A PROCEDURE TO DETERMINE THE IMPACT COEFFICIENT IN A LINEAR ELASTIC STRAIGHT BEAM CONSIDERING THE EXACT CURVATURE OF THE ELASTICA

João Augusto de Lima Rocha
Polytechnic School of the Federal University of Bahia, Salvador, Brazil
joarocha@ufba.br

Abstract. *The aim of this work is the impact coefficient calculation in the case of a straight beam with a punctual mass freely dropping in the middle of its span considering, not the simplified, but the exact curvature of the elastica. The generality of the theoretical formulation presented here permits other beam impact applications, of that this work represents only one possibility. The measure of the impact is based on the hypothesis that the kinetic energy of the incident dropping mass is totally absorbed by the beam in form of strain energy in the instant of the collision. The numerical approximated solution is obtained with the aid of a rapidly convergent Newton-Raphson iterative method and the Gauss- Legendre integration process. The material is considered linear elastic and the automatic solution is obtained with the aid of a code, in FORTRAN, specially prepared for this application. At the end a numeric example is presented as an illustration of this code.*

Keywords: impact, nonlinear impact, impact coefficient

1. Introduction

The purpose of this work is to develop a procedure for the impact coefficient calculation in the case of a linear elastic straight beam with a punctual mass freely dropping at its mid but considering the exact curvature of the *elastica*. The results obtained with help of this procedure are compared with others, calculated in basis the hypothesis of small rotation angles of transversal sections of the beam.

The novelty here is that the curvature of the beam is not considered as being simply the second derivative of the *elastica* - a simplification valid when the angles of sectional rotations are very small – but the mathematical expression curvature is the exact one.

In consequence of the formulation here proposed one obtains an integral equation that interconnects the derivative function of the *elastica* with the bending moment function. This contribution that permits a more refined solution of the impact problem on beams besides an extension to the analysis of more general loading cases. In fact, the example presented in this work is a mere illustration of the general formulation developed here.

For the numerical calculations is used a software, in FORTRAN, specially development for the finality of this work, with that one automates the Newton-Raphson Method for the search of a root of a certain function and the numerical integration realized by the Gauss-Legendre method.

2. Impact coefficient calculation on the hypothesis of small rotations of the transversal sections

Assuming that don't have energetic losses and the kinetic energy of the punctual mass *m*, acquired in the course of the drop that precedes the impact on the beam (see Fig. 1), is absorbed totally for it, at the shock instant, then there is a static and concentrated load, *P,* to that corresponds an potential elastic energy equivalent to the kinetic energy of the mass at the incidence moment. So

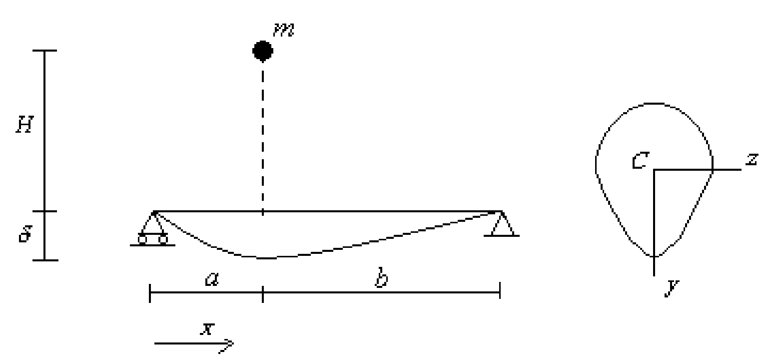

Figure 1. Incident mass on the symmetrical beam

$$mg(H+\delta)=U\ , \tag{1}$$

where δ is the deflection at the point where the punctual mass impact on the beam; H is the height of the mass, relative to the rest beam, and U is the potential elastic energy of the beam, loaded with a static equivalent vertical force.

In accordance with Timoshenko and Gere (1972), the expressions of δ and U, in this case, are:

$$\delta=\frac{Pa^2b^2}{3EIL} \quad \text{and}$$
$$U=\frac{P^2a^2b^2}{6EIL}, \tag{2}$$

where L is the length of the beam, E is the Young modulus of the material and I is the moment of inertia of the transversal area with respect to z axis (see Fig. 1).

