A MIXED MODEL AND AN ALGORITHM FOR VISCOPLASTICITY

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Abstract. *The main objective of this paper is to propose a mixed variational formulation (Hellinger-Reissner) and mixed finite element with a solution algorithm for elasto/viscoplasticity problems, where the rheologic phenomenon occurs only after the elastic phase. With the mixed formulation proposed, we present a finite element where we impose continuity and quadratic interpolation for velocities and geometry, but we allow an element discontinuity for stresses rates, stresses, viscoplastic strain rates, equivalent viscoplastic strain rates and accumulated viscoplastic strain. We present a solution algorithm for discrete problem based on Newton-Raphson method with global iterations for the constitutive equations and one step Euler scheme for the flow laws integration. The element discontinuity for the fields cited above has as consequence the constitutive matrix decoupled by triangle vertex. This fact is important for the computational viability, because we only have to invert a [3x3] or [4x4] matrix for stress or strain plane states, respectively. A numerical application is presented using the Von Mises criteria and the Perzyna constitutive model for isotropic hardening materials (softening). The result is compared with numerical solution in the literature.*

Keywords: *viscoplasticity, finite element, mixed model.*

1. Introduction

The main objective of this paper is to propose a mixed variational formulation (Hellinger-Reissner) and mixed finite element with a solution algorithm for elasto/viscoplasticity problems, where the rheologic phenomenon occurs only after the elastic phase.

As it is well known, viscoplasticity in metals is an important phenomenon when the absolute temperature exceeds one third of the absolute melting temperature. Nevertheless, because certain important materials also exhibit rate dependent deformation behavior at moderate temperatures, there exists an increasing interest in that deformation process not only under high temperatures. Also, the failure process in many engineering problems can be model by adopting the constitutive equation related to these time-dependent plasticity materials. As examples of these processes the propagation of Lüders bands and Portevin Le Chatelier effects in metals and shear banding and creep in geo-materials are mentioned (Heeres et al., 2002).

Viscoplasticity in its own right is the interest of this paper, but it is worth remembering that viscoplasticity can also be used to generate plasticity solutions. Particulary, it is very efficient as a regularized model to describe rate-independent plasticity in process in which the pure plasticity models usually fail. For example, for the above refereed processes, mainly in perfect plasticity materials, or in strain softening situation, in which strain localization is present (Díez et al., 1998, Sluys, 1998).

A triangular finite element with quadratic and continuous interpolations for velocities and geometry and linear discontinuous interpolations for stress rate and viscoplastic strain rate is proposed. The element, herein proposed in the elasto/viscoplasticity context, comes from large experience with it in limit analysis applications and thermo-elasticity in incompressible materials (Costa and Borges, 2002, Borges et al., 1995).

The outline of this paper is as follows. The variational principle to describe the infinitesimal elasto/viscoplasticity problems is proposed in Section **2**. The solution of the equation system, defined by the equilibrium, kinematics, constitutive equation and the flow laws are optimality conditions of an *inf-sup* mixed variational principle. In Section **3**, based on space discretization generated by the finite element method and on the variational formulation of Section **2**, the discrete mixed principle is presented. Finally, in Section **4** it is shown the solution algorithm for the discrete problem. In Section **5** a numerical application is presented to validate the formulation.

2. Mixed variational principle for infinitesimal elasto/viscoplasticity

The objective of this Section is to propose a variational principle to describe elasto/viscoplasticity problems. The field solution of the equation system, defined by equilibrium, kinematics, constitutive equations and flow laws (Costa, 2004 and Costa and Borges, 2003), are optimality conditions of an *inf-sup* mixed variational principle (Hellinger-Reissner).

The elasto/viscoplasticity problem consists in determining paths of displacement $\mathbf{u}(t)$, stress $\mathbf{T}(t)$ and strain $\mathbf{E}(t)$, developed in an elasto/viscoplasticity body during a load program. If, at a moment t of the process, one considers all state variable, total strain $\mathbf{E}$, viscoplastic strain $\mathbf{E}_{vp}$, internal variable $\boldsymbol{\chi}$ and temperature variation Θ fields as known, then from the constitutive relations, (Costa, 2004 and Costa and Borges, 2003), the dual variables $(\mathbf{T}, \boldsymbol{\mathcal{A}})$ might be able to be determined. Therefore, the next step will be obtaining the stress rate, strain rate, internal variable rate and temperature variation rate fields that occur in the body when it is submitted to variation in force system $\mathbf{F}$ or/and in the displacement constraints $\bar{\mathbf{u}}$, during a time interval dt.

In turn, at each moment, this problem consists in finding a stress rate field $\dot{\mathbf{T}} \in W'$, a kinematic hardening rate field $\dot{\boldsymbol{\chi}}_{kin} \in I\!R^n \,\mathrm{x}\, I\!R^n$, a isotropic hardening rate field $\dot{\boldsymbol{\chi}}_{iso} \in I\!R$, a strain rate field $\dot{\mathbf{E}} \in W$, a viscoplastic strain rate field $\dot{\mathbf{E}}_{vp} \in W$ and a velocity field $\mathbf{v} \in V$, such as the following equation system holds

$$(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \in \nabla_{(\mathbf{T},\mathcal{A})} D^c_{vp}(\mathbf{T}, \boldsymbol{\mathcal{A}}) \tag{1}$$

$$\dot{\mathbf{E}} = \mathcal{D}\,\mathbf{v} \tag{2}$$

$$\left\langle \dot{\mathbf{T}}\,,\, \mathcal{D}\,(\mathbf{v}^* - \bar{\mathbf{v}}) \right\rangle = \left\langle \dot{\mathbf{F}}\,,\, (\mathbf{v}^* - \bar{\mathbf{v}})\right\rangle \quad \forall\, \mathbf{v}^* \in V \tag{3}$$

