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THE IMPOSITION OF THE INJECTIVITY CONSTRAINT ON A FAMILY OF THREE-DIMENSIONAL ELASTICITY PROBLEMS

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Abstract. *There are problems in the classical linear theory of elasticity whose closed form solutions, while satisfying the governing equations of equilibrium together with well-posed boundary conditions, predict the existence of regions, often quite small, inside the body where material overlaps. Of course, material overlapping is not physically realistic, and one possible way to prevent it combines linear theory with the requirement that the deformation field be injective. A formulation of problems in elasticity proposed by Fosdick and Royer (2001) imposes this requirement through a Lagrange multiplier technique. The problems consist of determining the displacement field of a linear elastic body such that its potential energy is minimized subject to the constraint that the deformation field is locally invertible. An existence theorem for minimizers of plane problems is presented. In general, however, it is not certain that such minimizers exist. Here, it is shown that minimizers do exist for a family of three-dimensional problems. In classical linear elasticity, solutions to these problems yield stresses that are infinite at a point inside a body of anisotropic material for a range of material parameters. In addition, these solutions are not injective in a region surrounding this point, yielding unrealistic behavior such as overlapping of material. Applying the formulation of Fosdick and Royer on this family of problems, it is shown that the constitutive part of the stresses become finite everywhere and the injectivity constraint is preserved. These results are of interest not only in engineering design, but also in the development of a theory that, while still preserving the major underlying classical assumptions of linear elasticity theory, admits only those deformation fields that are locally invertible.*

Keywords: *Elasticity, Anisotropy, Overlapping, Singularity, Minimization Problem.*

1 Introduction

There are problems in the classical linear theory of elasticity whose closed form solutions, while satisfying the governing equations of equilibrium together with well-posed boundary conditions, are not locally injective, i.e., they are not a one-to-one mapping from the undeformed to the deformed configuration of the body. In such cases, there are regions, often quite small, where material overlaps. Typically, problems of this kind involve some sort of singularity, and strains exceeding level acceptable from the point of view of a linear theory occur around the singular points.

One such problem is considered by Ting (1999). This author investigates the equilibrium of a homogeneous sphere, which is radially compressed along its external contour by a uniformly distributed normal force. The requirement that the displacement field be rotationally symmetric with respect to the center of the sphere allows the derivation of a closed form solution that predicts overlapping of material in a certain region occupied by the linearly elastic sphere. Also, the stress field has a singularity at the center of the sphere.

One possible way to preserve injectivity consists of assuming a proper nonlinear elastic behavior of the material, such as the one considered by Aguiar and Fosdick (2001). Near the singular point, the *modified semi-linear material* proposed by these authors has an asymptotic behavior studied by Knowles and Sternberg (1975) that, while still singular, prevents material overlapping. Away from these points, this material behaves like the classical *semi-linear material* of John (1960).

An alternative approach, proposed by Fosdick and Royer (2001), combines the linear theory with the imposition of the injectivity constraint through a Lagrange multiplier technique. These authors consider the problem of determining the displacement field of a linearly elastic body such that its potential energy is minimized subject to the constraint that the deformation field is locally invertible. An existence theorem for minimizers of plane problems is presented. In general, however, it is not certain that such minimizers exist. Here, it is shown that minimizers do exist for a family of three-dimensional problems.

In particular, we study the rotationally symmetric sphere problem considered by Ting (1999) within the constrained minimization problem theory of Fosdick and Royer. Here, too, the requirement that the displacement field be rotationally symmetric with respect to the center of the sphere allows the derivation of a closed form solution. But, unlike the solution presented by Ting, our solution yields a deformation field that is both locally and globally injective, assuring therefore that material will not penetrate itself. Also, the constitutive part of the stresses are finite everywhere and the non-constitutive part, which is related to the Lagrange multiplier, is the only field with a logarithmic singularity at the center of the sphere.

In Section 2 we present preliminary developments that are useful throughout the paper. In particular, we present constitutive inequalities that are essential for the class of linearly elastic materials treated here to be physically sensible. We also present the closed form solution of the problem investigated by Ting (1999) and use this solution to illustrate what goes wrong in the classical linear theory when the injectivity constraint is not imposed. In Section 3 we present theoretical results obtained by Fosdick and Royer (2001), which fully characterize the solutions of minimimum potential energy problems in elasticity that must satisfy the injectivity constraint. In Section 4 we consider the solution of the rotationally symmetric sphere problem of Ting within the constrained minimization theory outlined in Section 3. We record some important results as propositions and refer the reader to a forthcoming paper (Aguiar, 2005) for the technical details regarding the proof of these propositions. In Section 5 we present some concluding remarks.

