EXACT SOLUTION OF THE SPTT MODEL FOR FULLY DEVELOPED CHANNEL FLOWS

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Abstract. *In this work we derive an exact solution for fully developed channel flows of viscoelastic fluids described by the linearized form of the SPTT (Simplified-Phan-Thien-Tanner) constitutive equation. We consider the governing equations for two-dimensional flows and employ the EVSS (Elastic-Viscous Stress-Splitting) transformation. We then apply the fully developed flow assumption and solve the resulting equations analytically and expressions for the velocity and the extra-stress tensor are given. We use this exact solution to validade a numerical method that simulates the flow in 2D channels using the SPTT constitutive equation.*

Keywords: *SPTT model, exact solution, channel flow, finite difference.*

1. Introduction

The non-linear viscoelastic constitutive equation Phan-Thien-Tanner (PTT) model ((Phan-Thien and Tanner, 1977); (Phan-Thien, 1978); (Xue et al., 1998); (Xue et al., 1999); (Alves et al., 2001) and (Pinho et al., 2003)) has been considered the more realistic model for polymer melts and concentrated solutions in comparison with other models. In the PTT constitutive equation, there is a parameter ξ that accounts for the slip between the molecular network and the continuum medium. In this work we study a particular case of the PTT Model - The SPTT Model (Simplified-Phan-Thien-Tanner Model) when $\xi = 0$ ((Phan-Thien and Tanner, 1977); (Oliveira and Pinho, 1999)). Viscoelastic flows in channels and pipes are important in many industrial processes (Tadmor and Gogos, 1979) and analytical solutions when avalaible are preferred as they can provide physical insight into flow phenomena. In addition, analytical solutions can be useful to validate numerical methods. Recently, Oliveira and Pinho (Oliveira and Pinho, 1999), (Alves et al., 2001), and Alves et al. (Alves et al., 2001) studied the various forms of the PTT constitutive equation and obtained exact solutions for duct flows. In their studies they solved the governing equations without using the EVSS transformation (Rajagopalan et al., 1990). In this work we consider the governing equations for two-dimensional flows governed by the SPTT constitutive equation. and assume fully developed flows. We use the ideas presented by Oliveira and Pinho (Oliveira and Pinho, 1999), to solve the resulting equations analytically and use the exact solution to validate the GENSMAC-SPTT method (see Paulo et al. (Paulo et al., 2004)).

2. Governing equations

The basic equations governing incompressible isothermal flows described by the SPTT model (Phan-Thien and Tanner, 1977) are the mass conservation equation, the equation of motion and the constitutive equation for the SPTT model which are given by

$$\nabla \cdot \mathbf{u} = 0 \,, \tag{1}$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \nabla(\mathbf{u}\mathbf{u}) \right] = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} \,, \tag{2}$$

$$f(\mathrm{tr}(\boldsymbol{\tau}))\boldsymbol{\tau} + \lambda \overset{\nabla}{\boldsymbol{\tau}} = 2\eta_P \mathbf{D} \,, \tag{3}$$

where t is the time, $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is the density, $\mathbf{g}$ is the gravitational field, $\boldsymbol{\tau}$ is the extra-stress tensor, which is related to the kinematic quantities by the constitutive equation Eq. (3), $\mathbf{D} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}\right]$ is the rate of deformation tensor, λ is the fluid relaxation time, η_P is the polymer-contributed viscosity, $(\overset{\triangledown}{\cdot})$ represents the following convected derivative:

$$\overset{\triangledown}{\boldsymbol{\tau}} = \frac{\partial \boldsymbol{\tau}}{\partial t} + \nabla(\mathbf{u}\boldsymbol{\tau}) - \boldsymbol{\tau}\,(\nabla\mathbf{u}) - (\nabla\mathbf{u})^{\mathrm{T}}\boldsymbol{\tau}\,,$$

and

$$f(\mathrm{tr}(\boldsymbol{\tau})) = 1 + \frac{\varepsilon\lambda}{\eta_P}\mathrm{tr}(\boldsymbol{\tau})\,,$$

where $\nabla\mathbf{u}$ is the velocity gradient tensor; ε is a positive parameter describing the SPTT model (Phan-Thien and Tanner, 1977).

To solve Eqs. (1)–(3), we employ the EVSS (Elastic-Viscous Stress-Splitting) formulation (Rajagopalan et al., 1990)

$$\boldsymbol{\tau} = 2\eta_N\mathbf{D} + \mathbf{S}\,, \tag{4}$$

where η_N is the Newtonian-contribuition viscosity, $\mathbf{S}$ is a non-Newtonian tensor which is the polymer-contribuition stress tensor.