$$(\dot{\mathbf{T}}, -\dot{\boldsymbol{\mathcal{A}}}) = \nabla_{(\dot{\mathbf{E}},\dot{\chi})} \mathcal{J}(\dot{\mathbf{E}}, \dot{\boldsymbol{\Theta}}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}}) \qquad \Longleftrightarrow \qquad (\dot{\mathbf{E}}, -\dot{\boldsymbol{\chi}}) = \nabla_{(\dot{\mathbf{T}},\dot{\mathcal{A}})} \mathcal{J}^c(\dot{\mathbf{T}}, \dot{\boldsymbol{\Theta}}, \dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\mathcal{A}}}) \tag{4}$$

The Equation (1) represents the flow laws defining the viscoplastic strain rate and the hardening rate parameters. The second Equation (2) represents the kinematic principle. The equilibrium principle is defined by Eq. (3). The fields $(\dot{\mathbf{T}}, \dot{\boldsymbol{\mathcal{A}}})$, solutions for this system, are associated with the total and viscoplastic strain rate fields and with the kinematic and isotropic hardening rate fields by the constitutive relation Eq. (4).

For a tridimensional continuum, under infinitesimal strain assumption, the tangent deformation operator $\mathcal{D}$, matches the symmetric part of the gradient ∇^s. The parameter $\bar{\mathbf{v}}$ is an element of the prescribed velocity field Γ_v. The potentials $\mathcal{J}$ and $\mathcal{J}^c$ and the dual dissipated function D^c_{vp} that defines the internal variable evolutive laws are defined in Costa (2004) and Costa and Borges (2003).

In Costa (2004) and Costa and Borges (2003) one can see that a solution for this system is also the solution for a mixed variational principle, defined in function of velocity and stress rate fields, which is denoted as the Hellinger-Reissner Principle, that is

Find $\mathbf{v} \in V$, $\dot{\mathbf{T}} \in W'$ such that

$$\begin{aligned}\widehat{\Pi}^{HR}(\mathbf{v}, \dot{\mathbf{T}}) = \inf_{\mathbf{v}^* \in V^0} \sup_{\dot{\mathbf{T}}^* \in W'} \Big[&-\frac{1}{2}\big\langle \dot{\mathbf{T}}^*, I\!\!D^{-1}\dot{\mathbf{T}}^*\big\rangle + \big\langle \dot{\mathbf{T}}^*, \mathcal{D}\mathbf{v}^*\big\rangle - \big\langle \dot{\mathbf{T}}^*, \dot{\mathbf{E}}_{vp}\big\rangle - \\ &-\big\langle \alpha\,\dot{\Theta}, \mathrm{tr}(\dot{\mathbf{T}}^*)\big\rangle + \frac{1}{2}\big\langle \mathbf{H}\dot{\boldsymbol{\chi}}, \dot{\boldsymbol{\chi}}\big\rangle + \big\langle \dot{\mathbf{T}}^*, \mathcal{D}\bar{\mathbf{v}}\big\rangle - \big\langle \dot{\mathbf{F}}, \mathbf{v}^*\big\rangle \Big]\end{aligned} \tag{5}$$

where $(\dot{\mathbf{E}}_{vp}, \dot{\boldsymbol{\chi}})$ complying with the flux law Eq. (1) and $V^0 = \{\mathbf{v}^* \in [H^1(\mathcal{B})]^3 \;/\; \mathbf{v}^*|_{\Gamma_u} = \mathbf{0}\}$.

3. Finite element models for mixed formulation

In this Section a brief description of the discretization procedure is presented for the purpose of characterizing the structure of the discrete viscoplastic problem arising from the mixed principle which was presented in Section **2.** Finite element models are considered for plane stress and plane strain conditions in bodies composed by materials obeying the von Mises yield criterion and Perzyna-like viscoplastic model. For the sake of brevity, only the isothermal process and isotropic hardening materials are considered.

3.1 Two-dimensional models

In two-dimensional models the deformation process can be described by means of the velocity field and a scale parameter $\dot{\bar{\epsilon}}_{vp}$ represented the equivalent viscoplastic strain rate

$$\mathbf{v} = [\,v_x \quad v_y\,]^T \qquad \text{and} \qquad \dot{\bar{\epsilon}}_{vp} = [\dot{\bar{\epsilon}}_{vp}] \tag{6}$$

For the model adopted $\dot{\bar{\epsilon}}_{vp} = -\dot{\boldsymbol{\chi}}_{iso}$ (Costa, 2004). Under plane stress condition the other comprised fields are

$$\dot{\mathbf{T}} = [\,\dot{T}_x \quad \dot{T}_y \quad \sqrt{2}\,\dot{T}_{xy}\,]^T \qquad \dot{\mathbf{E}} = [\,\dot{E}_x \quad \dot{E}_y \quad \sqrt{2}\,\dot{E}_{xy}\,]^T \qquad \dot{\mathbf{E}}_{vp} = [\,\dot{E}_{vpx} \quad \dot{E}_{vpy} \quad \sqrt{2}\dot{E}_{vpxy}\,]^T \tag{7}$$

and for the plane strain are

$$\dot{\mathbf{T}} = [\,\dot{T}_x\; \dot{T}_y\; \dot{T}_z\;\; \sqrt{2}\,\dot{T}_{xy}\,]^T \qquad \dot{\mathbf{E}} = [\,\dot{E}_x\; \dot{E}_y\; 0\;\; \sqrt{2}\,\dot{E}_{xy}\,]^T \qquad \dot{\mathbf{E}}_{vp} = [\,\dot{E}_{vpx}\; \dot{E}_{vpy}\; \dot{E}_{vpz}\; \sqrt{2}\dot{E}_{vpxy}\,]^T \tag{8}$$

where $\dot{\mathbf{T}}$, $\dot{\mathbf{E}}$ and $\dot{\mathbf{E}}_{vp}$ are vectors which represent stress rate, total strain rate and viscoplastic strain rate fields, respectively.