2 The Unconstrained Sphere Problem

We present a problem considered by Ting (1999) for which it is possible to derive a closed form solution that predicts overlapping of material in a certain region occupied by a linearly elastic body. Ting (1999) considers the equilibrium of a homogeneous sphere of radius ρ_e, which is radially compressed along its external contour by a uniformly distributed normal force per unit area p. The problem is three-dimensional so that, relative to the usual orthonormal spherical basis $(\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi)$, the stress and strain tensors are given by

$$\begin{aligned}\mathbf{T} &= \sigma_{\rho\,\rho}\,\mathbf{e}_\rho\otimes\mathbf{e}_\rho + \sigma_{\theta\,\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \sigma_{\phi\,\phi}\,\mathbf{e}_\phi\otimes\mathbf{e}_\phi + \sigma_{\rho\,\theta}\,(\mathbf{e}_\rho\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_\rho) + \\ &\quad \sigma_{\rho\,\phi}\,(\mathbf{e}_\rho\otimes\mathbf{e}_\phi + \mathbf{e}_\phi\otimes\mathbf{e}_\rho) + \sigma_{\theta\,\phi}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\phi + \mathbf{e}_\phi\otimes\mathbf{e}_\theta)\,, \end{aligned} \tag{1}$$

$$\begin{aligned}\mathbf{E} &= \epsilon_{\rho\,\rho}\,\mathbf{e}_\rho\otimes\mathbf{e}_\rho + \epsilon_{\theta\,\theta}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \epsilon_{\phi\,\phi}\,\mathbf{e}_\phi\otimes\mathbf{e}_\phi + \epsilon_{\rho\,\theta}\,(\mathbf{e}_\rho\otimes\mathbf{e}_\theta + \mathbf{e}_\theta\otimes\mathbf{e}_\rho) + \\ &\quad \epsilon_{\rho\,\phi}\,(\mathbf{e}_\rho\otimes\mathbf{e}_\phi + \mathbf{e}_\phi\otimes\mathbf{e}_\rho) + \epsilon_{\theta\,\phi}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\phi + \mathbf{e}_\phi\otimes\mathbf{e}_\theta)\,, \end{aligned} \tag{2}$$

respectively. These tensors are related to each other by the Generalized Hooke's Law, given by

$$\mathbf{T} = \mathbb{C}\,\mathbf{E}\,, \tag{3}$$

where $\mathbb{C}$ is the elasticity tensor, which is assumed to be symmetric and positive definite. Because of this, we can introduce the vectors

$$\begin{aligned}\mathbf{t}^T &= \{\sigma_{\rho\,\rho}, \sigma_{\theta\,\theta}, \sigma_{\phi\,\phi}, \sigma_{\theta\,\phi}, \sigma_{\phi\,\rho}, \sigma_{\rho\,\theta}\}\,, \\ \mathbf{e}^T &= \{\epsilon_{\rho\,\rho}, \epsilon_{\theta\,\theta}, \epsilon_{\phi\,\phi}, \epsilon_{\theta\,\phi}, \epsilon_{\phi\,\rho}, \epsilon_{\rho\,\theta}\}\,, \end{aligned} \tag{4}$$

and write the stress-strain law (3) in the alternative form

$$\mathbf{t} = \mathbf{C}\,\mathbf{e}\,, \tag{5}$$

where $\mathbf{C} = [c_{\alpha\,\beta}]\,, \alpha\,, \beta = 1, 2, \ldots, 6$, is a symmetric matrix whose components $c_{\alpha\,\beta}$ are the elastic stiffnesses of $\mathbb{C}$ in the spherical coordinate system. Following Ting (1999), we consider a *spherically uniform material*, for which the stiffnesses are constant and the elastic matrix $\mathbf{C}$ has the form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C_2} \end{bmatrix}\,, \tag{6}$$

where $\mathbf{C_1} = [c_{\alpha\,\beta}]\,, \alpha\,, \beta = 1, 2, 3$, with $c_{1\,3} = c_{1\,2}, c_{3\,3} = c_{2\,2}$, and $\mathbf{C_2} = [c_{\alpha\,\beta}]\,, \alpha\,, \beta = 4, 5, 6$. In view of (6), the stress-strain law (5) is uncoupled into one for normal stresses versus normal strains and one for shear stresses versus shear strains. Since the elasticity tensor $\mathbb{C}$ is symmetric and positive definite, the two 3×3 symmetric matrices, $\mathbf{C_1}$ and $\mathbf{C_2}$, must also be positive definite, which, in turn, requires that all principal submatrices of $\mathbf{C_1}$ and $\mathbf{C_2}$ must be positive definite[1]. In the case of matrix $\mathbf{C_1}$, we must have

$$c_{11} > 0\,, \qquad c_{22} > 0\,, \qquad -c_{22} < c_{23} < c_{22}\,, \qquad c_{11} > \frac{2\,c_{12}^2}{(c_{22} + c_{23})}\,. \tag{7}$$