Let $\beta = \dfrac{\eta_P}{\eta_0}$, with $\eta_0 = \eta_N + \eta_P$ being the total viscosity. Thus, $\eta_P = \beta\eta_0$ and $\eta_N = (1-\beta)\eta_0$. We remark that the Oldroyd-B model and the UCM model (Upper-Convected-Maxwell model) are special cases of the SPTT model; by making $\varepsilon = 0$ and $\lambda_2 = (1-\beta)\lambda$ we obtain the Oldroyd-B model and the UCM model is obtained taking $\varepsilon = 0$ and $\beta = 1$ (Bird et al., 1977).

Substituting Eq. (4) into Eqs. (2) and (3) we obtain:

$$\rho\left[\frac{\partial\mathbf{u}}{\partial t} + \nabla(\mathbf{u}\mathbf{u})\right] = -\nabla p + (1-\beta)\eta_0\nabla^2\mathbf{u} + \nabla\cdot\mathbf{S} + \rho\mathbf{g}\,, \tag{5}$$

$$f(\mathrm{tr}(\boldsymbol{\tau}))\mathbf{S} + \lambda\,\overset{\triangledown}{\mathbf{S}} = 2\eta_0\left[\beta - (1-\beta)f(\mathrm{tr}(\mathbf{S}))\right]\mathbf{D} - 2\lambda(1-\beta)\eta_0\,\overset{\triangledown}{\mathbf{D}}\,, \tag{6}$$

with $f(\mathrm{tr}(\mathbf{S})) = 1 + \dfrac{\lambda\varepsilon}{\beta\eta_0}\mathrm{tr}(\mathbf{S})$.

Therefore, we shall solve Eqs. (1), (5) and (6) for the dependent variables $\mathbf{u}$, p and $\mathbf{S}$.

We consider two-dimensional Cartesian flows. Then, Eqs. (1), (5) and (6) can be written in non-dimensional form as (see (Paulo et al., 2004)):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{7}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial u^2}{\partial x} - \frac{\partial(vu)}{\partial y} - \frac{\partial p}{\partial x} + (1-\beta)\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial S^{xx}}{\partial x} + \frac{\partial S^{xy}}{\partial y} + \frac{1}{Fr^2}g_x\,, \tag{8}$$

$$\frac{\partial v}{\partial t} = -\frac{\partial(uv)}{\partial x} - \frac{\partial v^2}{\partial y} - \frac{\partial p}{\partial y} + (1-\beta)\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \frac{\partial S^{xy}}{\partial x} + \frac{\partial S^{yy}}{\partial y} + \frac{1}{Fr^2}g_y\,, \tag{9}$$

$$\begin{aligned}\frac{\partial S^{xy}}{\partial t} = {} & -f(\mathrm{tr}(\mathbf{S}))\frac{1}{We}S^{xy} - \frac{\partial(uS^{xy})}{\partial x} - \frac{\partial(vS^{xy})}{\partial y} + \frac{\partial v}{\partial x}S^{xx} \\ & + \frac{\partial u}{\partial y}S^{yy} + \left[\beta - (1-\beta)f(\mathrm{tr}(\mathbf{S}))\right]\frac{2}{ReWe}D^{xy} \\ & - (1-\beta)\frac{2}{Re}\left[\frac{\partial}{\partial t}D^{xy} + \frac{\partial(uD^{xy})}{\partial x} + \frac{\partial(vD^{xy})}{\partial y} + \left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\frac{\partial u}{\partial x}\right]\,,\end{aligned} \tag{10}$$

$$\begin{aligned}\frac{\partial S^{xx}}{\partial t} = {} & -f(\mathrm{tr}(\mathbf{S}))\frac{1}{We}S^{xx} - \frac{\partial(uS^{xx})}{\partial x} - \frac{\partial(vS^{xx})}{\partial y} + 2\frac{\partial u}{\partial x}S^{xx} + 2\frac{\partial u}{\partial y}S^{xy} \\ + {} & \left[\beta - (1-\beta)f(\mathrm{tr}(\mathbf{S}))\right]\frac{2}{ReWe}D^{xx} - (1-\beta)\frac{2}{Re}\left[\frac{\partial}{\partial t}D^{xx} + \frac{\partial(uD^{xx})}{\partial x} + \frac{\partial(vD^{xx})}{\partial y}\right. \\ - {} & \left. 2\,(D^{xx})^2 - \left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right]\,,\end{aligned} \tag{11}$$

$$\begin{aligned}
\frac{\partial S^{yy}}{\partial t} &= -f(\mathrm{tr}(\mathbf{S}))\frac{1}{We}S^{yy} - \frac{\partial(uS^{yy})}{\partial x} - \frac{\partial(vS^{yy})}{\partial y} + 2\frac{\partial v}{\partial y}S^{yy} + 2\frac{\partial v}{\partial x}S^{xy} \\
&+ \left[\beta - (1-\beta)f(\mathrm{tr}(\mathbf{S}))\right]\frac{2}{ReWe}D^{yy} - (1-\beta)\frac{2}{Re}\left[\frac{\partial}{\partial t}D^{yy} + \frac{\partial(uD^{yy})}{\partial x} + \frac{\partial(vD^{yy})}{\partial y}\right. \\
&- \left. 2\left(D^{yy}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right], \qquad (12)
\end{aligned}$$

where $f(\mathrm{tr}(\mathbf{S})) = 1 + ReWe\frac{\varepsilon}{\beta}\mathrm{tr}(\mathbf{S})$; $Re = \rho UL/\eta_0$, $We = \lambda U/L$ and $Fr = U/\sqrt{L|g|}$ denote the Reynolds, the Weissenberg and the Froude numbers, respectively.