Because of the vector representation of the tensorial fields the deformation operators for plane stress and plane strain states are set as

$$\mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} \end{bmatrix} \quad \text{and} \quad \mathcal{D} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ 0 & 0 \\ \frac{1}{\sqrt{2}}\frac{\partial}{\partial y} & \frac{1}{\sqrt{2}}\frac{\partial}{\partial x} \end{bmatrix} \tag{9}$$

In the notation of these two-dimensional models and with the isotropic hardening hypothesis, the Von Mises yield function $f(\mathbf{T}, \bar{\epsilon}_{vp})$ is written as

$$f(\mathbf{T}, \bar{\epsilon}_{vp}) = \sqrt{\frac{3}{2}}\,\|\mathbf{S}\| - \sigma_Y - \mathcal{A}_{iso}(\bar{\epsilon}_{vp}) \qquad \|\mathbf{S}\| = \sqrt{\frac{1}{2}\,\mathbb{C}\,\mathbf{T}\cdot\mathbf{T}} \tag{10}$$

where σ_Y is the material yield limit in pure traction and $\mathbf{S}$ is the deviatoric part of the tensor $\mathbf{T}$. the function $\mathcal{A}_{iso}(\bar{\epsilon}_{vp})$ is given by a non-linear function (Simo and Hughes, 1997)

$$\mathcal{A}_{iso}(\bar{\epsilon}_{vp}) = (1-\beta)\,H\,\bar{\epsilon}_{vp}^{\,\xi} + (\sigma_\infty - \sigma_Y)\,(1 - \exp^{-\delta\,\bar{\epsilon}_{vp}}) \tag{11}$$

where $\beta \in (0,1)$, $\xi \geq 0$, $\sigma_\infty \geq 0$ and $\delta \geq 0$ are material properties. When $H > 0$ the parameter express the hardening material; when $H < 0$, the softening material. It is important to observe for $\delta = 0$ and $\xi = 1$ this law is reduced to a linear hardening law.

For plane stress and plane strain state the matrix $\mathbb{C}$ is set, respectively, as

$$\mathbb{C} = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 4/3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \mathbb{C} = \begin{bmatrix} 4/3 & -2/3 & -2/3 & 0 \\ -2/3 & 4/3 & -2/3 & 0 \\ -2/3 & -2/3 & 4/3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \tag{12}$$

For Mises criterion the yield function is regular and if, additionally, an exponential law for the Perzyna model is adopted (Alfano et al., 2001, Angelis, 2000 and Perzyna, 1998), the evolution relation can be written as

$$\dot{\mathbf{E}}_{vp} = \dot{\lambda}(\mathbf{T}, \bar{\epsilon}_{vp})\,\nabla_{\mathbf{T}} f(\mathbf{T}, \bar{\epsilon}_{vp}) \qquad \text{and} \qquad \dot{\bar{\epsilon}}_{vp} = \sqrt{\frac{2}{3}}\,\|\dot{\mathbf{E}}_{vp}\| \tag{13}$$

where

$$\dot{\lambda}(\mathbf{T}, \bar{\epsilon}_{vp}) = \eta \left(\frac{f^+(\mathbf{T}, \bar{\epsilon}_{vp})}{\sigma_Y} \right)^n \qquad \text{and} \qquad \nabla_{\mathbf{T}} f = \frac{1}{2}\sqrt{\frac{3}{2}}\,\frac{\mathbb{C}\,\mathbf{T}}{\|\mathbf{S}\|} \tag{14}$$

in which n is a material property and $\eta \in (0, \infty)$ represents the viscosity coefficient.

3.2 Mixed discretization

Here a general procedure for the discretization of the mixed formulation Eq. (5) is discussed and some the particular features of proposed mixed triangular are emphasized. A curved triangular mixed element, denoted V2T1, is proposed (Costa and Borges, 2002 and Borges et al., 1995), having six nodes intended for the C^0 quadratic interpolation of geometry and velocities and three nodes, at vertices, for the discontinuous linear interpolation of viscoplastic strain rates and stress rates. The viscoplastic strain rates are imposed locally in the element vertices by the collocation method. The adoption of the discontinuous linear interpolation of viscoplastic strain rates in continuum field is chosen for feasibility algebraic operations.

