ANALYSIS OF EXTREMES, OVERSHOOT AND UNDERSHOOT IN THE LINEAR CONTROL CONTINUOUS-TIME SYSTEM OF THIRD ORDER WITH REAL ZEROS AND POLES

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Abstract: *This paper analyses critical points, extreme, overshoot and undershoot that can occur in the step-response of a linear continuous-time and stable control system of third order with real zeros and poles. Sufficient conditions for the existence of extremes, necessary and sufficient conditions for overshoot and initial undershoot, and sufficient conditions for existence of the extended type r_u undershoot in the step-response of third order continuous time transfer functions, based on their poles and zeros, are presented. The authors also present a class of linear control stable continuous time-system of third order and of minimum phase that exhibits undershoot in the step response. Simple examples illustrate and complement the main results of this note.*

Keywords: *Extremes, critical points, step-response, poles, zeros, overshoot, undershoot*

1. Introduction

Automatic control has played a vital role in the advance of engineering and science. In addition to its extreme importance in space-vehicle systems, missile-guidance systems, robotic systems, automatic control has become an important and integral part of modern manufacturing and industrial processes (Franklin, 1991; Dorf, 1995; Ogata, 1997).

There exist some control problems, such as machine tool axis control and trajectory-following in robotics, where the step-response cannot exhibit local extrema (Rachid, 1995). Thus, the study of conditions to avoid extrema, overshoot and undershoot in the step response (and other possible types of extremes) is of continuing relevance in control theory. Recent works on the subject has described the connection between the nature of the overshoot and undershoot in the step-response of scalar continuous time systems and open right half plane zeros of the underlying transfer function (see e. g. Ogata, 1997; El-Khoury, 1993; Howell, 1997; Rachid, 1995; Leon de la Barra, 1994, 1994-a; Reis, 2004, 2004-a, 2003, 2002, 2001; Silva 2001).

Much work has been done to clarify the influence of the zeros on the transient part of a step-response (El-Khoury, 1993; Rachid, 1995; Howell, 1997; Goodwin, 1999; León de la Barra, 1994, 1994-a; Lin e Fang, 1997; Moore, 1990; Mita e Yoshida, 1981; Reis, 2001, 2002, 2003, 2004, 2004-a; Silva, 2001). These results are important, but in this moment, they cannot offer a complete relation between the relative locations of the poles and zeros of the plant and controllers and the existence of extremes (overshoot and undershoot). For example, the determination of the exact number of extremes remains an open problem (El-Khoury, 1993).

The authors presented necessary and sufficient condition for the existence of extremes, overshoot and undershoot in the step-response of second order continuous time transfer functions and same class the control systems of the third order, based on its real poles and zeros (Reis, 2001, 2002, 2003, 2004, 2004-a).

This paper presents sufficient conditions for analysis and classification of the extremes for a unit step-response applied in a linear stable continuous time system of third order with real zeros and poles. Necessary and sufficient conditions for the existence of overshoot and initial undershoot and sufficient condition for the existence the extended type r_u undershoot in this class of the control systems, are presented.

In the opinion of the authors, this note provides new insight about the correlation between poles and zeros of a scalar continuous-time transfer function and the nature of the extremes overshoot and undershoot in its step-response. These results to be presented do not constitute the final understanding of this connection, but they certainly complement, clarify and expand the previous research results, which have been subject of recent discussion in the literature.

The paper is organized as follows. Section 2 contains definitions and background material. In Section 3 the main results, which qualitatively correlate real zeros and extremes, overshoot and undershoot in the step-response, are presented. Examples are presented in the Section 4 and concluding remarks are given in Section 5.

2. Preliminares

In this paper consider a SISO linear control stable continuous-time system of third order with real zeros and poles, characterized by their continuous-time strictly proper transfer function *G(s)*:

$$G(s)=\frac{(T_1s+1)(T_2s+1)}{\prod_{j=1}^{3}(\tau_js+1)} \text{ or } G(s)=\frac{(T_1s+1)(-T_2s+1)}{\prod_{j=1}^{3}(\tau_js+1)} \text{ or } G(s)=\frac{(-T_1s+1)(-T_2s+1)}{\prod_{j=1}^{3}(\tau_js+1)}, \quad (1)$$

with $T_1, T_2 > 0$ are the real constants, τ_j for $j = 1, 2, 3$ are time constants of the system, $z_i = -\frac{1}{T_i}$, for $i = 1, 2$ are real zeros of the $G(s)$, $\lambda_j = -\frac{1}{\tau_j}$, for $j = 1, 2, 3$ are poles of the $G(s)$ and $\lambda_j \neq \lambda_i$, $\lambda_j \neq z_i, \forall\ i = 1, 2$ and $\forall\ j = 1, 2, 3$. Consider that the time t belongs to $[0, +\infty)$ and $\lambda_1 < \lambda_2 < \lambda_3 < 0$, $z_1 < z_2$. The following result provides a unit step response for the system (1). The proof of this lemma follows from the expansion in partial fraction of the $G(s)$.

