# MODELING AND CONTROL OF AN EULER-BERNOULLI BI-SUPPORTED BEAM

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**Abstract.** *This work will present the modeling of a bi-supported Euler-Bernoulli beam with the dynamics of the sensors incorporated in the beam, and the vibration control of the resulting system using a computer. The generalized Lagrange equations were used for the modeling and three assumed modes were adopted for discretization of the motion equation obtained. As control strategy, the techniques of pole allocation and LQR were applied. Experimental results using those techniques are presented, using a bi-supported beam assembled in the laboratory of the Mechanics and Aeronautics Department of the ITA*

***Keywords***: *modal control, flexible structures, control of structures.*

## 1. Introduction

Active control of vibrations, among other applications, is used in attitude control of satellites and spatial artifacts, reduction of internal vibrations of vehicles in general, increasing comfort and safety.

From the theoretical viewpoint, flexible systems request an infinite number of elastic modes to describe its dynamics; in practice, these systems can be modeled by using a finite number of such modes. In this work, we used the first three modes to model a flexible aluminum beam of 0.6 m long, 0.3 m wide and 0.003 m thick in a bi-supported configuration. Near the extremities of this beam two PZT (piezoelectric tablets type QP10N), were bonded, which together with a PZT accelerometer (B&K 4371), were included in the dynamics of the system. Figure 1 presents an outline of the system.

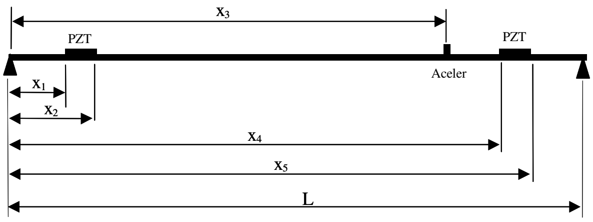


Fig 1 - Bi-supported beam

The transducers positions used in this experiment is given in Table 1

Table 1 position of transducers

| position | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | L |
|---|---|---|---|---|---|---|
| (m) | 0,05 | 0,10 | 0,49 | 0,47 | 0,51 | 0,60 |

In section 2, the modeling of an Euler-Bernoulli bi-supported beam is developed, obtaining the state equations of the flexible system; in section 3 the control strategies used are presented; in section 4, simulation results are presented; in section 5, experimental results are presented and finally in section 6 the work conclusions are presented.

## 2. Modeling of the System

The motion equations of this system (beam-transducers) were obtained through the Lagrange equations (Junkins and Kim, 1993), in which only the inertial properties and distributed rigidity of the beam and PZT transducers were considered. The damping of the system was not considered. The Lagrange of this system is given by $L=T\text{-}V$, where $T$ and $V$ are respectively the kinetic and potential energy of beam and transducers, as follows:

$$T = T_v + T_{ap} + T_{cp} + T_{ac} \tag{1}$$

$$V = V_v + V_{ap} + V_{cp} \tag{2}$$

where $T_v$, $T_{ap}$, $T_{cp}$, $T_{ac}$ are respectively the kinetic energy of the beam, disturbance PZT tablet, control PZT tablet and PZT accelerometer ; $V_v$, $V_{ap}$ *e* $V_{cp}$ are respectively the potential energy of the beam, PZT disturbance tablet and PZT control tablet.

The values of these energies, already properly discretized by the assumed mode technique through expression (3), below

$$w(x,t) = \sum_{i=1}^{n} \phi_i(x) q_i(t) \tag{3}$$

and mode shape functions

$$\phi_i(x) = sen \frac{i\pi x}{l} \tag{4}$$

are given by

$$\begin{aligned} T(t) &= \frac{1}{2}\int_0^l m_v \left(\frac{\partial w(x,t)}{\partial t}\right)^2 dx + \\ &+ \frac{1}{2}\int_0^l m_{ap} \left(\frac{\partial w(x,t)}{\partial t}\right)^2 [u(x-x_1) - u(x-x_2)]dx + \\ &+ \frac{1}{2}\int_0^l m_{cp} \left(\frac{\partial w(x,t)}{\partial t}\right)^2 [u(x-x_1) - (x-x_2)]dx + \\ &+ \frac{1}{2}\int_0^l m_{sa} \left(\frac{\partial w(x,t)}{\partial t}\right)^2 \delta(x-x_3)dx \end{aligned} \tag{5}$$