The equality of kinetic energy of the mass m and potential elastic energy statically stored in the beam is expressed, after the substitution of the Eqs (2) in Eq. (1), as

$$mg\left(H+\frac{Pa^2b^2}{3EIL}\right)=\frac{P^2a^2b^2}{6EIL}\ . \tag{3}$$

The following algebraic equation results of this expression:

$$P^2-2mgP-\frac{6mgEILH}{a^2b^2}=0\ , \tag{4}$$

whose solution is:

$$P=mg\left(1\pm\sqrt{1+\frac{6EILH}{a^2b^2mg}}\right). \tag{5}$$

Observing that the expression on the square root signal, on the right of Eq. (5), is greater or equal 1, and if P is assumed positive, then the only root physically meaningful of this equation is:

$$P=mg\left(1+\sqrt{1+\frac{6EILH}{a^2b^2mg}}\right). \tag{7}$$

For definition, the impact coefficient, $C_{I,}$ is a magnitude that multiplying the weight of the incident mass gives the static force P. Then:

$$C_I=1+\sqrt{1+\frac{6EILH}{a^2b^2mg}}\ . \tag{8}$$

In the particular case in that the mass collides on the beam at its middle point, i. e , $a=b=L/2$, one has for the value of the impact coefficient:

$$C_I=1+\sqrt{1+\frac{96EIH}{L^3mg}}\ . \tag{9}$$

3. Analysis of the *elastica* when the rotations of the beam sections are not small

The equation that connects the curvature of the *elastica* with the bending moment, in the more general case, is:

$$\frac{v''}{\left[1+(v')^2\right]^{3/2}} = \frac{-M}{EI}\,, \qquad (10)$$

where the left member corresponds to the curvature, in the point of x cartesian coordinate, of the curve $v = v(x)$ – the *elastica* - and the right member of the Eq. (10) contains the bending moment, also a x- function.

Making use of an auxiliary variable, t, one considers the following expression:

$$v' = \frac{dv}{dx} = tg\,t\,. \qquad (11)$$

Hence

$$v'' = t'\sec^2 t\,, \qquad (12)$$

where

$$t' = dt / dx.$$

In accordance with Eq. (11), one obtains:

$$\left[1+(v')^2\right]^{3/2} = \sec^3 t\,. \qquad (13)$$

Transporting the Eqs. (12) and (13) to the Eq. 10), then:

$$t'\cos t = -\frac{M}{EI},$$

or

$$\cos t\,dt = -\frac{M}{EI}dx\,, \qquad (14)$$

whose integration produces:

$$\operatorname{sen} t = \int \frac{-M}{EI}dx + C_1\,, \qquad (15)$$

where C_1 is an arbitrary real constant.

The Eq. (15) is valid to any straight beam with symmetric as in Fig. 1, under simple bending, when the material is linear elastic, and the being is situated o on the longitudinal symmetric plane of the beam.

Using the Eqs. (11) and (15), the problem will be placed in terms of the displacements $v(x)$. For this, it is convenient that Eq. (11) be placed in the following form:

$$(v')^2 = tg^2 t = \frac{\operatorname{sen}^2 t}{1-\operatorname{sen}^2 t}\,. \qquad (16)$$

With the theoretic basis showed above will possible to solve the impact problem presented here.

3. Solution for the particular case of a simple supported beam with a concentrated load on the middle span

In the example, a simple supported beam loaded at the middle span, as is showed in Fig. 2, $v = v(x)$ is the equation of the *elastica* and the boundary conditions, owing the load symmetry, are:

$$v(0) = 0,$$
$$v'(L/2) = 0. \tag{17}$$

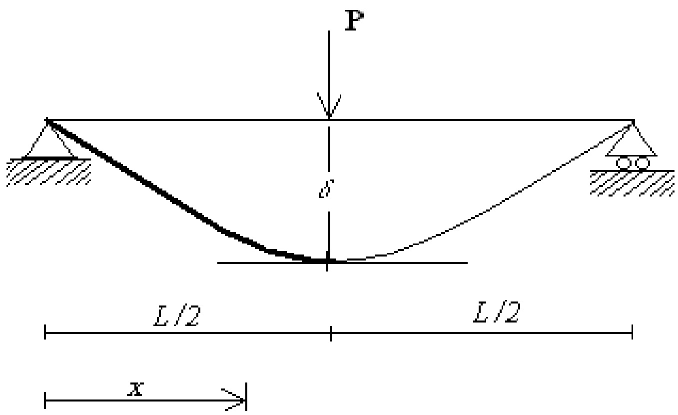

Figure 2. A simple supported beam with a load in the middle span

The function that represents the bending moments for the left half of the beam is:

$$M(x) = \frac{Px}{2}. \tag{18}$$

In accordance with the second of the boundary conditions in Eq. (17), considered Eq. (11), one have:

$$v'\big|_{x=L/2} = 0 \;\Rightarrow\; tg\, t\big|_{x=L/2} = 0 \Rightarrow \operatorname{sen} t\big|_{x=L/2} = 0\,,$$

that carries to Eq. (15) gives the real constant value C_1. So

$$C_1 = \left[\int \frac{M}{EI}dx\right]_{x=L/2}.$$

If de EI is constant then Eq. (15) transforms to:

$$\operatorname{sen} t = \int_x^{L/2} \frac{Px}{2EI}dx = \frac{P}{16EI}\left[L^2 - 4x^2\right]. \tag{19}$$