Lemma 2.1: The unit step response of the class linear control system with $G(s)$ as in (1) is given by:

$$y(t)=1+\sum_{i=1}^{3}c_ie^{\lambda_i t}, \quad (2)$$

where $c_1=-\frac{\lambda_2\lambda_3}{z_1z_2}\frac{(\lambda_1-z_1)(\lambda_1-z_2)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}$, $c_2=\frac{\lambda_1\lambda_3}{z_1z_2}\frac{(\lambda_2-z_1)(\lambda_2-z_2)}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}$ and $c_3=-\frac{\lambda_1\lambda_2}{z_1z_2}\frac{(\lambda_3-z_1)(\lambda_3-z_2)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}$.

For the discussion that follows, some definitions and results are necessary. Consider a SISO rational control system characterized by the continuous-time transfer function G(s). G(s) is also strictly proper (m < n), minimal (no pole-zero cancellations) with no zeros at the origin of the complex plane:

$$G(s)=K\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{j=1}^{n}(s-\lambda_j)}, \text{ where } K=\frac{\prod_{j=1}^{n}\lambda_j}{\prod_{i=1}^{m}z_i}, \quad (3)$$

$\lambda_1<\lambda_2<...<\lambda_n<0$ are the real poles of the $G(s)$ and z_i, for $i = 1, ..., m$, are the real zeros. It is convenient to classify the zeros of $G(s)$ in Eq. (3), in four different sets:

$M_1=\{z: G(z)=0,\ 0<z<+\infty\}$, $M_2=\{z: G(z)=0,\ \lambda_n<z<0\}$, $M_3=\{z: G(z)=0,\ \lambda_1<z<\lambda_n\}$ and

$$M_4=\{z: G(z)=0,\ -\infty<z<\lambda_1\}. \quad (4)$$

In addition, let m_i for $i = 1, 2, 3, 4$, denotes the number of zeros belonging to a given class M_i, such that $m = m_1 + m_2 + m_3 + m_4$. The following definitions are useful for addressing the latter problem.

Definition 1: (El-Khoury, 1993) The function $y(t)$ given by (2) has a ***local extremum at t^**** if there exists $\varepsilon > 0$ such that $[y(t^*) - y(t^* - \tau)][y(t^*) - y(t^* + \tau)] > 0\ \forall\tau \in (0,\varepsilon)$.

Definition 2: Let integer $\eta \geq 0$ be the ***total number of local extreme of y(t)***, for $t > 0$.

Since all poles of (1) are real and negative, $y(t)$ has a finite number of extremes for $t > 0$.

Definition 3: (El-Khoury, 1993) A ***pole bracket*** is the open interval $(\lambda_{i-1}, \lambda_i)$ between two distinct consecutive poles $\lambda_{i-1} < \lambda_i$ of $G(s)$.

There are $n - 1$ pole bracket when all the poles are distinct.

Definition 4: (El-Khoury, 1993) Let ***p be the number of poles brackets*** containing an odd number of zeros of $G(s)$.

Definition 5: (El-Khoury, 1993) Let x an integer. Then ***parity (x):= 0*** se x is even and ***parity (x):= 1*** se x is odd.

The analyses of the overshoot and undershoot in the control system is directly associated with the analyses of the extremes in the step-response. Therefore, the following theorem gives an upper and lower bounds for the number of extremes of the step-response in terms of the number and type of zeros in the transfer function.

Theorem 2.1: (El-Khoury, 1993) Let *G(s)* as in (3):

i) $m_1 + m_2 \le \eta \le m_1 + m_2 + m_3 - p;$

ii) *parity* $(\eta) =$ *parity* $(m_1 + m_2) =$ *parity* $(m_1 + m_2 + m_3 - p)$.

The following lemma gives the decomposition the system (3), in terms of the poles and zeros.

Lemma 2.1: (Rachid, 1995) The transfer function *G(s)* in (3) can always be rewritten under the form $G(s) = \prod_{i=1}^{r_1} G_{1i}(s) \prod_{j=1}^{r_2} G_{2j}(s)$, where $G_{1i} = \dfrac{1}{1+T_{1i}s}$, $G_{2i} = \dfrac{1+\tau_{2j}s}{1+T_{2j}s}$ and $r_1 + r_2 = n$.

The following theorem gives a sufficient condition to avoid extremes using the pole-zero configuration of the corresponding transfer-function (3).

Theorem 2.2: (Rachid, 1995) The step-response of system (3) has no extremum for $t > 0$ if arrangement of the *G(s)* in Lemma 2.1 can be made such that the following conditions hold $\tau_{2j} < T_{2j}$, for $j \in [1, r_2]$.

In this paper *G(s)* will be consider in the form time constants and therefore the final steady-state value of the *y(t)*, $y(\infty)$, will be equal to one. The following definition gives the concept of overshoot, in this case.

Definition 6: (Ogata, 1997) The ***overshoot*** M_p in percent is the maximum peak value of the step-response measured from unity and it is defined by $M_p = (y_m - 1).\ 100\%$, where y_m is maximum value the *y(t)* and considering that $y(\infty) = 1$.