$$\begin{aligned} V(t) &= \frac{1}{2}\int_0^l EI_v \left(\frac{\partial^2 w(x,t)}{\partial t}\right)^2 dx + \\ &\frac{1}{2}\int_0^l EI_{ap} \left(\frac{\partial^2 w(x,t)}{\partial t}\right)^2 [u(x-x_1) - u(x-x_2)]dx + \\ &\frac{1}{2}\int_0^l EI_{cp} \left(\frac{\partial^2 w(x,t)}{\partial t}\right)^2 [u(x-x_4) - u(x-x_5)]dx \end{aligned} \tag{6}$$

where $\phi_i(x)$ and $q_i(t)$ are modal amplitudes depending on space and time, respectively; $w(x,t)$ is the transverse displacement of the beam; $m_v$, $m_{ap}$ and $m_{cp}$ are respectively the linear mass density of the beam, PZT disturbance tablet, and PZT control tablet; the parameters $EI_v$, $EI_{ap}$ and $EI_{cp}$ are, respectively, bending stiffness coefficient of the beam, PZT disturbance tablet and PZT control tablet.

The control action of the system results from $M_{cp}$, that is the torque induced by the PZT control tablet; this control action (Wang and Rogers, 1989) is given by the expression below

$$\{F(x,t)\} = \frac{\partial^2 M_{cp}(x,t)}{\partial x^2} = M_{cp}[u'(x-x_4) - (x-x_5)] \tag{7}$$

In the expressions (5), (6) and (7) the *u(.)* symbol represents the unitary step function.

From expressions (5) and (6), after solving the temporal and spatial derivatives, the compact expressions of the kinetic and potential energies, respectively (Junkins e Kim, 1993), are obtained as follows:

$$T = \frac{1}{2}\dot{q}_i(t)Mq_i(t) \tag{8}$$

$$V(t) = q_i^T(t)Kq(t) \tag{9}$$

where *M* and *K* are, respectively, mass and rigidity matrix of the beam.

Substituting expressions (8), (9) and the left hand term of expression (7) in the Lagrange equations (Junkins and Kim, 1993 and Clark, Saunders and Gibbs, 1998), the motion equation of the flexible system is obtained:

$$M\ddot{q}(t) + Kq(t) = F^T(t) \tag{10}$$

The representation of the system in state space, $\dot{x} = Ax + Bu$ e $y = Cx$, is obtained from expression (10) (Clark Sounders e Gibbs, 1998), as follows:

$$[A] = \begin{bmatrix} [0]_{3x3} & [I]_{3x1} \\ [-M^{-1}K]_{3x3} & [0]_{3x1} \end{bmatrix} \tag{11}$$

$$[B] = \begin{bmatrix} [0]_{3x3} \\ [M^{-1}]_{3x3} \end{bmatrix} \tag{12}$$

$$[C] = \begin{bmatrix} [\Phi_{ac}]_{3x1} \\ [0]_{3x1} \end{bmatrix} \tag{13}$$

$$\{x\} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} \tag{14}$$

where *[A], [B]* and *[C]* are the matrices of the beam, control and observation, respectively; and *{x}* is the state vector of the system. In the output matrix $[\Phi_{ac}]$ expresses the spatial dependence of the accelerometer action.

## 3. Controller and Observers Deign

In this section two techniques of controller and observer design will be introduced. The first one consists of a controller and observer using poles allocation that will be presented in section 3.1. The second technique uses a LQR type controller and a state estimator through Kalman filter that will be presented in section 3.2.

Figure 2 shows the closed loop block diagram of the complete system, where both technique approaches for the beam control are used.