Hereinafter, the following notation is adopted: a superimposed hat is used to distinguish variables or parameters of the continuum model from their discrete counterparts. A discrete version for the flow laws are obtained by the collocation method, that is, a set of points are chosen in each element to enforce this rule. The points selection is addressed by the yield function behavior. It is because the convexity of the yield function $f(\widehat{\mathbf{T}}(x), \widehat{\bar{\epsilon}}_{vp}(x))$ and the piecewise linear interpolation assumed for the stress field $\widehat{\mathbf{T}}(x)$ and the accumulated viscoplastic strain $\widehat{\bar{\epsilon}}_{vp}(x)$ assure that the vertices of the triangles are the points in which the Mises yield function may be maximum. Therefore, the vertices are the natural chosen points to impose the flow laws. As a consequence, the vector $\dot{\mathbf{E}}^e_{vp}$ is assembled from three disjoint vectors $\dot{\mathbf{E}}^{ek}_{vp}$, which represents the viscoplastic strain rate at each vertex, and are determined by

$$\dot{\mathbf{E}}^e_{vp} = [\dot{\mathbf{E}}^{ek}_{vp}] \equiv \Lambda^e\, \mathcal{G}^e \qquad k = 1,2,3 \tag{15}$$

where the diagonal matrix Λ^e is defined by

$$\Lambda^e = \text{diag}\left[\Lambda^{ek}\right] \qquad \text{and} \qquad \Lambda^{ek} = \text{diag}\left[\dot{\lambda}(\mathbf{T}^{ek}, \bar{\epsilon}^{ek}_{vp})\right] \qquad k = 1,2,3 \tag{16}$$

with $\dot{\lambda}(\mathbf{T}^{ek}, \bar{\epsilon}^{ek}_{vp})$ defined by Eq. $(14)_1$ for Perzyna model. The vector $\mathcal{G}^e$ is defined by

$$\mathcal{G}^e = \left[\mathcal{G}^{ek}\right] \qquad \text{and} \qquad \mathcal{G}^{ek} = \nabla_{\mathbf{T}} f^+(\mathbf{T}^{ek}, \bar{\epsilon}^{ek}_{vp}) \qquad k = 1,2,3 \tag{17}$$

with the vector $\nabla_{\mathbf{T}} f(\mathbf{T}^{ek}, \bar{\epsilon}^{ek}_{vp})$ defined by Eq. $(14)_2$. The parameters $\mathbf{T}^{ek}$ and $\bar{\epsilon}^{ek}_{vp}$ are, respectively, the stress and the accumulated viscoplastic strain parameters at each vertex of element e. Notice that because one can regard inter-element stress and accumulated viscoplastic strain discontinuities and the coordinates x^k as coinciding with the vertices coordinates, the vector $\dot{\mathbf{E}}^{ek}_{vp} \in I\!R^{\hat{q}}$ is only dependent on a separate set, $\mathbf{T}^{ek}$, elementary vector $\mathbf{T}^e$ components and dependent on a separate set $\bar{\epsilon}^{ek}_{vp}$, elementary vector $\bar{\epsilon}^e_{vp}$ components.

Coherently with the flow laws imposed in the vertex, the vector $\dot{\bar{\boldsymbol{\epsilon}}}^e_{vp}$ is assembled from three disjoint scalars $\dot{\bar{\epsilon}}^{ek}_{vp}$, which represents the equivalent viscoplastic strain rates at each vertex

$$\dot{\bar{\boldsymbol{\epsilon}}}^e_{vp} = [\dot{\bar{\epsilon}}^{ek}_{vp}] \qquad \text{and} \qquad \dot{\bar{\epsilon}}^{ek}_{vp} = \sqrt{\frac{2}{3}}\, \|\dot{\mathbf{E}}^{ek}_{vp}\| = \sqrt{\frac{2}{3}}\, \|\Lambda^{ek}\, \mathcal{G}^{ek}\| \qquad k = 1,2,3 \tag{18}$$

where $\|\dot{\mathbf{E}}^{ek}_{vp}\|$ represents the module of the viscoplastic strain rate at each vertex of element e.

Finally, the hypothesis of isothermal process and isotropic hardening material with the substitution of the assumed interpolations in the continuum mixed principle Eq. (5) leads to its discrete version.

Find $\mathbf{v} \in \mathbb{R}^N$ e $\dot{\mathbf{T}} \in \mathbb{R}^q$ such that

$$\Pi^{HR}(\mathbf{v}, \dot{\mathbf{T}}) = \min_{\mathbf{v}^* \in \mathbb{R}^N} \max_{\dot{\mathbf{T}}^* \in \mathbb{R}^q} \left[-\frac{1}{2}\mathbb{D}^{-1}\dot{\mathbf{T}}^* \cdot \dot{\mathbf{T}}^* + \dot{\mathbf{T}}^* \cdot \mathbf{B}\mathbf{v}^* - \dot{\mathbf{F}} \cdot \mathbf{v}^* + \dot{\mathbf{T}}^* \cdot \bar{\mathbf{B}}\bar{\mathbf{v}} - \dot{\mathbf{T}}^* \cdot \mathbf{M}\dot{\mathbf{E}}_{vp} + \frac{1}{2}\mathbf{H}_{iso}\dot{\bar{\boldsymbol{\epsilon}}}_{vp} \cdot \dot{\bar{\boldsymbol{\epsilon}}}_{vp}\right] \tag{19}$$

where the flow laws are imposed at each element by Eqs. (15) and (18). The parameter N is the number of degrees of freedom in velocities, assuming that all rigid motions are ruled out by prescribed kinematic constraints. The deformation operator $\bar{\mathbf{B}}$ contains only the columns associated with the prescribed degrees. Additionally, the continuity for velocities and the inter-element discontinuity for stress rates, viscoplastic strain rates and equivalent viscoplastic strain rates are imposed by properly collecting the element vectors $\mathbf{v}^e$, $\dot{\mathbf{T}}^e$, $\dot{\mathbf{E}}^e_{vp}$ and $\dot{\bar{\boldsymbol{\epsilon}}}^e_{vp}$ in global vectors $\mathbf{v}$, $\dot{\mathbf{T}}$, $\dot{\mathbf{E}}_{vp}$ and $\dot{\bar{\boldsymbol{\epsilon}}}_{vp}$. The vectors $\dot{\mathbf{E}}^e_{vp}$ and $\dot{\bar{\boldsymbol{\epsilon}}}^e_{vp}$ are parameters, not model variables. The variable q represents the total number of stress parameters, consequently, $q = 3\,nel\ \hat{q}$, where nel is the total number of elements in the mesh.

