DYNAMIC ANALYSIS OF A ROTOR-BEARING SYSTEM BY FINITE ELEMENTS

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Abstract. *A rotating shaft supported by two hydrodynamic bearings is composed by two basic sub-system: (1) A flexible axis acted by an unbalanced rotating force and (2) Spring-damper supports which interacts with the shaft motion, represented by a hydrodynamic bearing. The forces produced by the dynamic pressure of the lubricant are obtained by the solution of the Reynolds equation for the fluid-film. Parameters associated to the speed of the axis and to the physical and geometric characteristics of the bearing, just like viscosity of the fluid, radial clearance, length and diameter of the bearing, are considered in the evaluation of the dynamic behavior of the axis. The matrix equations of motion for the axis are characterized by the occurrence of mass head offices, gyroscopic and a stiffness matrix. Rotordynamic systems are normally modeled as a finite set of interconnecting flexible-shaft elements, bearings and dampers, working fluid mechanisms and flexible disks. The Method of the Finite Elements can be applied for derivation of those equations. The movement of the axis should be described by six degrees of freedom for each node. For obtaining the time response of the vibrating axis, Newmark procedure has been employed for integrating the motion equations requested by the specification of the energies kinetics and potential elastic and for the application of the equations of Lagrange. Those equations should be simultaneously resolved. The dynamic pressure produced by the fluid motion is integrated along the circumferential direction of the bearing and the resulting forces are introduced in the force vector of the matrix equation of the axis. An iterative procedure has to be implemented, for the fluid forces have a dependency on the displacement of the axis at the supports position. Numeric solutions are obtained to describe the conditions of dynamic stability of the rotor-bearing system.*

Keywords: *Numerical Methods, Mechanical Vibrations, Rotordynamic, Fluid-film*

1. Introduction

Vibration considerations about parts revolving in a machine are not complete if one does not consider the elastic and damping effects of a present bearing. Most commonly a rotating shaft is analyzed under vibration criterions disregarding gyroscopic effects, rotating inertia and the bearings contributions to the system. A full model considering all real actions is troublesome and requires a adequate analysis of the pertinent sub-systems. Hydrodynamic bearings based on the solution of Reynolds's equation have been studied extensively in order to furnish accurate description of the forces produced by the fluid motion in the bearing. The main purpose of the present study is to carry out introductory considerations about the effect of damping and elastic forces at the support positions of a rotor. Rotor vibrations are strongly dependent on the rotor geometry, on the bearings type and on the excitation forces type. Usually the performances of mechanical systems are improved by methods, which lead to an increase of the dynamic stiffness. Rotating shafts perform a motion with a frequency different than the natural frequency of the non-rotating shaft, and the first critical speed is different from the first natural frequency. The subject is most interesting in the design of aeronautic turbines shafts. In this work the Finite Element Method is used, adopting beam elements with two nodal points and four degree of freedom for each node, two displacements and two rotations. The motion equations are obtained from Lagrange's equations, and describe the motion in two transverse planes. As a result, for each element, eight differential equations have to be accounted for. To obtain the mass and stiffness matrix, the software MATHEMATICA was used for the integrations of the kinetic and potential energy, and for the differentiation required by Lagrange's equations. A MATLAB computer code was written, to assembly the shaft element matrixes into global matrixes, and simulates the shaft response, for several dimensions, materials, angular velocities and external unbalanced loads. To obtain the time domain response, the integration procedure of Newmark was adopted "(Bode-Menezes,2003)".

2. Model of a Rotor-Bearing System

2.1 The Timoshenko Beam Elements

The Timoshenko beam theory includes the effect of transverse shear deformation. As a result, a plane normal the beam axis before deformation does not remain normal to the beam axial any longer after deformation. Let u and v be the axial and transverse displacements of a beam, respectively. Because of the transverse shear deformation, the slope of the beam β is different from dv/dx. Instead, the slope equals $(dv/dx) - \zeta$ where ζ is the transverse shear strain. As result, the displacement field in the Timoshenko beam can be written as "(Reddy1997)".