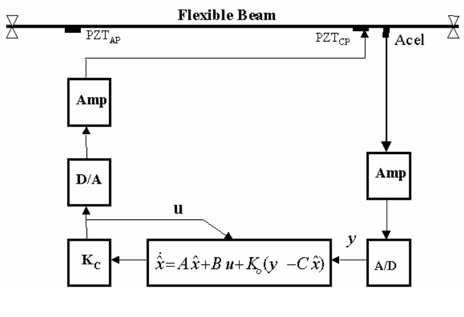


Fig 2- Block Diagram of the System

where:

*Amp* is the power amplifier;

*A/D* and *D/A* are the analogical to digital and digital to analogical converters, respectively;

$K_c$ and $K_o$ are the gains of the control and observation matrices, respectively;

$u$ is the control signal;

$\hat{x}$ is the state estimated value and

$y$ is the signal measured by the accelerometer.

### 3.1. Controller and Observer Design Using Pole Allocation Technique

The controller and observer gain matrices using pole allocation were obtained by expressions (Ogata, 1998):

$$Kc = [0...1][B \vdots AB \vdots \cdots \vdots A^5 B]^{-1} \Phi(A) \tag{15}$$

$$K_o = \Phi(A)[C \vdots CA \vdots \cdots \vdots CA^5 ][0 \cdots 1]^{-1} \tag{16}$$

where $\Phi(A)$ is the characteristic equation of the system that is satisfied by the matrix $A$ itself of the system (given by the Cayley-Hamilton theorem).

The observer expression is given by the principal block in Fig. 2 and is rewritten as:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_o( y - C\hat{x}) \tag{17}$$

The control expression is present in Fig. 2 under the form:

$$u = -K_c \hat{x} \tag{18}$$

From (17) and (18) results the final observer dynamics without the control signal, given by:

$$\dot{\hat{x}} = (A - BK_c - K_o C)\hat{x} + K_o y \tag{19}$$

This last expression after discretization is used in the control algorithm treated in section 4 of this paper.

### 3.2. Design of the LQR controller and of the State Estimator Using Kalman Filter

Let us consider the system in the form:

$$\dot{x} = Ax + Bu + Gw \tag{21}$$

The gain of the LQR optimal control law that minimizes the functional cost $J$, is given by equation:

$$J = E\left\{ \frac{1}{2} \int_{t_0}^{t_f} \left[ x^T Qx + u^T Ru \right] dt \right\} \tag{21}$$

where $Q$ is a real symmetric and positive semi defined matrix that defines the state cost, $R$ is a real symmetric and positive defined matrix that defines the control cost and $E$ is the mathematical expectancy operator, is given by (Skogestad and Postlethwaite, 1996)

$$u = -Kx \tag{22}$$

where $K$ is the control gain matrix:

$$K = R^{-1} B^T S \tag{23}$$

In this work the *steady state* version of the controller gain will used, in other words, the $S$ matrix is obtained through the solution of the expression:

$$0 = A^T S + SA + Q - SBR^{-1}B^T S \tag{24}$$

To estimate the states, a Kalman filter was used, considering the system with a Gaussian white noise $w(t)$ in the state, with a positive defined noise covariance matrix $P_w$, that means,

$$\dot{x} = Ax + Bu + Gw \tag{25}$$

where $G$ is the noise matrix of the state, with the output equation

$$y = Cx + Fv \tag{26}$$

that represents a sensor with gaussian white noise $v(t)$, and $F$ is the observation noise matrix

The estimator is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{27}$$

where $L$ is the gain matrix:

$$L = PC^T P_v \tag{28}$$

and $P_v$ is a positive defined matrix relative to the observation noise covariance. In this work the *steady state* version of the observer will be used, or in other words, a covariance matrix of the estimation error $P$ is given by the solution of the expression:

$$0 = AP + PA^T + GP_w G^T - PC^T P_v^{-1} CP \tag{29}$$

## 4. Simulation Results

In this section the simulation results of the model obtained in section 2 are presented, using a sampling time of $0.001$s. Figure 3 shows the open loop answer of the model excited with a $\pm 2$ *volt* double signal and $0.1s$ time duration. Figure 4 displays the model excited with the same double signal used upper and controlled by the *pole allocation* technique. Figure 5 shows the result attained by the control using the LQR technique is displayed. Figures 6 and 7 display the control signals generated using poles allocation and LQR, respectively