The matrices $\mathbb{D}^{-1}$, $\mathbf{B}$, $\mathbf{M}$, $\mathbf{H}_{iso}$ and the vector $\dot{\mathbf{F}}$ are assembled from elementary contributions of

$$\mathbb{D}^{-1^e} = \int_{\mathcal{T}^e} \mathbf{N}_T^T\, \widehat{\mathbb{D}}^{-1}\, \mathbf{N}_T\; d\mathcal{T} \qquad \mathbf{B}^e = \int_{\mathcal{T}^e} \mathbf{N}_T^T\, \mathcal{D}\, \mathbf{N}_v\; d\mathcal{T} \qquad \mathbf{M}^e = \int_{\mathcal{T}^e} \mathbf{N}_T^T\, \mathbf{N}_T\; d\mathcal{T} \tag{20}$$

$$\mathbf{H}^e_{iso} = \text{diag}\left[\left(\frac{d\mathcal{A}_{iso}}{d\bar{\epsilon}_{vp}}\right)^{ek}\right] \qquad k = 1,2,3 \qquad \dot{\mathbf{F}}^e = \int_{\mathcal{T}^e} \mathbf{N}_v^T\, \dot{\mathbf{b}}\; d\mathcal{T} + \int_{\Gamma^e_\tau} \mathbf{N}_v^T\, \dot{\mathbf{a}}\; d\Gamma_\tau \tag{21}$$

where $\widehat{\mathbb{D}}^{-1}$ is the constitutive matrix constrained to the bidimensional model for the stress or strain states (Costa, 2004 and Costa and Borges, 2003). The functions $\mathbf{N}_v(x)$ and $\mathbf{N}_T(x)$ are, respectively, the matrices of quadratic and linear shape functions. The parameters $\dot{\mathbf{b}}$ and $\dot{\mathbf{a}}$ are body and surface load rates, respectively.

It is worthwhile mention that although the global matrices $\mathbb{D}^{-1}$, $\mathbf{B}$, $\mathbf{M}$ and $\mathbf{H}_{\text{iso}}$ are defined, they do not explicitly assembled through the interactive process. Due to the elementary uncoupling of the problem, only the elementary matrices, Eqs. (20) and (21), are calculated. The elementary uncoupling of stress rate, viscoplastic strain rate and equivalent viscoplastic strain rate have important consequences on the structure of these elementary matrices, being fundamental in the computational feasibility of the discrete algorithm developed to solve this problem.

For $\dot{\mathbf{F}} \in I\!R^N$, $\mathbf{v} = \bar{\mathbf{v}} \in \Gamma_v$, it is easy to show that the $min - max$ principle, Eq. (19), is equivalent to the solution of the following system

Find $\dot{\mathbf{T}} \in I\!R^q$ and $\mathbf{v} \in I\!R^N$, such that

$$\begin{aligned}
&\mathbb{D}^{-1}\,\dot{\mathbf{T}} - \mathbf{B}\,\mathbf{v} - \bar{\mathbf{B}}\,\bar{\mathbf{v}} + \mathbf{M}\,\dot{\mathbf{E}}_{\text{vp}} = \mathbf{0}\\
&\mathbf{B}^T\dot{\mathbf{T}} - \dot{\mathbf{F}} = \mathbf{0}\\
&\dot{\mathbf{E}}_{vp} = \Lambda\,\mathcal{G}\\
&\dot{\bar{\boldsymbol{\epsilon}}}_{vp} = [\,\dot{\bar{\boldsymbol{\epsilon}}}^1_{vp} \ldots \dot{\bar{\boldsymbol{\epsilon}}}^e_{vp} \ldots \dot{\bar{\boldsymbol{\epsilon}}}^{nel}_{vp}\,]^T, \qquad \text{where} \quad \dot{\bar{\epsilon}}^{ek}_{vp} = \sqrt{\tfrac{2}{3}}\, \|\Lambda^{ek}\, \mathcal{G}^{ek}\| \quad k = 1,2,3
\end{aligned} \tag{22}$$

The flow laws are imposed by the adequate elementary assembling of the vectors, with $\dot{\boldsymbol{\epsilon}}^e_{vp}$ given by Eq. (18). The parameter $\dot{\mathbf{E}}_{vp}$ has the matrices Λ and $\mathcal{G}$ definided by

$$\Lambda = \text{diag}\,(\Lambda^e) \qquad \text{and} \qquad \mathcal{G} = [\mathcal{G}^e] \qquad\qquad e = 1, ..., nel \tag{23}$$

as Λ^e and $\mathcal{G}^e$ given by Eqs. (16) and (17), respectively.

The system above can be seen as a discrete version of continuum system Eqs. (1-4) for isotropic hardening materials applications in an isothermal process.

4. Solution algorithm

In this Section it is presented an algorithm solution, based on Newton-Raphson formula for the solution of the discrete problem defined by the system of Eq. (22). The algorithm has global iterations, with sub-increment for calculate the constitutive and equilibrium equations, Eqs. $(22)_1$ and $(22)_2$; and an one-step Euler scheme for integration of flow laws, Eqs. $(22)_3$ and $(22)_4$ (Sluys, 1998).