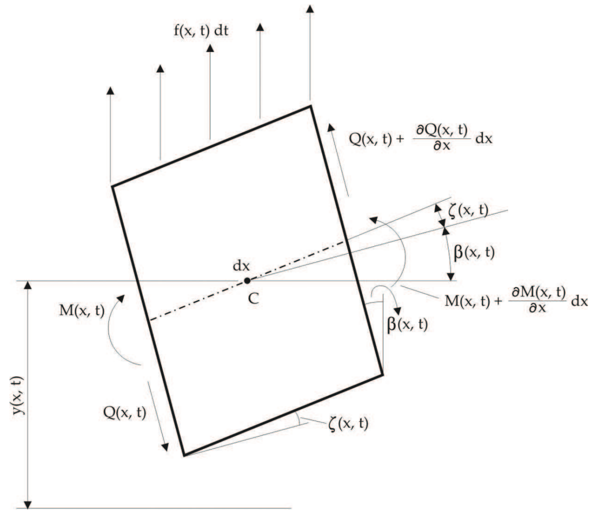

Figure 1. Equilibrium of an infinitesimal length of beam.

$$u\left(x,y\right) = -y\beta\left(x\right) \tag{1}$$

$$v\left(x,y\right) = v\left(x\right) \tag{2}$$

where β denotes rotation of a transverse normal about the z axis. According to the free body diagram of a beam element of length dx, "Fig. 1", $M(x,t)$ is the bending moment, $Q(x,t)$ is the shear transverse force, $\beta(x,t)$ is the deformation due the bending and $\zeta(x,t)$ is the deformation due the transverse shear. The strains and stresses in the Timoshenko beam theory are

$$\varepsilon_{xx} = y\frac{d\beta}{dx}, \quad 2\varepsilon_{xv} = \beta + \frac{dv}{dx}, \quad \sigma_{xx} = E\varepsilon_{xx}, \quad \sigma = 2G\varepsilon_{xy} \tag{3}$$

The balance of internal moments and transverse forces give the relations

$$M\left(x\right) = \int_A y\sigma_{xx}dA = EI\frac{d\beta}{dx} \tag{4}$$

$$Q\left(x\right) = k_s\int_A \sigma_{xy}dA = k_sAG\left(\beta + \frac{dv}{dx}\right) \tag{5}$$

and the equilibrium of moments and transverse forces over a typical element give

$$Q(x) - \frac{dM}{dx} = 0, \quad \frac{dQ}{dx} = -q(x) \tag{6}$$

where K_s denotes the shear correction coefficient for a circular section it is $9/10$. Using "Eq. (4)" and (5) in "Eq. (6)", we obtain the following equilibrium equations

$$EI\frac{\partial^2\beta}{\partial x^2} - k_sAG\left(\beta - \frac{\partial v}{\partial x}\right) = \rho I\frac{\partial\beta}{\partial t^2} \tag{7}$$

$$-k_s AG\left(\frac{\partial\beta}{\partial x}-\frac{\partial^2 v}{\partial x^2}\right)+q\left(x\right)=\rho A\frac{\partial^2 v}{\partial t^2} \tag{8}$$

3. Derivation of Elements Matrices

3.1 Lagrange's Equations

The equation o motion of a vibrating system can often be derived in a simple manner in terms of generalizes coordinates by the use of Lagrange's equations. Lagrange's equation can be stated as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i^{(n)},\quad i=1,2,...,n \tag{9}$$

where $\dot{q}_i=\partial q_i/\partial t$ is the generalized velocity and $Q_i^(n)$ is the nonconservative generalized force corresponding to the generalized coordinate q_i. The forces represented by $Q_i^(n)$ may be dissipative (damping) forces or other external forces which are not derivable from a potential function. For example, $F_x, F_y, andF_z$ represent the external forces acting on the mass of the system in the $x, y, andz$ directions, respectively, then the generalizes force $Q_i^(n)$ can be computed as follows:

$$Q_i^{(n)}=\sum\left(F_x\frac{\partial u}{\partial q_i}+F_y\frac{\partial v}{\partial q_i}+F_z\frac{\partial w}{\partial q_i}\right) \tag{10}$$