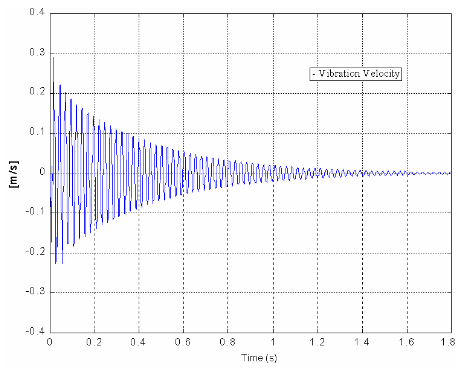


Fig 3 – Response of the model to the double excitation without control

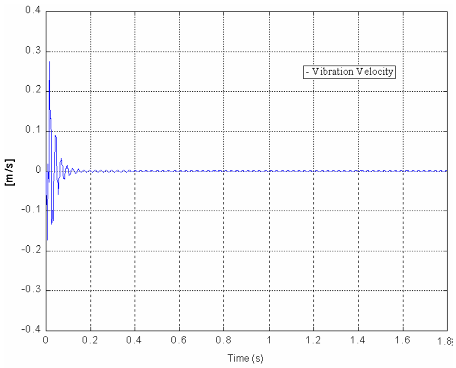


Fig 4 – Response of the model to the double excitation with pole allocation control

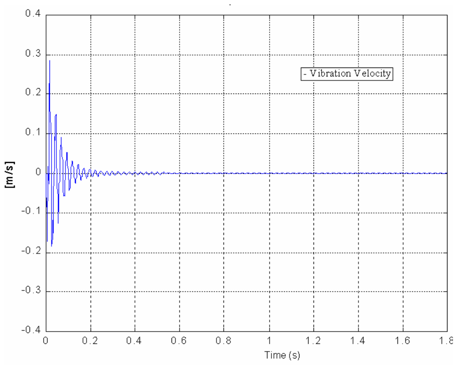


Fig 5 – Response of the model to the double excitation with LQR control

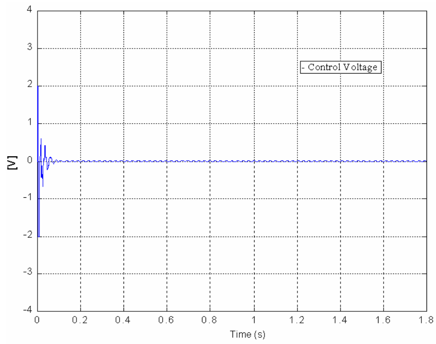


Fig 6 – Applied control generated by the pole allocation strategy

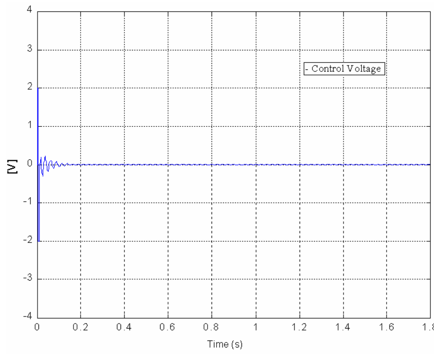


Fig 7 – Applied control generated by the LQR strategy

## 5. Experimental Results

Using a computer program two active vibration controls of the beam-transducer system were achieved, damping the first three vibration modes of this system; the first control was developed by the pole allocation technique through Ackermann's formula and the second one by the LQR technique. The higher vibration modes were neglected.

The experimental results were obtained using the control algorithms in a C language program on a Pentium 100, equipped with a *DT 300 AD/DA* board; this board is connected to the physical system (beam-transducers), mounted in the laboratory of the Mechanic and Aeronautic Department of the Aeronautic Technologic Institute (ITA).

Figure 8 shows the arithmetic average of the system outputs for sixteen pulses generated by an electromechnical actuator located in the center of the beam, with the control turned off. After this, with the pole allocation control activated the same experiment was repeated, obtaining the response presented in Fig. 9; the beam response for the same pulse conditions but now using a *LQR* control is presented in Fig. 10.

All recorded graphics presented in this work, as well as all signal treatments were obtained with the *Hp35665* analyzer supplied by Hewlett Packard.