Since uniqueness is guaranteed in classical linear elasticity, the displacement field must be rotationally symmetric with respect to the center of the sphere, i.e., $\mathbf{u}(\rho, \theta, \phi) = u(\rho)\,\mathbf{e}_\rho$. Thus, the strain components take the form[2]

$$\epsilon_{\rho\rho} = u'\,, \qquad \epsilon_{\theta\theta} = \epsilon_{\phi\phi} = \frac{u}{\rho}\,, \qquad \epsilon_{\rho\theta} = \epsilon_{\theta\phi} = \epsilon_{\phi\rho} = 0\,, \tag{8}$$

[1] See, for instance, Golub and van Loan (1996).
[2] See, for instance, Sokolnikoff (1956) for the general strain-displacement relations in spherical coordinates.

where $(\cdot)' \equiv d\,(\cdot)/d\,\rho$. It follows from (4-6) and (8) that

$$\begin{aligned} \sigma_{\rho\rho} &= c_{11}\,u' + 2\,c_{12}\,\frac{u}{\rho}\,, \\ \sigma_{\theta\theta} &= \sigma_{\phi\phi} = c_{12}\,u' + (c_{22}+c_{23})\,\frac{u}{\rho}\,, \qquad \sigma_{\rho\theta} = \sigma_{\theta\phi} = \sigma_{\phi\rho} = 0\,. \end{aligned} \tag{9}$$

Also, there is only one non-trivial equilibrium equation, given by

$$\frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{2}{\rho}\,(\sigma_{\rho\rho} - \sigma_{\theta\theta}) = 0\,, \tag{10}$$

which, because of (9), takes the form

$$u'' + 2\,\frac{u'}{\rho} - 2\,\gamma\,\frac{u}{\rho^2} = 0\,, \qquad \gamma \equiv \frac{c_{22}+c_{23}-c_{12}}{c_{11}}. \tag{11}$$

The general solution of (11) is

$$u(\rho) = \alpha^+\,\rho^{\lambda^+} + \alpha^-\,\rho^{\lambda^-}\,, \qquad \lambda^\pm \equiv \frac{1}{2}\,(-1 \pm 3\,\kappa)\,, \qquad \kappa \equiv \frac{1}{3}\sqrt{1+8\,\gamma}\,, \tag{12}$$

where $\alpha^\pm \in \mathbb{R}$.

Now, observe from the inequality (7.d) together with the definitions (11.b) and (12.b, c) that

$$(c_{11}\,\lambda^+ + 2\,c_{12})\,(c_{11}\,\lambda^- + 2\,c_{12}) = -2\,\left[c_{11}\,(c_{22}+c_{23}) - 2\,c_{12}^2\right] < 0\,. \tag{13}$$

It then follows from (12.b, c) that

$$c_{11}\,\lambda^+ + 2\,c_{12} > 0\,, \qquad -(c_{11}\,\lambda^- + 2\,c_{12}) > 0\,. \tag{14}$$

Also, the term on the left of the equal sign in (13) can be written as

$$-(\frac{c_{11}}{2})^2\,\{[3\,\kappa + (4\,\eta - 1)]\,\,[3\,\kappa - (4\,\eta - 1)]\}\,, \qquad \eta \equiv c_{12}/c_{11}\,,$$

which implies that κ is a positive, nonzero constant,

$$0 < \kappa < \infty\,. \tag{15}$$

If $0 < \kappa < 1/3$, we see from (12) that, unless $\alpha^+ = \alpha^- = 0$, the general solution of (11) is unbounded at $\rho = 0$. If $\kappa = 1/3$, then the imposition of the natural compatibility condition $u(0) = 0$ yields the trivial solution $u(\rho) = 0$ for $\rho \in (0, \rho_e)$.

If $\kappa > 1/3$, we consider (12) and impose both the natural compatibility condition $u(0) = 0$ and the boundary condition $\sigma_{\rho\rho}(\rho_e) = -p$, $p > 0$, to obtain

$$\alpha^+ = -q\,\rho_e^{1-\lambda^+}\,, \qquad \alpha^- = 0\,, \tag{16}$$

where

$$q \equiv \frac{p}{c_{11}\,\lambda^+ + 2\,c_{12}}\,. \tag{17}$$

It follows from (9), (12), and (16) that the nonzero stresses and displacements are given by, respectively,

$$\sigma_{\rho\rho} = -p\,\left(\frac{\rho}{\rho_e}\right)^{3\,(k-1)/2}\,, \quad \sigma_{\theta\theta} = \sigma_{\phi\phi} = -p\,\left(\frac{1+3\,\kappa}{4}\right)\,\left(\frac{\rho}{\rho_e}\right)^{3\,(k-1)/2}\,, \tag{18}$$

$$u = -q\,\rho_e\,\left(\frac{\rho}{\rho_e}\right)^{(3\,k-1)/2}\,. \tag{19}$$

Observe from (18) together with (15) that the nonzero stresses become singular at the center of the sphere for any $p \neq 0$ when $\kappa < 1$.

Since $p > 0$ and because of both the inequality (14) and the expression (17), it follows from (19) that all particles move towards the center of the sphere.