4.1 Time discretization

The solution of the discrete problem, Eq. (22), begins with the subdivision of the interest time interval, $t \in \mathcal{I} \subset I\!R^+$, in a finite number of steps Δt. If the Equation $(22)_2$ is considered valid at each time instant and if one can define the stress rate as a stress variation at a time interval Δt, the incremental form of the equilibrium condition is given by

$$\mathbf{B}^T \Delta \mathbf{T} \,=\, \mathbf{F}^{t+\Delta t} - \mathbf{B}^T \mathbf{T}^t \tag{24}$$

Resembling with the stress rate, one can define the velocity, the viscoplastic strain rate and the total strain rate as a variation of the displacement, viscoplastic strain and total strain at a time interval Δt, respectively.

The viscoplastic strain variation is estimated with the generalized Euler method

$$\Delta \mathbf{E}_{vp} = [(1-\gamma)\, \dot{\mathbf{E}}^t_{vp} \,+\, \gamma\, \dot{\mathbf{E}}^{t+\Delta t}_{vp}]\, \Delta t \tag{25}$$

where γ is the interpolation parameter for which $0 \leq \gamma \leq 1$. The viscoplastic strain rate at the end of the time interval is expressed in a limited Taylor series expansion as

$$\dot{\mathbf{E}}^{t+\Delta t}_{vp} = \dot{\mathbf{E}}^t_{vp} + \mathbf{G}^t\, \Delta \mathbf{T} + \mathbf{h}^t\, \Delta \bar{\epsilon}_{vp} \tag{26}$$

where the matrix $\mathbf{G}^t$ and the vector $\mathbf{h}^t$ represent the viscoplastic strain rate gradient with regard to the stress and the equivalent viscoplastic strain, respectively, that is,

$$\mathbf{G}^t = \nabla_{\mathbf{T}}\, \dot{\mathbf{E}}^t_{vp} \qquad\quad \text{and} \qquad\quad \mathbf{h}^t = \nabla_{\bar{\epsilon}_{vp}}\, \dot{\mathbf{E}}^t_{vp} \tag{27}$$

With some algebraic developments (Costa, 2004) one can lead to define the stress variation as

$$\Delta \mathbf{T} = I\!\!D^t_{vp}\, \mathbf{M}^{-1}\, [\, \mathbf{B}\, \Delta \mathbf{u} + \bar{\mathbf{B}}\, \Delta \bar{\mathbf{u}}\,] - \, \mathbf{q}^t \tag{28}$$

where

$$I\!\!D^t_{vp} = \left[\text{diag}\left(\widehat{I\!\!D}^{-1} + \gamma\, \Delta t\, \mathbf{G}^t\right)\right]^{-1} \qquad \text{and} \qquad \mathbf{q}^t \,=\, I\!\!D^t_{vp}\left(\dot{\mathbf{E}}^t_{vp} \Delta t \,+\, \gamma\, \Delta t\, \mathbf{h}^t \Delta \bar{\epsilon}_{vp}\right) \tag{29}$$

The matrix $I\!\!D^t_{vp}$ and the vector $\mathbf{q}^t$ are time dependents. The sequence of the algebraic substitutions associated with the discontinuous interpolation assumed for the stress field are fundamental to assure the nodal decoupled of the matrix $I\!\!D^t_{vp}$. This procedure have important consequences on the computational feasibility whereas this matrix have to be inverted at each new iteration. With this procedure, only $[3x3]$ or $[4x4]$ matrix is inverted for stress or strain plane states, respectively.

For assurance of the system symmetry, one can adopt the symmetric matrix

$$\widetilde{I\!\!D}^t_{vp} = \frac{1}{2}\, [I\!\!D^t_{vp}\, \mathbf{M}^{-1} + \mathbf{M}^{-1} I\!\!D^t_{vp}] \tag{30}$$

The displacement increment, $\Delta \mathbf{u}$, is defined by the solution of the system

$$\mathbf{K}^t_{vp}\, \Delta \mathbf{u} = \mathbf{F}^{t+\Delta t} - \mathbf{F}^t_i \tag{31}$$

where

$$\mathbf{K}^t_{vp} = \mathbf{B}^T \widetilde{I\!\!D}^t_{vp} \mathbf{B} \qquad \text{and} \qquad \mathbf{F}^t_i = \mathbf{B}^T(\mathbf{T}^t - \mathbf{q}^t) + \mathbf{B}^T I\!\!D^t_{vp}\, \mathbf{M}^{-1}\, \bar{\mathbf{B}}\, \Delta \bar{\mathbf{u}} \tag{32}$$

represent the viscoplastic stiffness matrix and the internal force at instant t, respectively.

4.1.1 Newton-raphson method for equilibrium calculation

The Table **1** presents an algorithm solution, based on Newton-Raphson formula (an incremental-iterative process) for the solution of the discrete problem (Sluys, 1998). The Table **2** presents the algorithm used in this work for time integration of flow laws (an one-step Euler scheme). Hereinafter, the superscript is used to identify the number of iteration at a fixed time and no more identify the time interval.