Transfer functions having neither poles nor zeros in the right-half *s* plane are ***minimum-phase transfer functions***, whereas those having poles and/or zeros in the right-half *s* plane are ***nonminimum-phase transfer functions***. Systems with minimum-phase transfer functions are called ***minimum-phase systems***, whereas those with nonminimum-phase transfer functions are called ***nonminimum-phase systems*** (Ogata, 1997). The nonminimum-phase systems with only zeros in the right-half *s* plane were originally identified by their initial inverse response to a step input, i.e., the initial response was in the opposite direction from the steady-state, are called systems with ***initial undershoot*** or ***type A undershoot***.

The literature classifies the undershooting phenomenon in the step-response by using the following terms: initial undershoot or type *A* undershoot, type *B* undershoot and type r_u undershoot, respectively. Type *B* undershoot occurs if there is no initial undershoot and there exists an open interval (a, b), $0 < a < b$, where the step-response has opposite sign to its steady state value (Mita e Yoshida, 1981). Type r_u undershoot provides a more detailed description of the undershooting phenomenon by specifying the number of times that the step-response displays undershoot for $t > 0$ (León de la Barra, 1994, 1994a; Silva 2001).The following theorem gives a necessary and sufficient condition for a stable control system to exhibit an initial undershooting step-response.

Theorem 2.3: (Vidysagar 1996) The control system (3) has initial undershoot if and only if its transfer function has an odd number of real right-half *s* plane zeros.

A more general characterization of multiple undershoot in the step-response is formalize for the notion of extended type r_u undershoot.

Definition 7: (Silva, 2001) Let *G(s)* be as in (3) and with nonzero *DC* gain *k*. Let *y(t)* be the corresponding time domain response to a positive unit step. Then it is said that *y(t)* displays extended type r_u undershoot for $t > 0$ if the following three part condition is met.

There exist exactly r_{ue} different values of *t*, say $0 < t_1 < t_2 < \ldots < t_{r_{ue}} < +\infty$, such that:

(i) $y(t_i)k < 0$, for $i = 1,\ldots,r_{ue}$,

(ii) $\left.\dfrac{dy(t)}{dt}\right|_{t=t_i} = 0$, for $i = 1,\ldots,r_{ue}$,

(iii) $k\left.\dfrac{d^2y(t)}{dt^2}\right|_{t=t_i} > 0$, or $\left.\dfrac{d^2y(t)}{dt^2}\right|_{t=t_i} = 0$, and there exist a positive constant δ_i such that $k\left.\dfrac{dy(t)}{dt}\right|_{t=t_i} < 0$ for all $t \in (t_i - \delta_i, t_i)$ and $k\left.\dfrac{dy(t)}{dt}\right|_{t=t_i} > 0$ for all $t \in (t_i, t_i + \delta_i)$, for $i = 1,\ldots,r_{ue}$.

The occurrence of the undershoot in the step-response *y(t)* is directly associated with the changes of signs of this function. The following theorem provides a lower bound on the number of changes of signs of the impulse response *g(t)* of *G(s)*, as a function of the number of real zeros located strictly on the right of its dominant pole(s).

Theorem 2.4: (León de la Barra, 1994) If $g(t) \leftrightarrow G(s)$ constitutes a Laplace Transform pair, with $G(s)$ be as in (3), but not necessarily nonminimum phase, the number of changes of sign of $g(t)$, for $t > 0$ will be lower bounded by the number of real zeros of $G(s)$ located strictly on the right of its dominant pole(s).

For the discussion that follows it is convenient to make the following analysis. It follows from (2) that:

$$y'(t) = 0 \Leftrightarrow c_1\lambda_1 + c_2\lambda_2 e^{(\lambda_2-\lambda_1)t} + c_3\lambda_3 e^{(\lambda_3-\lambda_1)t} = 0 \, . \tag{5}$$

In (5), define:

$$k_i = c_i\lambda_i \text{, for } i = 1,2,3 \text{ and } \beta_j = \lambda_j - \lambda_1 \text{, for } j = 2,3, \tag{6}$$

and after the substitution of (6) in (5), it follows that:

$$y'(t) = 0 \Leftrightarrow 1 + \frac{k_2}{k_1}e^{\beta_2 t} + \frac{k_3}{k_1}e^{\beta_3 t} = 0 \, . \tag{7}$$

In Eq. (7) define $\alpha_i = \dfrac{k_i}{k_1}$, for $i = 2, 3$. Therefore, by (7) it follows that $y'(t) = 0 \Leftrightarrow 1 + \alpha_2 e^{\beta_2 t} + \alpha_3 e^{\beta_3 t} = 0$. Let $f(t)$ be the auxiliary function, based on (7):

$$f(t) = 1 + \alpha_2 e^{\beta_2 t} + \alpha_3 e^{\beta_3 t} \, . \tag{8}$$

From (7) and (8), note that the critical points of $y(t)$ are points t_o in $[0, +\infty)$ for that $f(t_o) = 0$. Moreover, note that:

$$f(t) = \frac{y'(t)}{k_1 e^{\lambda_1 t}} \, . \tag{9}$$

The following section has as objective to make the analysis of the critical points of $y(t)$ in Eq. (2) and also, for the overshoot and undershoot in this function.