If $-u(\rho_e) = \rho_e$, the sphere is crushed to a single point at its center. We therefore assume that

$$\frac{-u(\rho_e)}{\rho_e} = q < 1, \tag{20}$$

which, because of (17), imposes a restriction on the values of the uniform pressure p that can be applied on the surface of the sphere in the classical linear theory of elasticity.

The restriction (20) applies to $\kappa \geq 1$ because, in this case, it follows from (19) and (20) that $-u(\rho)/\rho < q$. If $\kappa < 1$, then there is a core region, defined by

$$0 < \left(\frac{\rho}{\rho_e}\right)^{3(1-\kappa)/2} < q\,, \tag{21}$$

for which $u(\rho) < -\rho$. Since the deformation of the body is given by $\mathbf{f}(\mathbf{x}) = [\rho + u(\rho)]\,\mathbf{e}_\rho$ for each particle $\mathbf{x} = \rho\,\mathbf{e}_\rho$ of the sphere, we readily see that material penetrates itself in this central core. To find the radius ρ_c of this core, we set $u(\rho_c) = -\rho_c$ in (19) to obtain

$$\left(\frac{\rho_c}{\rho_e}\right)^{3\,(1-\kappa)/2} = q\,. \tag{22}$$

A solution for $\rho_c < \rho_e$ exists in view of the inequalities $\kappa < 1$ and (20).

Another interesting feature of the solution (19), which was not noted by Ting (1999), is that this solution yields a deformation field that is not locally injective when $\kappa < 1$. For instance, if

$$\left(\frac{\rho}{\rho_e}\right)^{3\,(1-\kappa)/2} < \lambda^+\,q\,, \tag{23}$$

then the determinant of the deformation gradient, which is given by

$$\det \nabla\mathbf{f} = \left[1 - \lambda^+\,q\left(\frac{\rho}{\rho_e}\right)^{3\,(\kappa-1)/2}\right]\left[1 - q\left(\frac{\rho}{\rho_e}\right)^{3\,(\kappa-1)/2}\right]^2\,, \tag{24}$$

is negative.

Thus, for $\kappa < 1$, the classical linear solution has no physical meaning and therefore should be rejected as a viable solution. The anomalous behavior of material overlapping provides, however, motivation to develop a pseudo-linear theory which respects the constraint that admissible deformations be at least locally invertible, i.e., that $\det \nabla\mathbf{f} > 0$.

3 The Constrained Minimization Problem

Let $\mathcal{B} \subset \mathbb{R}^3$ be the undistorted natural reference configuration of a body. Points $\mathbf{x} \in \mathcal{B}$ are mapped to points $\hat{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \equiv \mathbf{x} + \mathbf{u}(\mathbf{x}) \in \mathbb{R}^3$, where $\mathbf{u}(\mathbf{x})$ is the displacement of $\mathbf{x}$. The boundary $\partial\mathcal{B}$ of $\mathcal{B}$ is composed of two non-intersecting parts, $\partial_1\mathcal{B}$ and $\partial_2\mathcal{B}$, $\partial_1\mathcal{B} \cup \partial_2\mathcal{B} = \partial\mathcal{B}$, $\partial_1\mathcal{B} \cap \partial_2\mathcal{B} = \emptyset$, such that $\mathbf{u}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial_1\mathcal{B}$ and such that a dead load traction field $\bar{\mathbf{t}}(\mathbf{x})$ is prescribed for $\mathbf{x} \in \partial_2\mathcal{B}$. In addition, a body force $\mathbf{b}(\mathbf{x})$ per unit volume of $\mathcal{B}$ acts on points $\mathbf{x} \in \mathcal{B}$.

We consider the problem of minimum potential energy

$$\min_{\mathbf{v}\in\mathcal{A}_\varepsilon} \mathcal{E}[\mathbf{v}]\,, \tag{25}$$

where

$$\mathcal{E}[\mathbf{v}] \equiv \frac{1}{2}\,a[\mathbf{v},\mathbf{v}] - f[\mathbf{v}]\,. \tag{26}$$

In (26),

$$a[\mathbf{v},\mathbf{v}] \equiv \int_{\mathcal{B}} \mathbb{C}[\nabla_{Sym}\mathbf{v}]\cdot[\nabla_{Sym}\mathbf{v}]\,d\mathbf{x}\,, \qquad f[\mathbf{v}] \equiv \int_{\mathcal{B}} \mathbf{b}\cdot\mathbf{v}\,d\mathbf{x} + \int_{\partial_2\mathcal{B}} \bar{\mathbf{t}}\cdot\mathbf{v}\,\mathbf{dx}\,, \tag{27}$$

where $\mathbf{E} = \nabla_{Sym}\mathbf{v} \equiv \left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right]/2$ is the infinitesimal strain tensor field. The functional $\mathcal{E}[\cdot]$ is the total potential energy of classical linear theory of elasticity . Furthermore,

$$\mathcal{A}_\varepsilon \equiv \{\mathbf{v} : W^{1,2}(\mathcal{B}) \to \mathbb{R}^3 \big|\, \det(\mathbf{1} + \nabla\mathbf{v}) \geq \varepsilon > 0, \mathbf{v} = \mathbf{0} \text{ a.e. on } \partial_1\mathcal{B}\} \tag{28}$$

is the class of admissible displacement fields, and we recall from Section 2 that $\mathbb{C} = \mathbb{C}(\mathbf{x})$ is the elasticity tensor, which is positive definite and symmetric. We suppose that $\varepsilon > 0$ in (28) is sufficiently small.