If P is assumed positive, then the following condition is necessary:

$$0 \le \frac{PL^2}{16EI} \le 1.$$

The Eq. (19), if carries to Eq. (16) gives:

$$(v')^2 = \frac{c^2(L^2-4x^2)^2}{1-c^2(L^2-4x^2)^2}, \tag{20}$$

where

$$c = \frac{P}{16EI}. \tag{21}$$

4. Impact coefficient calculation in the case of great rotation angles of the sections

In this case, it will be assumed the hypothesis that, in the instant of the shock, the kinetic energy of the incident punctual mass, will be transferred integrally to the beam, of a same manner that occurred in the anterior case (small rotations). Then:

$$mg(H+\delta)=U \;, \tag{1, repeated}$$

Now, in accordance with Timoshenko and Gere (1972), the potential elastic energy, U, stored *statically* in the beam, considering the load symmetry in it, is:

$$U = 2\int_{x=0}^{x=L/2} \frac{M_R^2}{2EI}\,ds \;, \tag{23}$$

where $M_R = M_R(x)$ is the bending moment owing a static equivalent load, P, that applied at the impact point (see Fig. 2) produces a potential elastic energy on the beam, equal to the kinetic energy of the incident mass at the final of the vertical drop of length $H+\delta$.

The Unit-Load Method, in accordance with Timoshenko and Gere (1972), will be used to calculate the deflection δ in the middle span of the beam. Considering the loading symmetry of the beam again, one has:

$$\delta = 2\int_{x=0}^{x=L/2} \frac{M_U M_R}{EI}\,ds \;, \tag{24}$$

observing that Eqs. (23) and (24) are integrals calculated *on the arch of the elastica.* In accordance with the Unit-Load Method, the magnitude $M_U = M_U(x)$ represents the bending moments in the case where na unitary vertical load is applied on the mid of the beam, in the same direction and orientation of the load P. Carrying Eqs.(23) and (24) to Eq. (22), comes:

$$mgH + 2mg\int_{x=0}^{x=L/2} \frac{M_U M_R}{EI}\,ds - 2\int_{x=0}^{x=L/2} \frac{M_R^2}{2EI}\,ds = 0 \;. \tag{25}$$

Observing the Fig. 2, one concludes that:

$$M_U = \frac{x}{2} \quad \text{e} \quad M_R = \frac{Px}{2} \;. \tag{26}$$

After the substitution of Eq. 26) in the Eq. (25), one has:

$$b + 2mgaP - aP^2 = 0 \;, \tag{27}$$

where

$$a = \int_{x=0}^{x=L/2} x^2\,ds \tag{28a}$$

and

$$b = 4mgHEI \;. \tag{28b}$$

The purpose is to obtain the solution of Eq. (27) in terms of the variable P. That is to say that the magnitude a is a function of P. For one proves this it is enough to substitute, in Eq. (28a), the arch element, ds, of the *elastica*, for his expression in terms of dx. Thus:

$$a = \int_{x=0}^{x=L/2} x^2\,ds = \int_0^{L/2} x^2 (1+v'^2)^{1/2}\,dx \;.$$

Substituting in this equation the expression of v', gives for Eq. (20), considering Eq. (21), comes:

$$a = \int_0^{L/2} x^2 \left(1 + \frac{c^2 (L^2 - 4x^2)^2}{1 - c^2 (L^2 - 4x^2)^2} \right)^{1/2} dx = \int_0^{L/2} q(x, P) dx , \tag{29}$$

where

$$q(x, P) = \frac{16 E^2 I^2 x^2}{\left[256 E^2 I^2 - P^2 (L^2 - 4x^2)^2 \right]^{1/2}} . \tag{30}$$

As a is dependent of P, then Eq. (27) will be resolved with the aid of an approximated method. The Newton-Raphson Method will be used for this, being calculated the value of a, in each pass of the iterative procedure, with the help of the Gauss-Legendre Integration Method. So, the Eq. (27) will be placed in the following form:

$$f(P) = 0, \tag{31}$$

where

$$f(P) = b + 2mgaP - aP^2 . \tag{32}$$

For using the Newton-Raphson Method, it is necessary the calculation of the $f(P)$ derivative, i. e:

$$\frac{df}{dP} = 2a(mg - P) + P(2mg - P)\frac{da}{dP} . \tag{33}$$

On the other hand, the a derivative with relation to P, calculated after Eq. (29), is:

$$\frac{da}{dP} = \int_0^{L/2} \frac{\partial q(x, P)}{\partial P} dx , \tag{34}$$

where, in accordance with the Eqs. (21) and (29) :

$$\frac{\partial q}{\partial P} = \frac{16 EIPx^2 (L^2 - 4x^2)^2}{[256 E^2 I^2 - P^2 (L^2 - 4x^2)^2]^{3/2}} . \tag{35}$$