3. Necessary and sufficient conditions for analysis the extremes, overshoot and undershoot in the step-response *y(t)*

The following main results provide sufficient conditions for the analysis the extremes and necessary and sufficient conditions for overshoot and sufficient conditions for undershoot in the step-response $y(t)$ in Eq. (2).

Theorem 3.1: Let $y(t)$ be as in Eq. (2), the unit step-response of the control system (1). Then:
(i) If $z_1 \in M_4$ and $z_2 \in M_2$ or $z_1 \in M_3$ and $z_2 \in M_2$, then $y(t)$ presents one relative maximum in $t_o \in (0, +\infty)$;
(ii) If $z_1 \in M_4$ and $z_2 \in M_1$ or $z_1 \in M_3$ and $z_2 \in M_1$, then $y(t)$ presents one relative minimum in $t_1 \in (0, +\infty)$;
(iii) If $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$, then $y(t)$ presents zero extremes or one relative maximum in t_o and one relative minimum in t_1, where $t_o, t_1 \in (0, +\infty)$;
(iv) If $z_1, z_2 \in M_2$ or $z_1 \in M_2$ and $z_2 \in M_1$ or $z_1, z_2 \in M_1$, then $y(t)$ presents exactly one relative maximum in t_o and one relative minimum in t_1, where $t_o, t_1 \in (0, +\infty)$;
(v) If ($z_1, z_2 \in M_4$) or ($z_1 \in M_4$ and $z_2 \in M_3$) or ($z_1, z_2 \in M_3$ and $\lambda_1 < z_1 < \lambda_2 < z_2 < \lambda_3 < 0$ or $\lambda_1 < z_1 < z_2 < \lambda_2 < \lambda_3 < 0$), then $y(t)$ does not present relative extremes.

Theorem 3.2: Let $y(t)$ be as in Eq. (2), the unit step-response of the control system (1). Then:
(i) $y(t)$ may present overshoot if and only if ($z_1 \in M_4$ and $z_2 \in M_2$) or ($z_1 \in M_3$ and $z_2 \in M_2$) or ($z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$) or ($z_1, z_2 \in M_2$) or ($z_1 \in M_2$ and $z_2 \in M_1$) or ($z_1, z_2 \in M_1$);
(ii) If ($z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$) or ($z_1, z_2 \in M_2$), then $y(t)$ may present type r_{ue} undershoot extended;
(iii) If $z_1, z_2 \in M_1$, then $y(t)$ displays type r_u undershoot extended;
(iv) ($z_1 \in M_4$ and $z_2 \in M_1$) or ($z_1 \in M_3$ and $z_2 \in M_1$) or ($z_1 \in M_2$ and $z_2 \in M_1$) if and only if $y(t)$ displays initial undershoot.

Corollary 3.1: Let the step-response $y(t)$ be as in Eq. (2). Then it does not display overshoot neither undershoot if ($z_1, z_2 \in M_4$) or ($z_1 \in M_4$ and $z_2 \in M_3$) or ($z_1, z_2 \in M_3$ and $\lambda_1 < z_1 < \lambda_2 < z_2 < \lambda_3 < 0$ or $\lambda_1 < z_1 < z_2 < \lambda_2 < \lambda_3 < 0$).

The following results are necessary for the proofs of the Theorems 3.1, 3.2 and Corollary 3.1.

Lemma 3.1: Let $G(s)$ be as in Eq. (1). Then may occur the following possibilities for η, the number of the extremes the $y(t)$:

(i) η is exactly equal to one if and only if ($z_1 \in M_4$ and $z_2 \in M_2$) or ($z_1 \in M_3$ and $z_2 \in M_2$) or ($z_1 \in M_4$ and $z_2 \in M_1$) or ($z_1 \in M_3$ and $z_2 \in M_1$);

(ii) If ($z_1, z_2 \in M_2$) or ($z_1 \in M_2$ and $z_2 \in M_1$) or ($z_1, z_2 \in M_1$), then η is exactly equal to two;

(iii) If $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$, then η is equal to zero or η is equal to two;

(iv) If ($z_1, z_2 \in M_4$) or ($z_1 \in M_4$ and $z_2 \in M_3$) or ($z_1, z_2 \in M_3$ and $\lambda_1 < z_1 < \lambda_2 < z_2 < \lambda_3 < 0$ or $\lambda_1 < z_1 < z_2 < \lambda_2 < \lambda_3 < 0$), then η is equal to zero.