3. Exact solution for fully developed flows

If we consider fully developed flows then we have

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = v = 0\ ,\ u = u(y)\ \text{ and }\ \frac{\partial p}{\partial x} = \mathrm{dpdx}\ \text{(constant)}\ .$$

Let us employ a coordinate system which is centred in the symmetry line where we take $y = 0$ (see Fig. 1) and let us

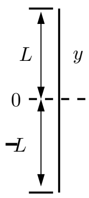


Figure 1. Definition of the domain for fully developed flow.

assume that the velocity $u(y)$ obeys the no-slip condition at $y = \pm L$, namely, $u(-L) = u(L) = 0$. In addition, we seek a solution $u(y)$ which satisfies

$$\frac{\partial u}{\partial y}(0) = 0\ . \qquad (13)$$

In this case, the mass conservation equation is trivialy satisfied and Eq. (8) together with Eqs. (10)-(12) reduce to

$$\frac{\partial S^{xy}}{\partial y} = \frac{\partial p}{\partial x} - (1-\beta)\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}\ , \qquad (14)$$

$$-\frac{1}{We}f(\mathrm{tr}(\mathbf{S}))S^{xx} + 2\frac{\partial u}{\partial y}S^{xy} + \frac{2}{Re}(1-\beta)\left(\frac{\partial u}{\partial y}\right)^2 = 0\ , \qquad (15)$$

$$-\frac{1}{We}f(\mathrm{tr}(\mathbf{S}))S^{yy} = 0\ , \qquad (16)$$

$$-\frac{1}{We}f(\mathrm{tr}(\mathbf{S}))S^{xy} + \frac{\partial u}{\partial y}S^{yy} + \left[\beta - (1-\beta)f(\mathrm{tr}(\mathbf{S}))\right]\frac{1}{ReWe}\frac{\partial u}{\partial y} = 0\ , \qquad (17)$$

where $\mathrm{tr}(\mathbf{S}) = S^{xx} + S^{yy}$ is the trace of the non-Newtonian tensor **S** and the function $f(\mathrm{tr}(\mathbf{S}))$ is given by

$$f(\mathrm{tr}(\mathbf{S})) = 1 + ReWe\frac{\varepsilon}{\beta}(S^{xx} + S^{yy})\ . \qquad (18)$$

To solve the system of equations given by Eqs. (14)-(18) we proceed as follows: First, we integrate Eq. (14) in the interval $[0,\ y]$ to obtain:

$$S^{xy}(y) - S^{xy}(0) = \frac{\partial p}{\partial x}y - \left[(1-\beta)\frac{1}{Re}\frac{\partial u(y)}{\partial y} - (1-\beta)\frac{1}{Re}\frac{\partial u(0)}{\partial y}\right]. \qquad (19)$$

Now, since we have $\frac{\partial u}{\partial y}(0) = 0$ (from Eq. (13)), one can see that the solution of the constitutive equations Eqs. (15)-(17) at $y = 0$ is $S^{xy}(0) = S^{xx}(0) = S^{yy}(0) = 0$. Thus, Eq. (19) reduces to

$$S^{xy}(y) = \frac{\partial p}{\partial x}y - (1-\beta)\frac{1}{Re}\frac{\partial u(y)}{\partial y}\ . \qquad (20)$$

The components of the extra-stress tensor $S^{xx}(y)$, $S^{yy}(y)$ and $\dfrac{\partial u(y)}{\partial y}$ will be obtained from Eqs. (15)-(17), as follows:

From Eq. (16) we obtain

$$S^{yy} = 0 \tag{21}$$

so that Eq. (18) reduces to

$$f(S^{xx}) = 1 + Re\, We\, \frac{\varepsilon}{\beta} S^{xx} \tag{22}$$

and introducing Eqs. (22) and (20) into Eqs. (15) and (17) we obtain (after some simplifications)

$$Re\frac{\varepsilon}{\beta}\,(S^{xx})^2 + \frac{1}{We}\; S^{xx} - 2\frac{\partial p}{\partial x}\, y\, \frac{\partial u}{\partial y} = 0\ , \tag{23}$$

$$Re\frac{\varepsilon}{\beta}\frac{\partial p}{\partial x}\, y\, S^{xx} - \frac{\beta}{ReWe}\frac{\partial u}{\partial y} + \frac{1}{We}\frac{\partial p}{\partial x}\, y = 0\ . \tag{24}$$