where $u, v, andw$ are the displacements of the mass the journal bearing system in the $x, y, andz$ directions, respectively. If the conservative system, $Q_i^(n)=0$. Thus the equations of motion of the vibrating system can be derived, provided the energy expressions are available. The problem of a beam in torsion is in everything similar to the discussed in this section. The similarity begins in the expression of the energies and of the virtual work and it finishes essentially in an equation differential of motion of similar mathematical presentation. Where has $u(x,t)$ is displacement axial, $p(x,t)$ is the external shipment, $m(x)$ is the mass and $EA(x)$ in the axial problem, we were in the problem torsion with $\theta(x,t)$, $T(x,t)$, $I_p(x)$ and $GJ(x)$, respectively where θ is the angular displacement of the beam around your axis, starting from the condition non deformed, $T(x,t)$ is the distributed torque and outward applied at the beam, $I_p(x)$ denote the polar moment of inertia of the section transverse and $GJ(x)$ it is torsion stiffness ($J=I_p$ for a circular cross section) of the beam. The kinetic energy of the beam in a certain instant for unit of length can be expressed as

$$T_x=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx \tag{11}$$

The elastic strain energy instantaneous of the beam in axial solicitation can be written as

$$V_x=\frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{12}$$

The kinetic energy torsion of the beam in a certain instant can be expressed as

$$T_\theta=\frac{1}{2}\int_0^L \rho I_p\left(\frac{\partial \theta}{\partial t}\right)^2 dx \tag{13}$$

The strain energy can be expressed as

$$V_\theta=\frac{1}{2}\int_0^L GJ_p\left(\frac{\partial \theta}{\partial x}\right)^2 dx \tag{14}$$

For us to appreciate the consequences of the inclusion promoted by Timoshenko, let us consider the bending angular total $y(x,t)$ of a traverse section of the referred beam the a point of the elastic axis it can be defined for

$$\frac{\partial y(x,t)}{\partial x} = \beta(x,t) + \zeta(x,t) \quad (15)$$

The kinetic energy T of the beam element is given by "(Meirovitch,1997)"

$$T_y = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial y}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l \rho I \left(\frac{\partial \beta}{\partial t}\right)^2 dx \quad (16)$$

$$T_z = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial w}{\partial t}\right)^2 dx + \frac{1}{2}\int_0^l \rho I \left(\frac{-\partial \gamma}{\partial t}\right)^2 dx \quad (17)$$

The first term in "Eqs. (16) and (17)" gives the translational kinetic energy an the second term the rotational kinetic energy. The strain energy V of the uniform rotating beam element of length L is given by

$$V_y = \frac{1}{2}\int_0^l EI \left(\frac{\partial \beta}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l k_s GA \left(\frac{\partial v}{\partial x} - \beta\right)^2 dx \quad (18)$$

$$V_z = \frac{1}{2}\int_0^l EI \left(\frac{-\partial \gamma}{\partial x}\right)^2 dx + \frac{1}{2}\int_0^l k_s GA \left(\frac{\partial w}{\partial x} - \gamma\right)^2 dx \quad (19)$$

The first term in "Eqs. (18) and (19)" represents the flexural strain energy, the second term the shear strain energy.

3.2 Gyroscopic Effects

"Timoshenko in 1918" showed, using the principle of angular momentum, that under certain conditions, not only the unbalancing forces should be taking in account when obtaining the critical speeds of rotating beams. Then

$$M_y = \frac{d}{dt}\left(J_a \frac{d\beta}{dt} - I_p \Omega \gamma\right) \quad (20)$$

$$M_z = \frac{d}{dt}\left(J_a \frac{d\gamma}{dt} - I_p \Omega \beta\right) \quad (21)$$

The kinetic energy functionals due the gyroscopic effect are "(Timoshenko 1918)"

$$T_y = \frac{1}{2}\int_0^L I_p \Omega (-\gamma) \left(\frac{\partial \beta}{\partial t}\right) dx \quad (22)$$

$$T_z = \frac{1}{2}\int_0^L I_p \Omega (-\beta) \left(\frac{-\partial \gamma}{\partial t}\right) dx \quad (23)$$

3.3 Model Description

The general model is shown in "Fig. 2" consists of a rotor, treated as elastic and continuous, to which a rigid disk is coupled . Additionally, at support positions, hydrodynamic bearing can be considered. Presently, gyroscopic has not been taken into account. The equations of the hydrodynamic forces produced by the oil film, due to pressure increase promoted by the fluid motion, may be formulated and solved numerically. The rotor revolves about the x-axis with an angular velocity ω counter-clockwise. The oil film forces in the x and y direction, F_r and F_t, are obtained from the oil pressure distribution, which is obtained from the Reynolds equation. The general Reynolds equation governing the flow of the squeeze film damper is well known as "Hamrock, 1994".