Proof: Since $G(s)$ has two real zeros of multiplicity equal to one, can occur that z_1 and z_2 belongs to the same class M_i, for $i = 1, 2, 3, 4$ or z_1 and z_2 belongs to the distinct classes. For proving item (i) assume that ($z_1 \in M_4$ and $z_2 \in M_2$) or ($z_1 \in M_3$ and $z_2 \in M_2$) or ($z_1 \in M_4$ and $z_2 \in M_1$) or ($z_1 \in M_3$ and $z_2 \in M_1$). If ($z_1 \in M_4$ and $z_2 \in M_2$), then $m_2 = 1$ and $m_1 = m_3 = p = 0$. By Theorem 2.1, $\eta = 1$. The proofs for other relative positions are similar. Conversely, now assume that $\eta = 1$. By Theorem 2.1, it follows that ($m_1 + m_2 = 1$ and $m_1 + m_2 + m_3 - p = 1$). If $m_1 + m_2 = 1$, since z_i have equal to one multiplicity for $i = 1, 2$, then $m_1 = 1$ or $m_2 = 1$ and $m_3 = p$. Therefore it can occur ($m_1 = 1$ or $m_2 = 1$ and $m_3 = p = 0$) or ($m_1 = 1$ or $m_2 = 1$ and $m_3 = p = 1$). If $m_1 = 1$ or $m_2 = 1$ and $m_3 = p = 0$, it follows that $z_2 \in M_2$ or M_1 and $z_1 \in M_4$. Therefore $z_1 < \lambda_1 < \lambda_2 < \lambda_3 < z_2 < 0$ or $z_1 < \lambda_1 < \lambda_2 < \lambda_3 < 0 < z_2$. If ($m_1 = 1$ or $m_2 = 1$ and $m_3 = p = 1$), it follows that $z_2 \in M_2$ or $z_2 \in M_1$ and $z_1 \in M_3$, and then $\lambda_1 < z_1 < \lambda_2 < \lambda_3 < z_2 < 0$ or $\lambda_1 < \lambda_2 < z_1 < \lambda_3 < z_2 < 0$ or $\lambda_1 < z_1 < \lambda_2 < \lambda_3 < 0 < z_2$ or $\lambda_1 < \lambda_2 < z_1 < \lambda_3 < 0 < z_2$. Therefore the proof of the (i) is concluded. The proofs of the (ii) and (iii) can be obtained using a similar way. The proof of the (iv) follows directly of the Theorem 2.1 or 2.2.

The following lemma provides necessary and sufficient conditions for analysis the extremes in the function $f(t)$ in (8). The proof is made by the direct analysis of the derivative of $f(t)$.

Lemma 3.2: The function $f(t)$ in Eq. (8) presents an unique critical point if and only if ($z_1, z_2 \in M_2$) or ($z_1 \in M_2$ and $z_2 \in M_1$) or ($z_1, z_2 \in M_1$) or ($z_1, z_2 \in M_3$ and $\lambda_2 + \lambda_3 < z_1 + z_2$) or ($z_1 \in M_4$ and $z_2 \in M_2$ and $\lambda_2 + \lambda_3 > z_1 + z_2$) or ($z_1 \in M_4$ and $z_2 \in M_1$ and $\lambda_2 + \lambda_3 > z_1 + z_2$) or ($z_1 \in M_3$ and $z_2 \in M_1$ and $\lambda_2 + \lambda_3 > z_1 + z_2$) or ($z_1 \in M_3$ and $z_2 \in M_2$ and $\lambda_2 + \lambda_3 > z_1 + z_2$);

Lemma 3.3: Assume that $z_1 < z_2 < 0$. Then:

(i) $\alpha_3 < 0$ and $k_1 > 0$ if and only if ($z_1 \in M_3$ and $z_2 \in M_2$);

(ii) $\alpha_3 > 0$ and $k_1 < 0$ if and only if ($z_1 \in M_4$ and $z_2 \in M_2$);

(iii) If ($\alpha_3 < 0$ and $k_1 > 0$) or ($\alpha_3 > 0$ and $k_1 < 0$), then the step-response $y(t)$ presents an unique point of relative maximum in t_o.

(iv) $\alpha_3 > 0$ and $k_1 > 0$ if and only if ($z_1, z_2 \in M_2$) or ($z_1, z_2 \in M_3$) or ($z_1, z_2 \in M_4$);

(v) If $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$, then the step-response $y(t)$ presents zero extremes or one relative maximum in t_o and one relative minimum in t_1, where $t_o, t_1 \in (0, +\infty)$;

(vi) If $z_1, z_2 \in M_2$, then the step-response $y(t)$ presents one relative maximum in t_o and one relative minimum in t_1.

(vii) If $\alpha_3 < 0$ and $k_1 < 0$ if and only if ($z_1 \in M_4$ and $z_2 \in M_3$);

(viii) If ($z_1, z_2 \in M_4$) or ($z_1 \in M_4$ and $z_2 \in M_3$) or ($z_1, z_2 \in M_3$ and $\lambda_1 < z_1 < \lambda_2 < z_2 < \lambda_3 < 0$ or $\lambda_1 < z_1 < z_2 < \lambda_2 < \lambda_3 < 0$), then the step-response $y(t)$ does not present relative extremes.