Equations (23) and (24) form a (2× 2)-nonlinear system for the unknowns S^{xx} and $\dfrac{\partial u}{\partial y}$ which can be solved giving

$$S^{xx} = \frac{2}{\beta} ReWe \left(\frac{\partial p}{\partial x}\right)^2 y^2; \tag{25}$$

$$\frac{\partial u}{\partial y} = \frac{Re}{\beta}\frac{\partial p}{\partial x} y + 2Re^3 We^2 \frac{\varepsilon}{\beta^3}\left(\frac{\partial p}{\partial x}\right)^3 y^3\ . \tag{26}$$

The velocity $u(y)$ is found by integrating Eq. (26) in y and imposing $u(-L) = u(L) = 0$ yielding

$$u(y) = \frac{Re}{\beta}\frac{\partial p}{\partial x}(y^2 - L^2) + Re^3 We^2 \frac{\varepsilon}{2\beta^3}\left(\frac{\partial p}{\partial x}\right)^3 (y^4 - L^4)\ . \tag{27}$$

Therefore, $u(y)$ is obtained from Eq. (27) and having calculed $\dfrac{\partial u}{\partial y}$ from Eq. (26) the extra-stress components S^{xy}, S^{xx} are computed via Eqs. (20) and (25), respectively.

4. An application for the exact solution

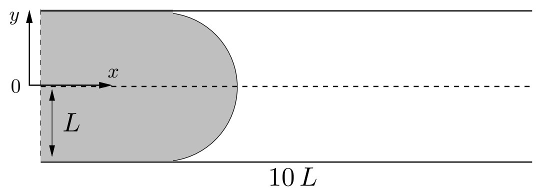


Figure 2. Definition of the computational domain for 2D channel flow.

The analytic solution obtained in this work has been implemented in the GENSMAC-PTT simulation system developed by Paulo et al. (Paulo et al., 2004) to simulate two-dimensional free surface flows governed by the PTT constitutive equation (Xue et al., 1998). In the code GENSMAC-PTT (Paulo et al., 2004) we set $\xi = 0$ so that the PTT Model reduces to SPTT Model. Thus we shall simulate the flow in a 2D channel and analyse the numerical solution at the cross-section of the channel (at $x = 5\ L$):

We consider two-dimensional Cartesian flows in a channel formed by two parallel plates which are at a distance L from the symmetry axis $(y = 0)$ and having a length of $10L$ (see Fig. 2). At the channel entrance we impose the analytic values of the velocity field $u(y)$ as well the analytic values of the extra-stress components, S^{xy}, S^{yy} and S^{xx} obtained in Section 3 as given by Eqs. (27), (20), (21) and (25), respectively. At the channed exit we impose homogeneous Neumann conditions for both the velocity field and the extra-stress components, namely:

$$\frac{\partial u}{\partial x} = 0\ ,\ \frac{\partial v}{\partial x} = 0\ ,\ \frac{\partial S^{xy}}{\partial x} = 0\ ,\ \frac{\partial S^{xx}}{\partial x} = 0\ ,\ \frac{\partial S^{yy}}{\partial x} = 0\ .$$

On the channel walls the velocity field satisfies the no-slip condition while the extra-stress components are computed following the ideas of Tomé et al. (Tomé et al., 2002) and are given by:

$$
\begin{aligned}
S^{xy}(x,y,t+\delta t) &= \left\{1+\varepsilon\frac{\delta t^2}{2}\left(\frac{\partial u(x,y,t+\delta t)}{\partial y}\right)^2\right\}^{-1}\left\{e^{-\frac{1}{We}\delta t}S^{xy}(x,y,t)\right.\\
&-\left[\beta+\varepsilon\frac{\delta t}{2}\left[e^{-\frac{1}{We}\delta t}\left(S^{xx}(x,y,t)+2We\beta\left(e^{\frac{1}{We}\delta t}-1\right)\left(\frac{\partial u(x,y,t^*)}{\partial y}\right)^2\right.\right.\right.\\
&\left.\left.\left.+\delta t\frac{\partial u(x,y,t)}{\partial y}S^{xy}(x,y,t)\right)\right]+(\varepsilon-1)\frac{\delta t}{2}\left[e^{-\frac{1}{We}\delta t}S^{yy}(x,y,t)\frac{\partial u(x,y,t+\delta t)}{\partial y}\right]\right]\\
&+\beta e^{-\frac{1}{We}\delta t}\left(e^{\frac{1}{We}\delta t}-1\right)\frac{\partial u(x,y,t^*)}{\partial y}\\
&\left.+e^{-\frac{1}{We}\delta t}\left[\beta-\varepsilon\frac{\delta t}{2}S^{xx}(x,y,t)-(\varepsilon-1)\frac{\delta t}{2}S^{yy}(x,y,t)\right]\frac{\partial u(x,y,t)}{\partial y}\right\}, \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
S^{xx}(x,y,t+\delta t) &= e^{-\frac{1}{We}\delta t}\left[S^{xx}(x,y,t)+2We\beta\left(e^{\frac{1}{We}\delta t}-1\right)\left(\frac{\partial u(x,y,t^*)}{\partial y}\right)^2\right.\\
&\left.+\delta t\frac{\partial u(x,y,t)}{\partial y}S^{xy}(x,y,t)\right]+\delta t\frac{\partial u(x,y,t+\delta t)}{\partial y}S^{xy}(x,y,t+\delta t)\,, \qquad (29)
\end{aligned}
$$

