x) = i\omega G(x)\theta(x) + h(x)m_1 \tag{38}$$

where

$$h(x) = [Z_I - G(x)]W_I(x)S \tag{39}$$

4. The algorithm for evaluation of the frequency response of the non-uniform rod

In this section we present a recursive algorithm that is able to accurately describe the dynamic response of the non-uniform rod. Due to the lack of space we will focus on the basic features of this algorithm which relies on the exact solutions of Eq. (5) along each homogeneous piece of the rod. It may be shown that the proposed algorithm is based on a discrete form of the Riccati transformation, which is a key ingredient in the invariant imbedding approach (Dieci, 1988; Keller, 1982; Chin, 1984).To evaluate the torsional stiffness of the rod illustrated in Fig. 2, we start from the section at $x = 0$, where the reflection coefficient or the global impedance of the rod is known.

For example for a free surface:

$G_I = 0$ and $R_I = - Z_2 / Z_I$

For a clamped surface:

$G_I^{-1} = 0$ and $R_I = - 1$

Next we evaluate the impedance G_2 at the interface $x = L_I$ and the coefficient S_I. We then proceed to the next interface to evaluate G_3 and S_2 and so on until the end of the rod. This procedure is sumarized in the following algorithm:

Given G_I, m_I and ω
Repeat from j = 1 to $\boldsymbol{N}$
$R_j = \left(Z_I^{(j)} - G_j\right)\left(G_j - Z_2^{(j)}\right)$
$s_j = \left(Z_I^{(j)} - G_j\right)^{-1}$
$H_j = W_I^{(j)}(L_j)R_j\left(W_2^{(j)}(L_j)\right)^{-1}$
$h_j = \left(Z_I^{(j)} - G_j\right)W_I^{(j)}(L_j)s_j$
$G_{j+1} = \left(Z_I^{(j)}H_j + Z_2^{(j)}\right)\left(H_j + 1\right)^{-1}$
$m_{j+1} = h_j(L_{j+1})m_j$
End

At each interface between the uniform pieces of the rod, the global impedance relates the twisting moment with the angular velocity. As illustrated in Fig. 2, we label each uniform piece of the rod by an index $j = 1, \ldots , N$. The interface between pieces are also labeled by j, running from 1 to $N + 1$. For instance, if there is a concentrated torque at $x = 0$, we represent it by m_I.

For each frequency ω we evaluate the angular displacement of the rod at $x = L$ through the following relationship:

$$M_{N+1} = i\omega G_{N+1}\theta_{N+1} - m_{N+1}$$

In order to evaluate the angular displacement at the other interfaces, we have to march backwards:

Repeat from j = $\boldsymbol{N}$ to 1
$\theta_2^j = \left(1 + H_j\right)^{-1}\theta_{j+1} - \left(1 + H_j\right)^{-1}W_{Ij}s_j m_j$
$\theta_j = \left(I + R_j\right)W_2^{-1}\theta_2^j + s_j m_j$
End

The angular displacement $\theta(x)$ along the uniform pieces of the rod can be determined through the above algorithm by subdividing each uniform piece, creating as many interfaces as necessary, to completely describe the mode of vibration of the rod. The pieces of the rod with variable cross-section are also subdivided in small pieces of uniform cross-section.

5. Example Results

The results of the proposed method were compared to the results of the method presented by Li, 2001. The same rod used by L1, 2001, shown in Fig. 3, was used to verify the method proposed in this paper. The natural frequencies of a five step cantilever non-uniform rod with five rigid discs were determined and compared to the results obtained by Li, 2001.

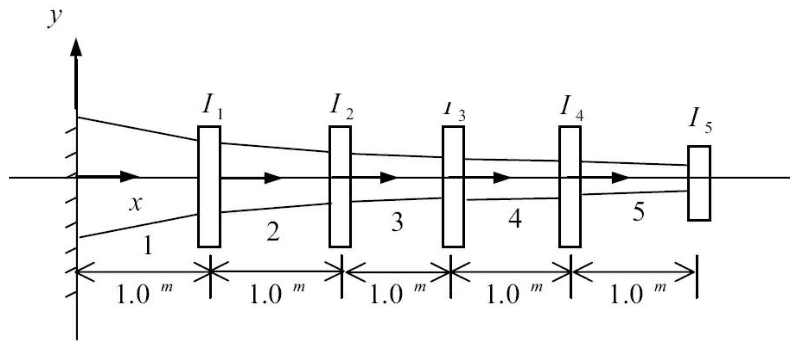

Figure 3. Non-uniform rod used by Li, 2001, to verify the proposed method.