Proof: For the proof of the item (i), by virtue of Eq. (6) and definition of the α_3, follow that:

$$\alpha_3 < 0 \Leftrightarrow \frac{c_3\lambda_3}{c_1\lambda_1} < 0 \Leftrightarrow \frac{(\lambda_3 - z_1)(\lambda_3 - z_2)}{(\lambda_1 - z_1)(\lambda_1 - z_2)} < 0 \Leftrightarrow \frac{\lambda_3^2 - (z_1 + z_2)\lambda_3 + z_1 z_2}{\lambda_1^2 - (z_1 + z_2)\lambda_1 + z_1 z_2} < 0 \text{ and}$$

$$k_1 > 0 \Leftrightarrow c_1\lambda_1 > 0 \Leftrightarrow (\lambda_1 - z_1)(\lambda_1 - z_2) > 0 \Leftrightarrow \lambda_1^2 - (z_1 + z_2)\lambda_1 + z_1 z_2 > 0 .$$

Thus $\alpha_3 < 0$ and $k_1 > 0$ if and only if $\lambda_1^2 - (z_1 + z_2)\lambda_1 + z_1 z_2 > 0$ and $\lambda_3^2 - (z_1 + z_2)\lambda_3 + z_1 z_2 < 0$ if and only if $\lambda_1 < z_1 < \lambda_2 < \lambda_3 < z_2$ or $\lambda_1 < \lambda_2 < z_1 < \lambda_3 < z_2$. The proof for (ii) follows the same ideas. For (iii), assume that ($\alpha_3 < 0$ and $k_1 > 0$). By (i), ($z_1 \in M_3$ and $z_2 \in M_2$). If ($\alpha_3 > 0$ and $k_1 < 0$), by (ii) ($z_1 \in M_4$ and $z_2 \in M_2$). In view of Lemma 3.1, $\eta = 1$ and by Lemma 3.2, $f(t)$ presents unique critical point. By Eq. (9), it follows that $f(0) > 0$ as $k_1 > 0$. Since $\alpha_3 < 0$, $f(t) \to -\infty$ for $t \to +\infty$. Figure 1 gives the form of the graphic of the $f(t)$. Now, from (9), since $k_1 > 0$ and $f(t) > 0$ if $t < t_o$ then $y'(t) > 0$ if $t < t_o$. Furthermore, as $k_1 > 0$ and $f(t) < 0$ if $t > t_o$, $y'(t) < 0$ if $t > t_o$. Hence $y(t)$ has one relative maximum point in t_o. The proof for $\alpha_3 > 0$ and $k_1 < 0$ is analogue. The proof of the (iv) also follows the same ideas the proof of (i). For (v), assume that $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$. By Lemma 3.1, $\eta = 0$ or $\eta = 2$. In view of Lemma 3.2, if $z_1, z_2 \in M_3$ and $\lambda_2 + \lambda_3 < z_1 + z_2$, then $f(t)$ presents one unique critical point. Assume $\eta = 2$. Since $\alpha_3 > 0$ and $k_1 > 0$, it follows that $f(0) > 0$ and $f(t) \to +\infty$ for $t \to +\infty$. Figure 2 shows the form of the graphic of $f(t)$. In view of (9) and analysis of the graphic of $f(t)$ shows that $y(t)$ has one relative maximum in t_o and one relative minimum in t_1. For (vi), if $z_1, z_2 \in M_2$, by Lemma 3.1, $\eta = 2$ and by Lemma 3.2, $f(t)$ has one unique critical point. The graphic of the $f(t)$ have the form shows in Fig. 2. Therefore, $y(t)$ has one relative maximum in t_o and one relative minimum in t_1.The proof for (vii) also follows the same ideas the proof of (i). For (viii), if $(z_1, z_2 \in M_4)$ or $(z_1 \in M_4$ and $z_2 \in M_3)$ or $(z_1, z_2 \in M_3$ and $\lambda_1 < z_1 < \lambda_2 < z_2 < \lambda_3 < 0$ or $\lambda_1 < z_1 < z_2 < \lambda_2 < \lambda_3 < 0)$, by Lemma 3.1, $\eta = 0$, therefore the step-response $y(t)$ does not have extremes.

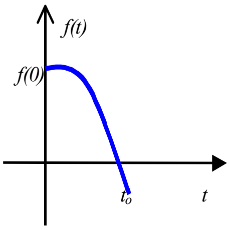

Figure 1. Illustration of the plot of $f(t)$ in (8) for $\alpha_3 < 0$ and $k_1 > 0$.

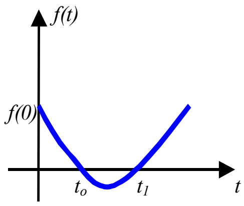

Figure 2. Illustration of the plot of $f(t)$ in (8) for $\alpha_3 > 0$ and $k_1 > 0$.

Lemma 3.4: Assume that $z_1 < 0 < z_2$. Then:

(i) $\alpha_3 > 0$ and $k_1 > 0$ if and only if $(z_1 \in M_4$ and $z_2 \in M_1)$;
(ii) $\alpha_3 < 0$ and $k_1 < 0$ if and only if $(z_1 \in M_3$ and $z_2 \in M_1)$;
(iii) If $(\alpha_3 > 0$ and $k_1 > 0)$ or $(\alpha_3 < 0$ and $k_1 < 0)$, then the step-response $y(t)$ has one unique relative minimum in t_o;
(iv) $\alpha_3 > 0$ and $k_1 < 0$ if and only if $z_1 \in M_2$ and $z_2 \in M_1$;
(v) If $\alpha_3 > 0$ and $k_1 < 0$, then the step-response $y(t)$ has one relative maximum in t_o and one relative minimum in t_1.