$$S^{yy}(x,y,t+\delta t) = e^{-\frac{1}{We}\delta t}S^{yy}(x,y,t)\,. \qquad (30)$$

where $t^* \in (t, t+\delta t)$ (see (Paulo et al., 2004).

We shall start the simulation with the cannel empty and then fill it gradually so that initially, we shall have a free surface flow within the channel. On the free surface of the fluid the boundary conditions are given by (see Batchelor (Batchelor, 1967))

$$p = 2(1-\beta)\frac{1}{Re}\left[\left(\frac{\partial u}{\partial x}{n_x}^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)n_xn_y+\frac{\partial v}{\partial y}{n_y}^2\right)\right]+S^{xx}{n_x}^2+2S^{xy}n_xn_y+S^{yy}{n_y}^2\,, \qquad (31)$$

$$(1-\beta)\frac{1}{Re}\left[2\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)n_xn_y+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(n_y^2-n_x^2)\right]+(S^{xx}-S^{yy})n_xn_y+S^{xy}(n_y^2-n_x^2)=0\,, \qquad (32)$$

where $\mathbf{n} = (n_x,\ n_y)$ is the local unity normal vector to the surface.

Details of the application of the boundary conditions on rigid boundaries and on the free surface can be found in Tomé et al. (Tomé et al., 2002).

To solve Eqs. (7)-(12) and Eqs. (28)-(30) we employ the procedure used by Paulo et al. (Paulo et al., 2004) for the PTT-Model. This procedure is similar to the numerical technique, GENSMACVISCO, developed by Tomé et al. (Tomé et al., 2002) for a fluid Oldroyd-B. In these numerical techniques the equations are solved by the finite difference method on a staggered grid.

To simulate the problem described above we used the following input data:

$\varepsilon = 0.1,\ \beta = 0.8,\ \dfrac{\partial p}{\partial x} = -1.5,\ Re = 1.0$ and $We = 0.5$.

With these data, the scaling parameters were obtained as follows:

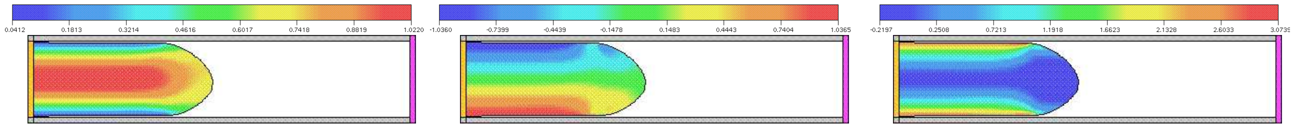
$L = 1.0\,m$ (half of the channel width), $U = Umax = 1.019897\,ms^{-1}$ (calculated using Eq. (27)),

$\nu = \dfrac{\eta_0}{\rho} = 1.019897\,m^2s^{-1}$ (value used so that $Re = 1$) and $\lambda = 0.490245\,s$ (computed so that $We = 0.5$).

To analyse the convergence of GENSMAC-SPTT on this problem we employed three meshes:

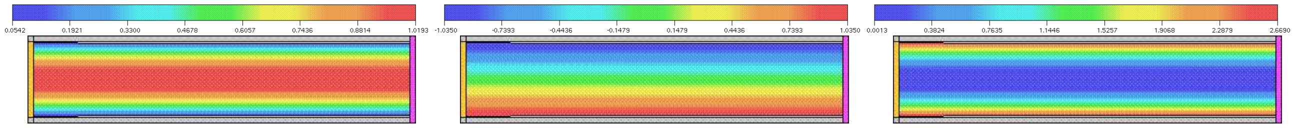
Mesh 1 ($\mathbf{M1}$) : 10×50 cells ($\delta x = \delta y = 0.2$),
Mesh 2 ($\mathbf{M2}$) : 20×100 cells ($\delta x = \delta y = 0.1$),
Mehs 3 ($\mathbf{M3}$) : 40×200 cells ($\delta x = \delta y = 0.05$) .