If the local injectivity constraint $\det(\mathbf{1} + \nabla\mathbf{v}) \geq \varepsilon > 0$ was not present in the definition of $\mathcal{A}_\varepsilon$, the resulting problem would fall in the general class of problems considered in the classical linear theory of elasticity, which are well understood. In this case, however, there is no guarantee that the solution will be locally injective, i.e., that $\det(\mathbf{1} + \nabla\mathbf{v}) > 0$. For instance, recall from Section 2 that the solution of the rotationally symmetric sphere problem yields a deformation field that is not locally injective in a central core of the sphere.

Fosdick and Royer (2001) fully characterize the solutions of the minimization problem (25). In particular, they show that there exists a solution to this problem which does not violate the injectivity constraint $\det(\mathbf{1}+\nabla\mathbf{v}) \geq \varepsilon > 0$ and derive first variation conditions for a minimizer $\mathbf{u} \in \mathcal{A}_\varepsilon$ of $\mathcal{E}[\cdot]$. We record here some steps in this derivation for later use in Section 4.

For this, we define

$$\mathcal{V} \equiv \{\mathbf{v} : W^{1,2}(\mathcal{B}) \to \mathbb{R}^3 : \mathbf{v}\big|_{\partial_1\mathcal{B}} = \mathbf{0}\}\,. \tag{29}$$

We then obtain the first variation of $\mathcal{E}[\cdot]$ at $\mathbf{u}$ in the form

$$< D\mathcal{E}[\mathbf{u}], \mathbf{v} > \equiv a[\mathbf{u}, \mathbf{v}] - f[\mathbf{v}]\,, \qquad \forall\, \mathbf{v} \in \mathcal{V}\,, \tag{30}$$

where $a[\cdot,\cdot]$ and $f[\cdot]$ are defined in (27).

On the other hand, it can be shown that there exists a scalar Lagrange multiplier field $\lambda : \mathcal{L}^2(\mathcal{B}) \to \mathbb{R}$ such that the first variation (30) has the equivalent representation

$$< D\mathcal{E}[\mathbf{u}], \mathbf{v} > = \int_{\mathcal{B}} \lambda(\mathrm{cof}\,\nabla\mathbf{f}) \cdot \mathbf{v}\, d\mathbf{x}\,, \qquad \forall\, \mathbf{v} \in \mathcal{V}\,, \tag{31}$$

where $\mathrm{cof}\,\nabla\mathbf{f}$ is the cofactor of the deformation gradient and we recall from above that $\mathbf{f}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$.

Defining

$$\mathcal{B}_> \equiv \text{int}[\{\mathbf{x} \in \mathcal{B} : \det\nabla\mathbf{f} > \varepsilon\}]\,, \qquad \mathcal{B}_= \equiv \text{int}[\{\mathbf{x} \in \mathcal{B} : \det\nabla\mathbf{f} = \varepsilon\}]\,, \tag{32}$$

where int$[\cdot]$ denotes the interior of a set, the necessary first variation conditions for the existence of a minimizer are given by

- The Euler-Lagrange equations

$$\mathrm{Div}\,\mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{in}\, \mathcal{B}_>\,, \quad \mathrm{Div}(\mathbf{T} - \varepsilon\lambda(\nabla\mathbf{f})^{-T}) + \mathbf{b} = \mathbf{0}\,, \quad \lambda \geq 0\,, \quad \text{in}\, \mathcal{B}_=\,, \tag{33}$$

  together with the boundary conditions

$$\mathbf{T}\,\mathbf{n} = \bar{\mathbf{t}} \quad \text{on}\, \partial\mathcal{B}_> \cap \partial_{\mathbf{2}}\mathcal{B}\,, \quad (\mathbf{T} - \varepsilon\lambda(\nabla\mathbf{f})^{-\mathbf{T}})\,\mathbf{n} = \bar{\mathbf{t}} \quad \text{on}\, \partial\mathcal{B}_= \cap \partial_{\mathbf{2}}\mathcal{B}\,, \tag{34}$$

  where $\mathbf{n}$ is a unit normal to $\partial_2\mathcal{B}$.