Proof: The proof also follows the same ideas the proof of the Lemma 3.3.

Lemma 3.5: Assume that $0 < z_1 < z_2$. Then:

(i) $\alpha_3 > 0$ and $k_1 > 0$ if and only if $\lambda_1 < \lambda_2 < \lambda_3 < 0 < z_1 < z_2$;
(ii) If $\alpha_3 > 0$ and $k_1 > 0$, then the step-response $y(t)$ has one relative maximum in t_o and one relative minimum relative in t_1.

Proof: The proof also follows the same ideas the proof of the Lemma 3.3.

Proof of the Theorem 3.1: It follows directly from Lemmas 3.3, 3.4 and 3.5.

Proof of the Theorem 3.2: For (i) assume that $y(t)$ presents overshoot. Then there exists $t_0 \in (0, +\infty)$ such that t_0 is a relative maximum of the $y(t)$. In view of Theorem 3.1, $(z_1 \in M_4$ and $z_2 \in M_2)$ or $(z_1 \in M_3$ and $z_2 \in M_2)$ or $(z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0)$ or $(z_1, z_2 \in M_2)$ or $(z_1 \in M_2$ and $z_2 \in M_1)$ or $(z_1, z_2 \in M_1)$. Conversely, now assume that $(z_1 \in M_4$ and $z_2 \in M_2)$ or $(z_1 \in M_3$ and $z_2 \in M_2)$ or $(z_1, z_2 \in M_2)$ or $(z_1 \in M_2$ and $z_2 \in M_1)$ or $(z_1, z_2 \in M_1)$. Then from Lemma 3.1, η is exactly equal to one or η is exactly equal to two. If $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$ then from Lemma 3.1 $\eta = 0$ or $\eta = 2$. It is interesting the case $\eta = 2$. In view of Theorem 3.1, in those cases, $y(t)$ has one relative maximum in t_o. Can occur the following form for the graphic the step-response $y(t)$ show in Fig. 3. Observe that in forms (a) and (c), $y(t)$ displays overshoot and the proof of the (i) is concluded. For the proof of (ii), assume that $(z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0)$ or $(z_1, z_2 \in M_2)$. In view of Theorem 3.1, if $z_1, z_2 \in M_2$, then $y(t)$ has exactly one relative maximum and one relative minimum. If $z_1, z_2 \in M_3$ and $\lambda_1 < \lambda_2 < z_1 < z_2 < \lambda_3 < 0$, then by Lemma 3.1, $y(t)$ can have zero extremes or one relative maximum and one relative minimum. It is interesting to observe that, when $\eta = 2$, in both cases, the following forms of the step-response $y(t)$, showed in Fig. 3, can occur. Observe that in the forms (a) and (b) $y(t)$ displays extended type r_u undershoot. Therefore the proof of the (ii) is concluded.

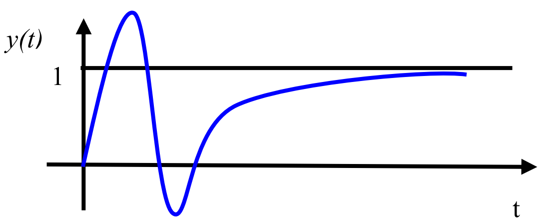

Figure 3: Illustration of the form (a) described in the proof of Theorem 3.2.

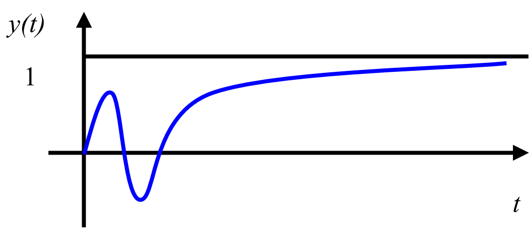

Figure 3: Illustration of the form (b) described in the proof of Theorem 3.2.

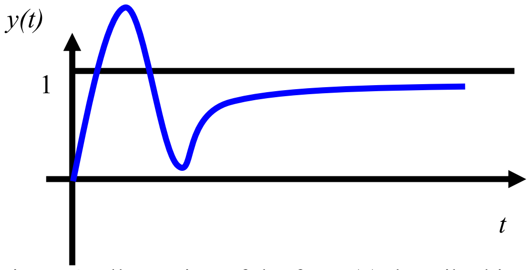

Figure 3: Illustration of the form (c) described in the proof of Theorem 3.2.

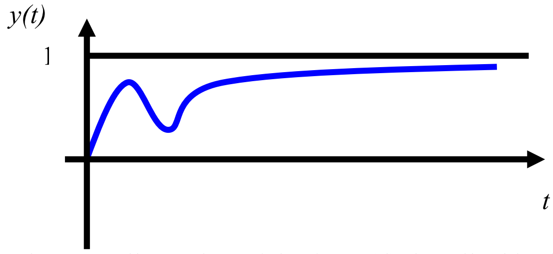

Figure 3: Illustration of the form (d) described in the proof of Theorem 3.2.