The GENSMAC-SPTT simulated the channel flow for these three meshes and the results are as follows: At time $t = 0$ the channel is empty and the fluid is injected in at the channel entrance until the channel is completely full and the steady state is achieved. Figure 3 shows the contours of the velocity and the extra-stress components at the time $t = 6.0$s on mesh **M3**. We can see that at time $t = 6.0$s the channel is being filled and the contours are not parallel which indicates the transient state of the flow.



a) b) c)

Figure 3. Channel flow simulation - $Re = 1, We = 0.5$. Contour lines of the velocity and components of the extra-stress tensor at the time $t = 6.0$s on the finner mesh **M3**: a) $u(y)$, b) $S^{xy}(y)$ and c) $S^{xx}(y)$.



a) b) c)

Figure 4. Channel flow simulation - $Re = 1, We = 0.5$. Contour lines of the velocity and components of the extra-stress tensor at the time $t = 100$ s on the finner mesh **M3**. a) $u(y)$, b) $S^{xy}(y)$ and c) $S^{xx}(y)$.

Figure 4 displays the contours of the velocity and the extra-stress components at the time $t = 100$s on the finner mesh **M3**. We can see that the channel is completely full and the contour lines are all parallel indicating that the steady state has being achieved at that time.

We point out that at the steady state, the solution within the channel should be the exact solution which is imposed at the channel entrance. To verify this fact, we calculated the values of the variables u, S^{xy} and S^{xx} at the middle of the channel ($x = 5$m) using the results obtained on the three meshes at the time $t = 100$s and compared with the exact solution imposed on the channel entrance. These values are shown in Fig. 5, Fig. 6 and Fig. 7. Indeed, we can observe in Fig. 5, Fig. 6 and Fig. 7 that the numerical solutions obtained using the three meshes are in good agreement with the respective analytic solution. Moreover, Fig. 7 shows that as the mesh is refined the numerical solution converges to the exact solution.

a) b) c)

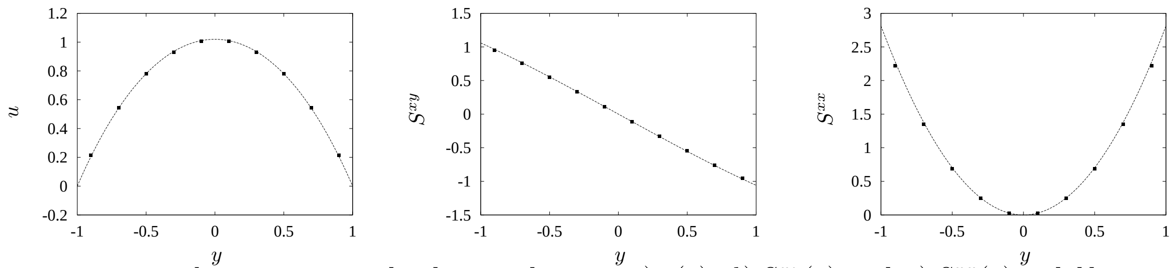


Figure 5. Comparison between numerical and exact solutions: *a)* $u(y)$, *b)* $S^{xy}(y)$ and *c)* $S^{xx}(y)$. Solid lines represent the analytic solutions and the symbols represent the numerical solutions. Results obtained on mesh **M1**.

Indeed, to demonstrate the convergence of GENSMAC-SPTT, Tab. 1 shows the $\mathbf{L_2}$ norms of the errors between the exact solution ($ExSol$) and the numerical solutions ($NumSol$) obtained on each mesh by Eq. (33).

$$E(NumSol) = \frac{\sum(ExSol - NumSol)^2}{\sum ExSol^2}\,. \tag{33}$$

We can see in the Tab. 1 that the errors decrease as the mesh is refined. These results demonstrate that GENSMAC-SPTT converges as the mesh is refined.

Table 1. Relative l_2-norm of the errors between the exact and the numerical solutions.

$Mesh$	$E(u)$	$E(S^{xx})$	$E(S^{xy})$
$M1$	$4.6389\,10^{-5}$	$5.7827\,10^{-4}$	$1.1457\,10^{-4}$
$M2$	$3.5943\,10^{-6}$	$3.7575\,10^{-5}$	$7.5740\,10^{-6}$
$M3$	$2.4271\,10^{-7}$	$2.3358\,10^{-6}$	$4.8307\,10^{-7}$

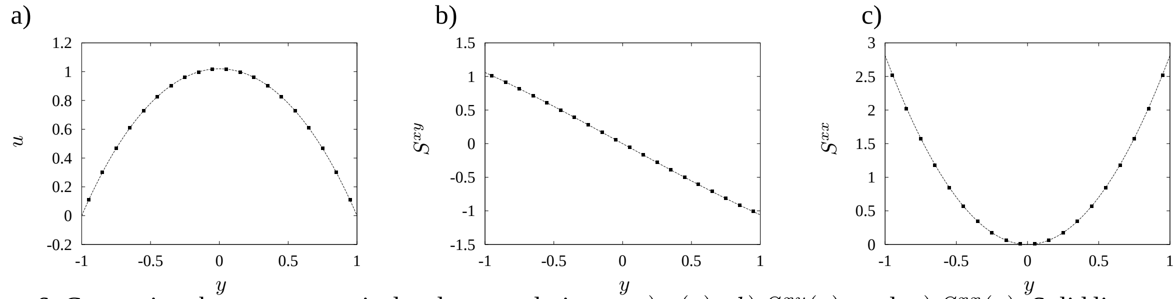


Figure 6. Comparison between numerical and exact solutions: $a)\ u(y)$, $b)\ S^{xy}(y)$ and $c)\ S^{xx}(y)$. Solid lines represent the analytic solutions and the symbols represent the numerical solutions. Results obtained on mesh **M2**.