- Jump conditions across $\Sigma \equiv \bar{\mathcal{B}}_> \cap \bar{\mathcal{B}}_=$, which is assumed to be sufficiently smooth:

$$(\mathbf{T} - \varepsilon\lambda(\nabla\mathbf{f})^{-T})\big|_{\Sigma\cap\bar{\mathcal{B}}_=}\,\mathbf{n} = \mathbf{T}\big|_{\Sigma\cap\bar{\mathcal{B}}_>}\,\mathbf{n}\,, \tag{35}$$

  where $\mathbf{n}$ is a unit normal to Σ and where $\Sigma \cap \bar{\mathcal{B}}_=$ and $\Sigma \cap \bar{\mathcal{B}}_>$ mean that the evaluations are understood as limits to the dividing interface Σ from within $\mathcal{B}_=$ and $\mathcal{B}_>$, respectively.

In the next section we use the variation conditions (33-35) in the derivation of a solution for the rotationally symmetric sphere problem presented in Section 2 that belongs to the set of admissible displacement fields $\mathcal{A}_\varepsilon$, i.e., a solution for which $\det\nabla\mathbf{f} > 0$.

4 The Constrained Sphere Problem

We consider the solution of the rotationally symmetric sphere problem of Ting discussed in Section 2 for the material parameter $\kappa \in (0,1)$, defined by (12.c) and (11.b), within the constrained minimization theory outlined in Section 3.

Here, we set $\mathbf{b} = 0$ in (33) and assume that the displacement field is rotationally symmetric with respect to the center of the sphere, i.e., $\mathbf{u}(\rho,\theta,\phi) = u(\rho)\,\mathbf{e}_\rho$. Thus, the strain and stress components take the form (8) and (9), respectively. Also, the deformation gradient, defined by $\nabla\,\mathbf{f} \equiv \mathbf{1} + \nabla\,\mathbf{u}$, and its inverse are given by, respectively,

$$\begin{aligned} \nabla\,\mathbf{f} &= (1+u')\,\mathbf{e}_\rho \otimes \mathbf{e}_\rho + (1+u/\rho)\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (1+u/\rho)\,\mathbf{e}_\phi \otimes \mathbf{e}_\phi\,, \\ (\nabla\,\mathbf{f})^{-1} &= (1+u')^{-1}\,\mathbf{e}_\rho \otimes \mathbf{e}_\rho + (1+u/\rho)^{-1}\,\mathbf{e}_\theta \otimes \mathbf{e}_\theta + (1+u/\rho)^{-1}\,\mathbf{e}_\phi \otimes \mathbf{e}_\phi\,. \end{aligned} \tag{36}$$

We have denoted the region occupied by the sphere in its natural state by $\mathcal{B} \subset \mathbb{R}^3$, and we take the subregions in which the constraint of local injectivity is active ($\det \nabla \mathbf{f} = \varepsilon$) and non-active ($\det \nabla \mathbf{f} > \varepsilon$) to be denoted, respectively, by

$$\mathcal{B}_= = \{\mathbf{x} = \rho\, \mathbf{e}_\rho \in \mathcal{B} : 0 < \rho < \rho_a\}\,, \qquad \mathcal{B}_> = \{\mathbf{x} = \rho\, \mathbf{e}_\rho \in \mathcal{B} : \rho_a < \rho < \rho_e\}\ , \tag{37}$$

for some $\rho_a \in [0, \rho_e]$, yet to be determined.

In $\mathcal{B}_=$, it follows from (36) that $\det \nabla \mathbf{f} = (1 + u')\,(1 + u/\rho)^2 = \varepsilon$, which yields

$$\varepsilon = \frac{1}{3\,\rho^2}\,\frac{d(\rho + u)^3}{d\,\rho} \qquad \text{in } \mathcal{B}_= \,. \tag{38}$$

The solution of this equation, subjected to the compatibility condition $u(0) = 0$, is given by

$$u(\rho) = -(1 - \varepsilon^{1/3})\,\rho \qquad \text{in } \mathcal{B}_= \,. \tag{39}$$

In view of both (9) and (36), the only terms not identically zero in the Euler-Lagrange equations (33) are

$$\begin{aligned} 0 &= \frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{2}{\rho}\,(\sigma_{\rho\rho} - \sigma_{\theta\theta}) \qquad \text{in } \mathcal{B}_> \,, \\ 0 &= \frac{\partial\left[\sigma_{\rho\rho} - \varepsilon\,\lambda\,(1 + u')^{-1}\right]}{\partial \rho} + \\ &\quad \frac{2}{\rho}\left\{\sigma_{\rho\rho} - \sigma_{\theta\theta} - \varepsilon\,\lambda\,\left[(1 + u')^{-1} - (1 + u/\rho)^{-1}\right]\right\} \qquad \text{in } \mathcal{B}_= \,. \end{aligned} \tag{40}$$

Substituting (9) in (40.a), we obtain

$$u'' + 2\,\frac{u'}{\rho} - 2\,\gamma\,\frac{u}{\rho^2} = 0 \qquad \text{in } \mathcal{B}_> \,, \tag{41}$$

where γ is given by (11.b). The general solution of (41) is of the form (12), where $\alpha^\pm \in \mathbb{R}$ are determined below from continuity conditions on $\rho = \rho_a$.