Table 1. Time integration of the equilibrium equation

For each load or displacement increment:

$j = 0$

$\mathbf{F} = \mathbf{F}^{t+\Delta t} = \mathbf{F}^t + \Delta\mathbf{F} \qquad \Delta\mathbf{u}^0 = \mathbf{0} \qquad \mathbf{K}^0_{vp} = \mathbf{K}^t_{vp} \qquad \mathbf{F}^0_i = \mathbf{F}^t_i \qquad \mathbf{T}^0 = \mathbf{T}^t$

$\dot{\mathbf{E}}^0_{vp} = \dot{\mathbf{E}}^t_{vp} \qquad \bar{\epsilon}^0_{vp} = \bar{\epsilon}^t_{vp} \qquad \Delta\bar{\epsilon}^0_{vp} = \Delta\bar{\epsilon}^t_{vp} \qquad \mathbf{R}^0 = \mathbf{F} - \mathbf{F}^0_i$

Repeat

 Solve the linear system: $\mathbf{K}^j_{vp}\,\delta\mathbf{u}^{j+1} = \mathbf{R}^j$

 $\zeta = 1$

 Repeat

  Compute increment: $\Delta\mathbf{u}^{j+1} = \Delta\mathbf{u}^j + \zeta\,\delta\mathbf{u}^{j+1}$

  For each element *e*

   Compute strain increment: $\Delta\mathbf{E}^{j+1} = \mathbf{B}\,\Delta\mathbf{u}^{j+1} + \bar{\mathbf{B}}\,\Delta\bar{\mathbf{u}}$

   Compute new matrices: $\mathbb{D}^{j+1}_{vp}$, $\mathbf{q}^{j+1}$ and $\mathbf{K}^{j+1}_{vp}$ (Table **2**)

   Compute stress increment: $\Delta\mathbf{T}^{j+1} = \mathbb{D}^{j+1}_{vp}\,\mathbf{M}^{-1}\Delta\mathbf{E}^{j+1} - \mathbf{q}^{j+1}$

   Compute total stress: $\mathbf{T}^{j+1} = \mathbf{T}^0 + \Delta\mathbf{T}^{j+1}$

   Update at each node *k*

    Yield function: $f^{j+1} = f(\mathbf{T}^{j+1}, \bar{\epsilon}^0_{vp})$

    If $f^{j+1} > 0$ **then**

     Constitutive function: $\dot{\lambda}^{j+1} = \dot{\lambda}(\mathbf{T}^{j+1}, \bar{\epsilon}^0_{vp})$

     Viscoplastic strain rate: $\dot{\mathbf{E}}^{j+1}_{vp} = \dot{\lambda}^{j+1}\,\nabla_{\mathbf{T}} f^{j+1}$

     Equivalent viscoplastic strain rate: $\dot{\bar{\epsilon}}^{j+1}_{vp} = \sqrt{\frac{2}{3}}\,\|\dot{\mathbf{E}}^{j+1}_{vp}\|$

     Accumulate viscoplastic strain increment: $\Delta\bar{\epsilon}^{j+1}_{vp} = \left[(1-\gamma)\,\dot{\bar{\epsilon}}^0_{vp} + \gamma\,\dot{\bar{\epsilon}}^{j+1}_{vp}\right]\Delta t$

    End-If

   Next node *k*

  Next element *e*

  Compute internal force vector: $\mathbf{F}^{j+1}_i = \mathbf{B}^T(\mathbf{T}^{j+1} - \mathbf{q}^{j+1}) + \mathbf{B}^T\mathbb{D}^{j+1}_{vp}\mathbf{M}^{-1}\bar{\mathbf{B}}\,\Delta\bar{\mathbf{u}}$

  $\zeta = 0,7\,\zeta$

 Until $\left(\mathbf{R}^{j+1}/\mathbf{R}^j\right) < 1$

 $j = j + 1$

Until $|\mathbf{F} - \mathbf{F}^j_i|_2 \leq \epsilon\,|\mathbf{F} - \mathbf{F}^1_i|_2$

5. Numerical application

The problem considers a slab subject to traction with a circular imperfection (Díez et al., 1998). In order to induce the strain localization in the specimen, a geometric imperfection must be introduced.

The rectangular slab with a small circular hole is submitted to an increasing prescribed displacement $\bar{\mathbf{u}}(t)$ considering the plane strain problem. The geometry of the problem, with $R/H = 0.111$ and $L/H = 2.111$, the loading conditions and the mesh adopted are showed in Fig. 1. For symmetry reasons the analysis is performed for one quarter of the section with appropriate boundary conditions.

The mechanical properties of the material are: elastic modulus $E = 2.10^{11}Pa$, Poison's ratio $\nu = 0.3$, yield limit $\sigma_Y = 2.10^8 Pa$, Perzyna linear behavior and softening material, $H_{iso} = -2.10^{10}Pa$. The imposed displacement has

Table 2. Time integration of the constitutive equation

For each element e

$\Delta\mathbf{E}^e = \mathbf{B}^e \Delta\mathbf{u}^e \qquad \mathbf{T}^e = [\mathbf{T}^e]^0 \qquad \dot{\mathbf{E}}^e_{vp} = [\dot{\mathbf{E}}^e_{vp}]^0 \qquad \bar{\epsilon}^e_{vp} = [\bar{\epsilon}^e_{vp}]^0 \qquad \Delta\,\bar{\epsilon}^e_{vp} = [\Delta\,\bar{\epsilon}^e_{vp}]^0$

Compute trial stress state: $\mathbf{T}^e_{trial} = \mathbf{T}^e + \mathbb{D}\,\Delta\mathbf{E}^e$

For each node k

If $f(\mathbf{T}^{ek}_{trial}, \bar{\epsilon}^0_{vp}) < 0$

Then $\mathbb{D}^{ek}_{vp} = \widehat{\mathbb{D}}$

Else $\mathbb{D}^{ek}_{vp} = [\widehat{\mathbb{D}}^{-1} + \gamma\,\Delta t\,\mathbf{G}^{ek}]^{-1}$

End-If

$\mathbf{q}^{ek} = \mathbb{D}^{ek}_{vp}\left(\dot{\mathbf{E}}^{ek}_{vp} + \gamma\,\mathbf{h}^t \Delta\bar{\epsilon}^0_{vp}\right)\Delta t$