The mechanical properties and dimensions of the rod are:

$G = 8{,}3 \times 10^{10}$ N/m^2;
$\rho = 7{,}8 \times 10^6$ kg/m^3;
$J_i(x) = \alpha\, e^{-\beta_i x/l_i}$
where,

$\alpha_1 = 1{,}60 \times 10^{-8}$ m^4	$\beta_1 = 0{,}1$;
$\alpha_2 = 1{,}38 \times 10^{-8}$ m^4	$\beta_2 = 0{,}05$;
$\alpha_3 = 1{,}28 \times 10^{-8}$ m^4	$\beta_3 = 0$;
$\alpha_4 = 1{,}20 \times 10^{-8}$ m^4	$\beta_4 = 0{,}05$;
$\alpha_5 = 1{,}0 \times 10^{-8}$ m^4	$\beta_5 = 0{,}08$;

$L_i = 1$ m (i = 1, 2, 3, 4);
$I_i = 0{,}32$ kgm^2 (i = 1, 2, 3, 4);
$I_5 = 0{,}16$ kgm^2

The comparison of results is shown in Table 1. Figure 4 shows the frequency response of the angular displacement at the right end of the rod obtained through the proposed method.

Table 1. Comparison of results.

Method	ω_1 (rad/sec)	ω_2 (rad/sec)	ω_3 (rad/sec)
Proposed method	21,44	57,61	90,11
Li, 2001	19,29	57,72	95,55

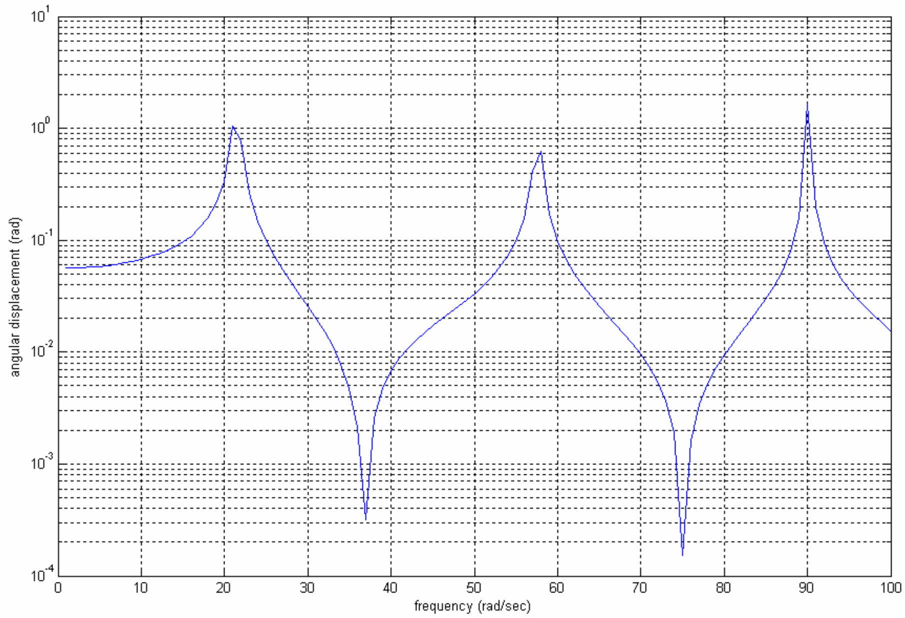

Figure 4. Frequency response of the rod shown in Fig. 3.

6. Conclusions

A theory usually applied to model the wave propagation in layered media was employed to model the torsional vibration of non-uniform rods. An algorithm based on the discrete form of the Riccati transformation was developed to determine the natural frequencies and modes of vibration of the rod. A computer program was implemented according to this algorithm for solving problems of torsional vibration of rods with circular cross-section of arbitrary geometry. The computational procedure is fast, simple and does not require a root-finding technique.

The authors are the only responsible for the printed material included in this paper.

7. References

Braga, A.M.B., de Barros, L.P.F. and Gama, A.L., 2000 “Models for the high frequency response of active piezoelectric composite beams”, Applied Mechanics and Engineering 5 47–61, 2000.

Chin, R.C.Y., Hedstorm, G.W. and Thigpen, L.,1984, “Matrix methods in synthetic seismograms”, Geophys. J.R. astr. Soc. 77, 483-502.

Dieci, L., Osborne, M.R., and Russel, R.D.,1988 “A Riccati transformation method for solving linear BVPs I: Theoretical aspects”, SIAM J. Numer. Anal., Vol. 25, No 5, october.

Dieci, L., Osborne, M.R., and Russel, R.D.,1988 “A Riccati transformation method for solving linear BVPs II: Computational aspects”, SIAM J. Numer. Anal., Vol. 25, No 5, october.

Keller, H.B. and Lentine, M., 1982, “Invariant imbedding, the box scheme and an equivalence between them”, SIAM J. Numer. Anal., Vol. 19, No 5, october.

Li, Q.S., 2001 “Torsional vibration of multi-step non-uniform rods with various concentrated elements”, Journal of Sound and Vibration 260 (2003) 637–651.

Wu J.S. and Yang, I.H., 1995, “Computer method for torsion-and-flexure-coupled forced vibration of shafting system with damping”, Journal of Sound and Vibration 1995, 180 (3), 417 – 435.