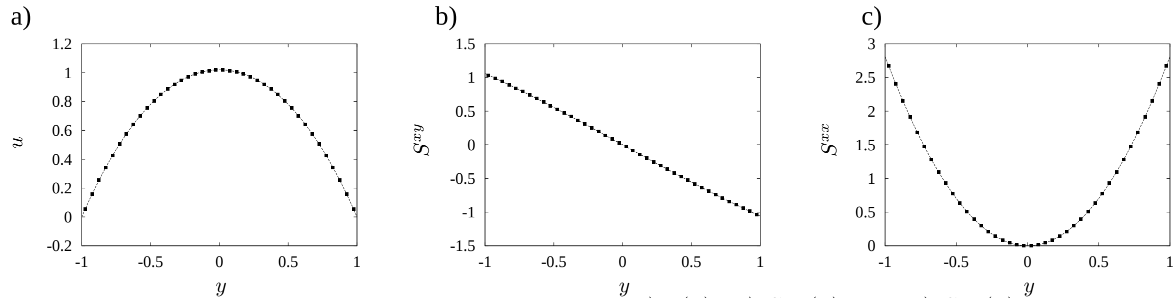


Figure 7. Comparison between numerical and exact solutions: $a)\ u(y)$, $b)\ S^{xy}(y)$ and $c)\ S^{xx}(y)$. Solid lines represent the analytic solutions and the symbols represent the numerical solutions. Results obtained on mesh **M3**.

5. Implementation comments

The equations describing the GENSMAC-SPTT have been implemented into the *Freeflow2D* code (Oliveira and Castelo, 1999). The Eqs. (7)-(12) and Eqs. (28)-(30) are solved by the finite difference method on a staggered grid (see Fig. (8) a)) with cell dimensions $\delta x \times \delta y$. The pressure and the components of the non-Newtonian extra stress are located at cell centres (i, j) while the velocity u and v are staggered by $(i + 1/2, j)$ and $(i, j + 1/2)$, respectively. A scheme for identifying the fluid region and the free surface is employed. To effect this the cells in the mesh can be of several types, namely: cells Full of fluid (F), Surface cells (S), Empty cells (E), Inflow cells (I), Outflow cells (O) and Boundary cells (B). The F-cell is required to contain fluid and to have no E-cell face in contact with any of its faces while S-cells are defined to contain fluid and to have at least one face in contact with an E-cell face. I-cells define an inflow boundary and O-cells define an outflow boundary. B-cells define rigid boundaries where the no-slip condition is imposed. Figure (8) b)) displays the types of cells within the mesh.

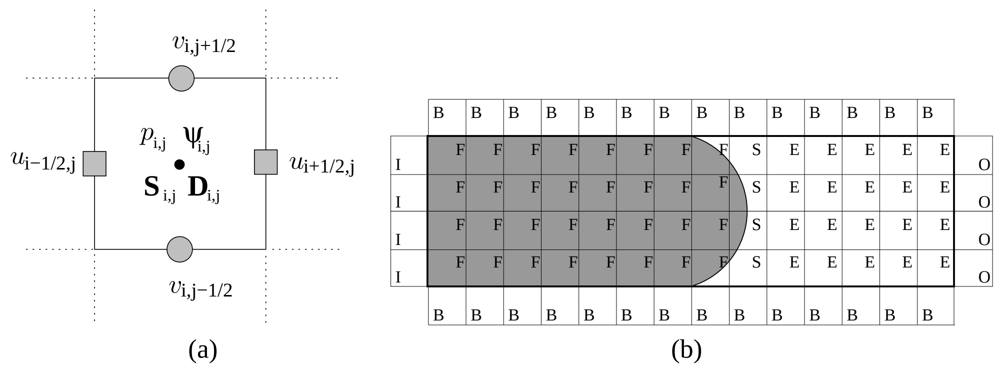


Figure 8. Staggered grid (a) and types of cells in the computational domain (b).

The time derivatives are approximated by the explicit Euler method which is first order. The pressure gradient and the linear terms of the momentum equations are approximated by central differences. For the convective terms we employ a high order upwind method. In this work the CUBISTA method – Convergent and Universally Bounded Interpolation Scheme for the Treatment of Advection (Alves et al., 2003) is employed to approximate the convective terms. The terms involving the divergent of the non-Newtonian extra stress are approximated by central differences.