It is observed that both controllers attenuated the beam vibrations to levels lower than *7mm/s* in approximately *0.4s*, while the response of the beam to the same excitation with the open loop shows that 1.8 seconds are needed to attain the same attenuation. Figures 11 and 12 present controls signals generated by pole allocation and LQR controls respectively.

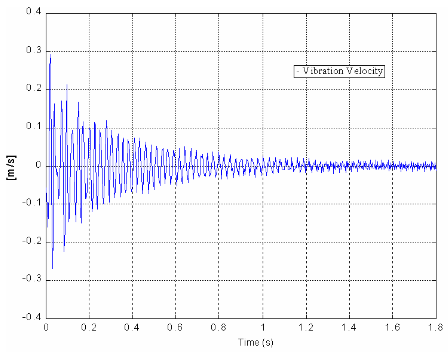


Figure 8 – Beam response to pulses without control

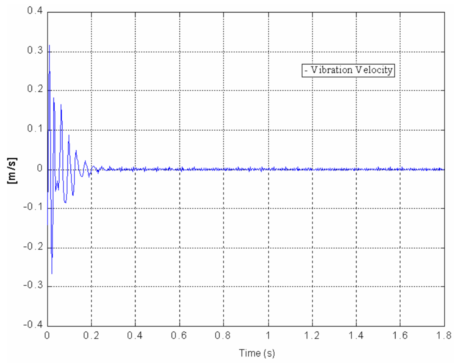


Figure 9 – Beam response to pulses with pole allocation control activated

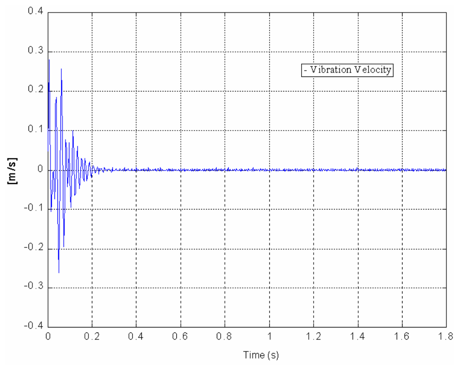


Figure 10 - Beam response to pulses with LQR control activated

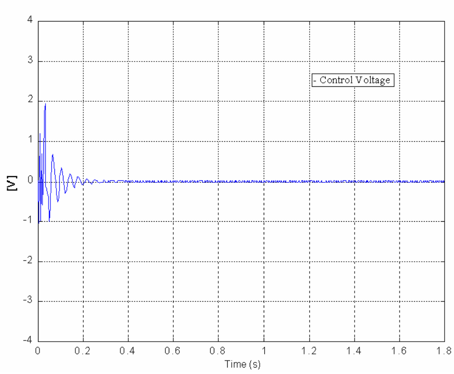


Figure 11 - Signal control generated by pole allocation strategy applied to

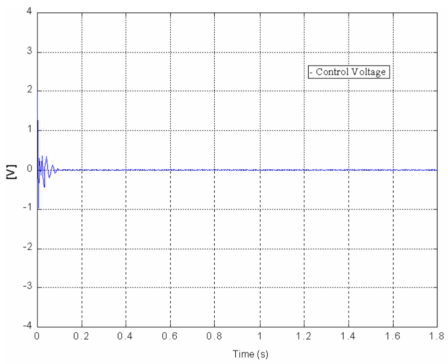


Figure 12 - Signal control generated by LQR strategy applied to the PZT

## 6. Conclusion

The experimental results show that the model obtained is very close to the real model; they can be seen comparing Figs. 3 and 8 that show reasonable similarity. We emphasize that the model and the beam were excited with the same *±2 volts* double signal with *0.1s* time duration.

In this work, both techniques presented very close results, as shown in Figs. 9 and 10; it was observed that the LQR control presents a larger control facility rather than pole allocation control only during the design, when we need introduce the control parameters. The advantage of using LQR control, therefore, is the speed to obtain its design, while in the case of the pole allocation controller, several trials were necessary until a controller with a performance compatible with the LQR controller performance was obtained.

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