To prove (iii) assume that $z_1, z_2 \in M_1$. Define a function:

$$\tilde{G}(s) = \frac{G(s+a)}{(s+a)} = -\frac{\lambda_1\lambda_2\lambda_3}{z_1 z_2}\frac{(s-(z_1-a))(s-(z_2-a))}{(s-(\lambda_1-a))(s-(\lambda_2-a))(s-(\lambda_3-a))(s-(-a))},$$

where $a > 0$ is chosen sufficiently small such that $\tilde{G}(s)$ does not have zeros at the origin of the complex plane and satisfies the condition $z - a > 0$. Then $\tilde{g}(t) = L^{-1}\left[\tilde{G}(s)\right] = y(t)e^{-at}$. Observe that the dominant pole the $\tilde{G}(s)$ is $\lambda = -a$. Therefore $\tilde{m}_2 = \tilde{m}_3 = \tilde{p} = 0$ and $\tilde{m}_1 = 2$. In view of Theorem 2.4, the number of changes of sign of $\tilde{g}(t) = y(t)e^{-at}$ for $t > 0$, is at least two. Now, the step-response $y(t)$ and $\tilde{g}(t)$ have the same number of changes of sign for any $a > 0$. Recall that $Y(s) = \frac{G(s)}{s}$. Hence $y(t)$ has at least two changes of sign, for $t > 0$. By Theorem 2.3, $y(t)$ does not have initial undershoot and by Lemma 3.1, η is exactly equal to two. Hence $y(t)$ displays extended type r_u undershoot for z_1, $z_2 \in M_1$ and this shows (iii). The proof of (iv) follows directly from Theorem 2.3.

Proof of the Corollary 3.1: Follow by Theorem 3.1.

4. Example

Let $G_1(s)$ and $G_2(s)$ given for $G_1(s) = \frac{(10s+1)(20s+1)}{(0.33s+1)(0.5s+1)(s+1)}$ and $G_2(s) = \frac{(-10s+1)(-s+1)}{(0.3s+1)(0.5s+1)(s+1)}$.
$G_1(s)$ has the following zeros: $z_1 = -0.1$, $z_2 = -0.2$ and poles $\lambda_1 = -3$, $\lambda_2 = -2$ e $\lambda_3 = -1$, whereas $G_2(s)$ to have zeros $z_1 = 0.1$, $z_2 = 1$ and poles $\lambda_1 = -3$, $\lambda_2 = -2$ e $\lambda_3 = -1$ satisfying the Theorems 3.1 and 3.2. Figure 4 – (a) and (b) shows the graphics of the step-response in these systems. Note that both systems display overshoot and extended type r_u undershoot. For $G_1(s)$, the extended type r_u undershoot occurs because $G_1(s)$ has one negative zero. It is important because in literature undershooting phenomenon is associated with positive zeros. For $G_2(s)$, the overshoot occurs because this function has positive zeros.

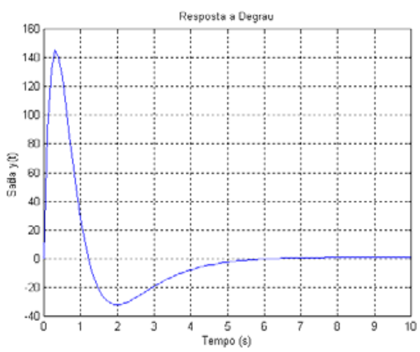

Figure 4 – (a): Step-response for $G_1(s)$.

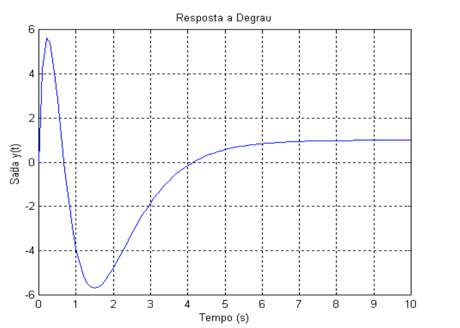

Figure 4 – (b): Step-response for $G_2(s)$.

5. Conclusions

This paper presented sufficient conditions for analysis and classification of the extremes for a unit step-response and for linear continuous-time control stable system of third order with real zeros and poles.

Necessary and sufficient conditions for the existence of overshoot and initial undershoot and sufficient conditions for the existence the extended type r_u undershoot for this class of the control systems are presented.

It is proved that the existence of undershoot can also occurs when $G(s)$ is of minimum-phase (Theorem 3.2). It is important because in literature the undershooting phenomenon was associated with nonminimum-phase systems. It is also proved that the overshooting phenomenon occurs in nonminimum-phase systems with only zeros in the right-half *s* plane (Theorem 3.2).

In the opinion of the authors, this note provides new insight into the correlation between poles and zeros of a scalar continuous-time transfer function and the nature of the extremes, overshoot and undershoot in its step-response. The proposed results do not constitute the final understanding of this connection, but they certainly complement, clarify and expand the previous research results, which have been subject of recent discussion in the literature.

6. Acnowlegements

The authors acknowledge support by FAPESP, Fundação de Apoio à Pesquisa do Estado de São Paulo, Brazilian Research Funding Agency and CNPq.

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