For the simulations presented in this work, a Pentium Xeon IV 2.66 GHz with hyper threading CPU, 4GB memory was used. The CPU time spent on mesh $\mathbf{M}1$ was approximately 10 minutes while for the finer mesh $\mathbf{M3}$ the CPU time was about 24 hours.

6. Concluding remarks

This paper was concerned with the calculation of the exact solution for fully developed channel flows of viscoelastic fluids described by the linearized form of the SPTT (Simplified-Phan-Thien-Tanner) constitutive equation. The governing equations for two-dimensional flows were considered and the EVSS (Elastic-Viscous Stress-Splitting) transformation was employed.By applying the fully developed flow assumption, the resulting equations were solved analytically. Expressions for the velocity field and for the components of the extra-stress tensor were given by Eqs. (27), (20), (21) and (25). An application for the analytic solution (see Eqs. (27), (20), (21) and (25)) was given by simulating the problem of the flow in a 2D channel governed by SPTT Model and comparing the numerical results with the respective analytic solution. The numerical results were obtained with the GENSMAC-SPTT method which was implemented into the *Freeflow2D* code. The behaviour of the variables, u, S^{xy} and S^{xx} was analysed at the cross-section of the channel (middle of channel) and the numerical results were compared with the analytic solutions and very good agreement was obtained. In addition, mesh refinement was performed which showed the convergence of the numerical method.

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8. References

Alves, M. A., Oliveira, P. J., and Pinho, F. T., 2003, "A Convergent and Universally Bounded Interpolation Scheme for the Treatment of Advection", International Journal for Numerical Methods in Fluids, Vol. 41, pp. 47–75.

Alves, M. A., Pinho, F. T., and Oliveira, P. J., 2001, "Study of Steady Pipe and Channel Flows of a Single-Mode Phan-Thien–Tanner Fluid", Journal of Non-Newtonian Fluid Mechanics, Vol. 101, pp. 55–76.

Batchelor, G. K., 1967, "An introduction of fluid dynamics", Cambridge University Press, Cambridge.

Bird, R. B., Armstrong, R. C., and Hassager, O., 1977, "Dynamics of Polymetric Liquids", Vol. 1, Wiley, New York. Fluid Mechanics.

Oliveira, J. and Castelo, A., 1999, "Desenvolvimento de um Sistema de Simulação de Escoamentos de Fluidos com Superfícies Livres Bidimensionais", Master's thesis, ICMC - USP de São Carlos.

Oliveira, P. J. and Pinho, F. T., 1999, "Analytical Solution for Fully Developed Channel and Pipe Flow of Phan-Thien–Tanner Fluids", Journal of Fluid Mechanics, Vol. 387, pp. 271–280.

Paulo, G. S., Tomé, M. F., Federson, F. M., Castelo, A., and Cuminato, J. A., 2004, "Numerical Solution of Viscoelastic Free Surface Flows Using the PTT Constitutive Equation", Proceedings of 10th Brazilian Congress of Thermal Engineering and Sciences, Rio de Janeiro, Brazil, CD-ROM.

Phan-Thien, N., 1978, "A Nonlinear Network Viscoelastic Model", Journal of Rheology, Vol. 22, No. 3, pp. 259–283.

Phan-Thien, N. and Tanner, R. I., 1977, "A New Constitutive Equation Derived from Network Theory", Journal of Non-Newtonian Fluid Mechanics, Vol. 2, pp. 353–365.

Pinho, F. T., Alves, M. A., and Oliveira, P. J., 2003, "Benchmark Solutions for the Flow of Oldroyd-B and PTT Fluids in Planar Contractions", Journal of Non-Newtonian Fluid Mechanics, Vol. 110, pp. 45–75.

Rajagopalan, D., Armstrong, R. C., and Brown, R. A., 1990, "Finite Element Methods for Calculation of Steady Viscoelastic Flow Using Constitutive Equations with a Newtonian Viscosity", Journal of Non-Newtonian Fluid Mechanics, Vol. 36, pp. 159–192.

Tadmor, Z. and Gogos, C. G., 1979, "Principles of Polymer Melts and Solutions", Vol. 1, Wiley, Boston. Fluid Mechanics.

Tomé, M. F., Mangiavacchi, N., Castelo, A., Cuminato, J. A., and McKee, S., 2002, "A Finite Difference Techique for Simulating Unsteady Viscoelastic Free Surface Flows", Journal of Non-Newtonian Fluid Mechanics, Vol. 106, pp. 61–106.

Xue, S. C., Phan-Thien, N., and Tanner, R. I., 1998, "Three-dimensional Numerical Simulation of Viscoelastic Flows through Planar Contractions", Journal of Non-Newtonian Fluid Mechanics, Vol. 74, pp. 195–245.

Xue, S. C., Tanner, R. I., and Phan-Thien, N., 1999, "Three-dimensional Numerical Simulation of Viscoelastic Flows - Predictability and Accuracy", Computer Methods in Applied Mechanics and Engineering, Vol. 180, pp. 305–331.