Of course, $u(\rho)$ must be continuous at $\rho = \rho_a$ and the boundary and appropriate jump conditions, as recorded generally in, respectively, (34) and (35), must be appended. These conditions are expressed as

$$\begin{aligned} u(\rho_a^-) &= u(\rho_a^+)\,, \\ \sigma_{\rho\rho}(\rho_e) &= -p\,, \\ \sigma_{\rho\rho}(\rho_a^-) - \varepsilon\,\lambda(a)\ \left[1 + u'(\rho_a^-)\right]^{-1} &= \sigma_{\rho\rho}(\rho_a^+)\,, \end{aligned} \tag{42}$$

where $\sigma_{\rho\rho}$ is given by (9.a).

Substituting both (9) and (39) in (40.b), we obtain the first order differential equation

$$0 = \frac{\alpha}{\rho} - \lambda'(\rho)\,, \qquad \alpha \equiv 2\,c_{11}\varepsilon^{-2/3}(1 - \varepsilon^{1/3})\,(-1 + \gamma)\,, \tag{43}$$

which must hold in the region $\mathcal{B}_=$.

Proposition 1: In $\mathcal{B}_=$, let

$$\lambda(\rho) = \alpha\,\log(\rho/\hat{\rho}) \tag{44}$$

be the solution of (43) for some $\hat{\rho} \in \mathbb{R}$ to be determined consistent with $\lambda(\rho) \geq 0$. Then[3],

$$\hat{\rho} = \rho_a \,. \tag{45}$$

Note from (44) that λ has a logarithmic singularity at the origin, which is a weaker singularity than the stress singularity of the unconstrained problem discussed in Section 2.

[3] We refer the reader to Aguiar (2005) for the proof of all propositions in this paper.

Next, we use the continuity of both traction and displacement on $\rho = \rho_a$, expressed by, respectively, (42.a, c), to find $\alpha^{\pm}$. Imposing these continuity conditions, we arrive at the vector equation

$$-\left\{ \begin{array}{c} 1+2\,\eta \\ 1 \end{array} \right\} (1-\varepsilon^{1/3})\,\rho_a = \begin{bmatrix} \lambda^+ + 2\,\eta & \lambda^- + 2\,\eta \\ 1 & 1 \end{bmatrix} \left\{ \begin{array}{c} \alpha^+ \rho_a^{\lambda^+} \\ \alpha^- \rho_a^{\lambda^-} \end{array} \right\} , \tag{46}$$

where $\eta \equiv c_{12}/c_{11}$. The solution of this equation is given by

$$\alpha^{\pm} = \frac{(\mp 1-\kappa)}{2\,\kappa}\,(1-\varepsilon^{1/3})\,\rho_a^{3\,(1\mp\kappa)/2} \,. \tag{47}$$

To arrive at (47), we used the definitions (12.b).

We still need to find ρ_a in (47). For this, we impose the traction condition (42.b). Substituting (9.a), (12.a), and (47) in (42.b), we obtain the algebraic equation

$$\begin{aligned} 0 = g(\zeta) \quad \equiv \quad & \left(\lambda^+ + 2\,\eta\right) \left(\frac{1+\kappa}{2\,\kappa}\right) \zeta^{\,3(\kappa-1)/2} - \\ & \left(\lambda^- + 2\,\eta\right) \left(\frac{1-\kappa}{2\,\kappa}\right) \zeta^{\,-3(\kappa+1)/2} - \frac{p}{c_{11}\,(1-\varepsilon^{1/3})} \,, \end{aligned} \tag{48}$$

where $\zeta \equiv \rho_e/\rho_a$ and $\eta \equiv c_{12}/c_{11}$.

Proposition 2: There exists a unique $\rho_a \in [0, \rho_e]$ that satisfies the algebraic equation $g(\rho_e/\rho_a) = 0$, where g is given by (48), provided that $p \le p_0 \equiv c_{11}\,(1+2\,\eta)\,(1-\varepsilon^{1/3})$. If $p > p_0$, then $\mathcal{B}_> = \emptyset$ and $\mathcal{B}_= = \mathcal{B}$.

We now need to verify that $\det \nabla \mathbf{f} \ge \varepsilon$ in $\mathcal{B}$. In $\mathcal{B}_=$, this is certainly the case.

Proposition 3: In $B_>$, the injectivity constraint $\det \nabla \mathbf{f} \ge \varepsilon$ holds for $0 < \kappa < 1$.

Proposition 4: At $\rho = \rho_a$, $\det \nabla \mathbf{f}$ is continuous and has continuous first derivative with respect to ρ.