Substituting the last expression in Eq. (34), one has:

$$\frac{da}{dP} = \int_0^{L/2} \frac{16 EIPx^2 (L^2 - 4x^2)^2}{[256 E^2 I^2 - P^2 (L^2 - 4x^2)^2]^{3/2}} dx , \tag{36}$$

that will permit the calculation of the $f(P)$ derivative, in Eq. (33), with the aid of the Eqs. (29) and (36), by means of the iterative Newton-Raphson Method:

$$P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1}).} \quad n = 1, \ldots \tag{37}$$

5. Calculation of the deflection at the middle and the rotation angles at the supports

The deflection δ, in the mid of the bean will be calculated with the help of Eqs. (16) and (17), being the derivative v' calculated only in points of the the domain [0, L/2], reason for that the positive signal was adopted for the expression under the square root in the equation below . Then, of Eq. (16), comes :

$$v' = tg\,t \quad \Rightarrow \quad v' = \frac{\operatorname{sen} t}{\cos t} = \frac{\operatorname{sen} t}{[1-\operatorname{sen}^2 t]^{1/2}}. \tag{38}$$

Substituting the Eq. (19) in the last expression, one has:

$$v' = \frac{c(L^2-4x^2)}{[1-c^2(L^2-4x^2)^2]^{1/2}}, \tag{39}$$

The integration of this expression gives:

$$v = \int \frac{c(L^2-4x^2)}{[1-c^2(L^2-4x^2)^2]^{1/2}}\,dx + C_2, \tag{40}$$

where C_2 is an arbitrary real constant.

The numeric value of this constant, for the case in question, one obtains with the aid of the first of the boundary conditions in Eq. (17). Then, Eq. (40) turn into

$$v = \int_0^x \frac{c(L^2-4x^2)}{[1-c^2(L^2-4x^2)^2]^{1/2}}\,dx, \tag{41}$$

Thus, of Eq. (41), one calculates the deflection at the mid of beam of the Fig.2.

The rotation angles at the supports, both equals, are obtained of Eq. (40), with x = 0. Then, considering also Eq. (21), comes:

$$v' = \frac{c(L^2-4x^2)}{[1-c^2(L^2-4x^2)^2]^{1/2}},$$

and

$$v'(0) = v'(L) = \frac{PL^2}{[256E^2I^2 - P^2L^4]^{1/2}}. \tag{42}$$

Naturally, since in the case studied , the load P is associated to impact, one substitute it, in Eq. (41) and (42), for:

$$P = C_i mg .$$

The value of C_i - the impact coefficient -, in this problem, is obtained by means of the P value that was calculated by means of the iterative process of the Eq. (37).

In the sequence, some results are presented. They were obtained with the use of a code specially developed for this work. For the finality of comparison, two ways were used for to calculate the impact coefficients: 1) under the hypothesis of small rotation and 2)under the hypothesis of great rotations of the transversal sections of the beam.

The results, referenced in the Fig. 2, are represented in the Tab. 1. In it the greatest value of the two ones, in each case, is assigned with *red* (L=2.00 m) and *blue* (L=20.0 m).

Tab. 1 Impact coefficients. Rectangular beam. Constant section: *b*=0.2m; *h*=0.4m. Mass = 500 Kg material: E=20 GPa

L (m)	Model	H (m)								
		0.00	0.01	0.10	0.30	0.50	1.00	3.00	5.00	10.0
2.00	SR (1)	2.00	23.871	73.263	126.16	162.57	229.50	396.76	511.93	723.56
	GR (2)	2.00	32.169	93.484	152.76	190.70	243.95	378.82	461.99	601.37
20.0	SR (1)	2.00	2.2237	3.4942	5.0820	6.2062	8.2945	13.555	17.188	23.871
	GR (2)	2.00	2.6749	3.7499	5.2574	6.2283	7.8920	11.571	13.835	17.621

1- Hypothesis of small rotations; 2- Hypothesis of great rotations;

6. Conclusions

The results of Tab. 1 reveals significant differences between the two models used in the impact coefficient calculation. In the first model, the curvature of the *elastica* is calculated as the second derivative (small rotation angles) and in the second model, the curvature is exact (great rotation angles). In the first model, the software that calculate the impact coefficient is simple, direct and exact. But the software in the second model is more exigent since the process of localization the root associated to significant solution, via Newton-Raphson Method, includes a choice process in that the external interference is necessary..

7. Acknowledgements

The financial support of FAPESB and NNST/SESI/Ba, Bahia State, Brazil, is acknowledged.

8. Reference

Timoshenko, S. P; Gere, J. M. " Mechanics of Materials", International Edition, D. Van Nostrand, 1972, 552p.

9. Responsibility notice

João Augusto de Lima Rocha is the only responsible for the printed material included in this paper.