Next node k

Assemble elementar matrix and vector: $\mathbb{D}^e_{vp} = [\text{diag}(\mathbb{D}^{ek}_{vp})] \qquad \mathbf{q}^e = [\mathbf{q}^{ek}]$

Compute symmetric part: $\widetilde{\mathbb{D}}^e_{vp} = \dfrac{\mathbb{D}^e_{vp}\,\mathbf{M}^{-1^e} + \mathbf{M}^{-1^e}\,\mathbb{D}^{e\;T}_{vp}}{2}$

Compute elementar stiffness: $\mathbf{K}^e_{vp} = \mathbf{B}^T\,\widetilde{\mathbb{D}}^e_{vp}\,\mathbf{B}$

Assemble elementar matrix $\mathbf{K}^e_{vp}$ in global matrix $\mathbf{K}_{vp}$

Next element

been given in 20 single steps $\Delta\mathbf{u} = 0.0065mm$ up to a final displacement $\mathbf{u}_{max} = 0.13mm$. The imposed velocity is $v = 1m/s$.

The force-displacement curve is plotted in Fig. 1, where the force is the sum of the nodal reactions on the bounded right edge. In the graphics, Fig. 1, the dashed line represents the numerical values obtained by Díez et al. (1998). The discrete points represent numerical values obtained by the proposed mixed model. The softening behavior is observed. As both works present numerical solutions for the problem, the results are considered satisfactory, because different numerical models and different finite element meshes are compared.

The Figure 2 shows the countour plots of the accumulated viscoplastic strain field, $\bar{\epsilon}_{vp}$, obtained for $\mathbf{u} = 0.065$; 0.091 and $0.117mm$, respectively. The collapse mechanism in this example is formed by the strain localization band and almost all the deformation is localized along the band.

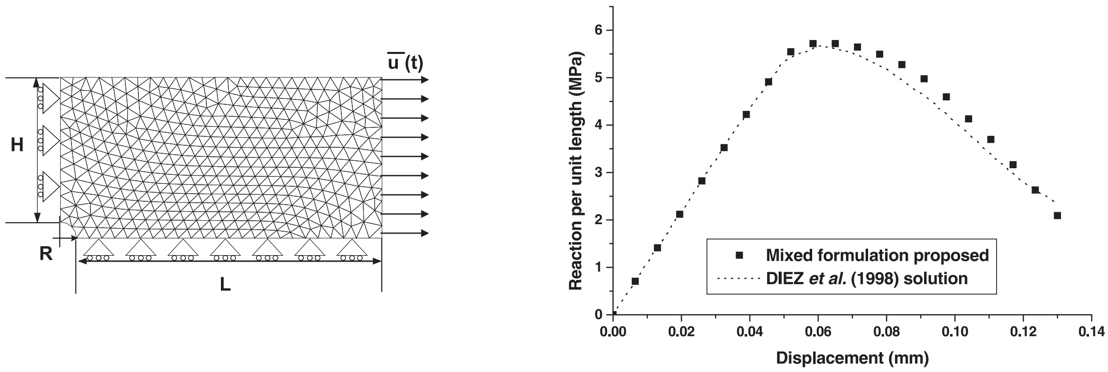

Figure 1. Geometry of the problem, loading conditions, mesh adopted and force-displacement curve

6. Conclusions

A mixed model and an algorithm to deal with elasto/viscoplasticity problems was proposed. This mixed (*min-max*) principle is written as function of stress rates and velocities. The viscoplastic strain rate and the hardening rate are obtained from the flow rule and appears in the functional only as parameters.

A mixed discretization process, based on a triangular mixed element, is proposed. The element holds C^0 quadratic

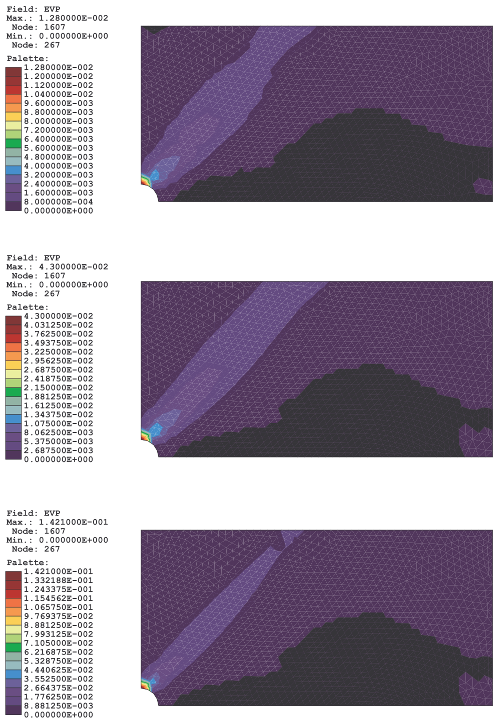

Figure 2. Contour plot of the $\bar{\epsilon}_{vp}$ at prescribed displacements $\mathbf{u} = 0.065;\ 0.091$ and $0.117mm$

interpolation for geometry and velocities and an element discontinuity for stresses rates, stresses, viscoplastic strain rates, equivalent viscoplastic strain rates and accumulated viscoplastic strain.

The results indicate the viability of the presented mixed methodology. More advanced applications need to be carried out in order to consolidate the formulation as an effective procedure for elasto/viscoplasticity. These advanced model must include adaptivity meshes and other cycle load programs. Such advanced features are now being preformed and will be subject of a future report.

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8. Responsibility notice

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