Using both (39) and (12) with (47) , we find the final expression for the displacement field $\mathbf{u} = u(\rho)\,\mathbf{e}_\rho$, which is given by

$$u(\rho) = \begin{cases} -\left(1-\varepsilon^{1/3}\right)\rho & \text{in} \quad \mathcal{B}_= \,, \\ \frac{(1-\varepsilon^{1/3})\,\rho_a}{2\,\kappa} \left[-(1+\kappa)\,(\frac{\rho}{\rho_a})^{-(1-3\,\kappa)/2} + (1-\kappa)\,(\frac{\rho}{\rho_a})^{-(1+3\,\kappa)/2}\right] & \text{in} \quad \mathcal{B}_> \,, \end{cases} \tag{49}$$

where, according to Proposition 2, ρ_e/ρ_a is the unique solution of the algebraic equation (48).

Using (49), we can easily obtain the expression

$$\det \nabla \mathbf{f}(\rho) = \varepsilon \qquad \text{in}\; \mathcal{B}_= \,, \qquad \det \nabla \mathbf{f}(\rho) = \left[1+\chi_1\left(\frac{\rho}{\rho_a}\right)\right] \left[1-\chi_2\left(\frac{\rho}{\rho_a}\right)\right]^2 \qquad \text{in}\; \mathcal{B}_> \,, \tag{50}$$

where $\mathbf{f}(\rho) = [\rho + u(\rho)]\,\mathbf{e}_\rho$ and

$$\begin{aligned} \chi_1\,(\xi) \quad & \equiv \quad \frac{(1-\varepsilon^{1/3})}{4\,\kappa\,\xi^{\,3/2}} \left[(1+\kappa)\,(1-3\,\kappa)\,\xi^{\,3\,\kappa/2} - (1-\kappa)\,(1+3\,\kappa)\,\xi^{\,-3\,\kappa/2}\right] , \\ \chi_2\,(\xi) \quad & \equiv \quad \frac{(1-\varepsilon^{1/3})}{2\,\kappa\,\xi^{\,3/2}} \left[(1+\kappa)\,\xi^{\,3\,\kappa/2} - (1-\kappa)\,\xi^{\,-3\,\kappa/2}\right] , \qquad \xi \equiv \frac{\rho}{\rho_a} \,. \end{aligned} \tag{51}$$

The solution (49) yields the deformation of the sphere, which is, in fact, globally injective. To see this, recall from Section 2 that a globally injective deformation satisfies the inequality $-u(\rho)/\rho < 1$ for all $\rho \in (0, \rho_e)$. This is certainly the case for $\rho \in (0, \rho_a)$. To show that this is also the case for $\rho \in (\rho_a, \rho_e)$, observe from both (49) and (51.b) that $-u(\rho)/\rho = \chi_2(\rho/\rho_a)$. Since $\chi_2(1) = 1 - \varepsilon^{1/3} < 1$, $\chi_2'(\xi) < 0$ for all $\xi \in [1, \infty)$, $\chi_2'(1) = 0$, and $\lim_{\xi\to\infty} \chi_2(\xi) = \lim_{\xi\to\infty} \chi_2'(\xi) = 0$, we find that $\chi_2(\xi)$ is a monotonically decreasing function of ξ that attains its maximum value, which is smaller than one, at $\xi = 1$ and tends asymptotically to zero from above as $\xi \to \infty$. Therefore $-u(\rho)/\rho < 1$ for $\rho \in (\rho_a, \rho_e)$.

One interesting feature of this solution, which follows from Proposition 5 below, is that the sphere is stiffer in its response when compared to the classical solution of Ting. Recall from Section 2 that this unconstrained solution allows interpenetration of material to take place.

Proposition 5: If $1/3 < \kappa < 1$, then $u(\rho)/u_T(\rho) < 1$ for all $\rho \in (\rho_a, \rho_e)$, where $u(\rho)$ is given by (49) and $u_T(\rho)$ is given by (19) together with (17).

5 Conclusion

We investigated a family of three-dimensional problems in a class of constrained minimization problems considered by Fosdick and Royer (2001). A constrained problem in this class consists of finding a minimizer $\mathbf{u}$ for the total potential energy $\mathcal{E}$ of classical linear theory of elasticity over a set $\mathcal{A}_\varepsilon$ of admissible displacement fields that satisfy the local injectivity constraint $\det(\mathbf{1} + \nabla\mathbf{u}) - \varepsilon \geq 0$ for a sufficiently small $\varepsilon \in \mathbb{R}$.

This family of problems is characterized by the requirement that the displacement field be rotationally symmetric with respect to a point inside the body. For the case of a sphere, we derived a closed form solution, which yields a deformation field that is both locally and globally injective, preventing therefore that material overlapping occurs. The constitutive part of the stresses are bounded everywhere inside the sphere and the non-constitutive part has a logarithmic singularity at its center. The constrained solution derived here is valid for $0 < \kappa < 1$ and yields a stiffer material response than the one obtained by Ting (1999) in the context of the classical linear theory.

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