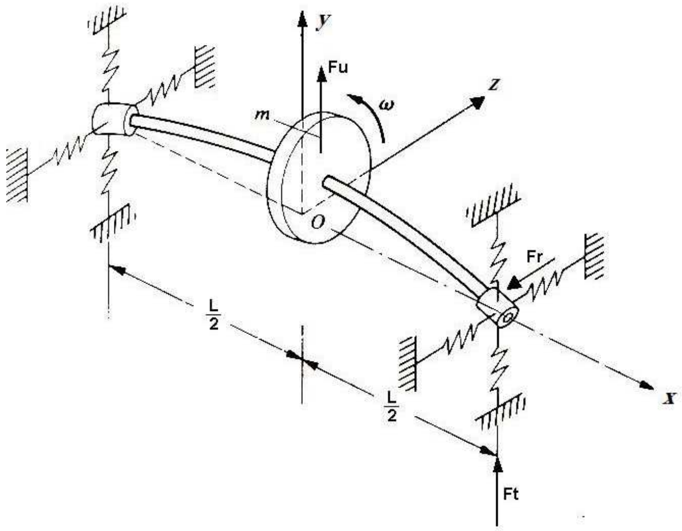

Figure 2. Theoretical models for the analysis of flexible rotor.

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right)=6\rho\left\{\frac{\partial p}{\partial x}\left[\left(w_a+w_b\right)h\right]+2\frac{\partial h}{\partial x}\right\} \tag{24}$$

In "Eq. (24)", the following assumptions were made: (i) the fluid inertia terms in Navier-Stokes equations have been neglected due to their small magnitude; (ii) the flow of lubricant is laminar; (iii) the fluid is Newtonian; (iv) no slip exists at the fluid-solid interface; (v) the flow in the radial direction has been neglected. Thus the flow of lubricant is two-dimensional; (vi) the inclination of one surface relative to the other is so small that the sine of the angle of inclination can be set equal to the angle and the cosine can be set equal to unity. The general Reynolds equation given in "Eq. (24)" can be applied to any section of the oil film and in this paper only the dynamically loaded infinitely wide-journal-bearing solution will be presented. The film thickness can be described as $h = c\left(1+\varepsilon\cos\phi\right)$

$$w_y' = r_b\int_0^{\phi_m}\cos\phi\frac{dp}{d\phi}d\phi, \quad w_z' = r_b\int_0^{\phi_m} sen\phi\frac{dp}{d\phi}d\phi \tag{25}$$

Replacing "Eq. (24) in Eqs. (25)" for to F_r and F_t gives the oil film forces of the squeeze film damper about the damper center in the radial and tangential directions, which are obtained by integrating the oil film pressure distribution along and normal to the line of the centers of the journal over the positive pressure values region as "(Bode-Menezes, 2003)"

$$F_r = 12\eta\left(\frac{r_a}{c}\right)^2 Lr_b\frac{\left[\frac{\partial\varepsilon}{\partial t}\sin^2\phi-\varepsilon\sin\phi\cos\phi\left(\omega-\frac{\omega_a-\omega_b}{2}\right)-\frac{3\varepsilon^2}{2+\varepsilon^2}\left(\omega-\frac{\omega_a-\omega_b}{2}\right)\sin\phi\right]d\phi}{\left(1+\varepsilon\cos\phi\right)^3} \tag{26}$$

$$F_t = 12\eta\left(\frac{r_a}{c}\right)^2 Lr_b\int_0^{\phi_m}\frac{\left[\frac{\partial\varepsilon}{\partial t}\sin\phi\cos\phi-\varepsilon\cos^2\phi\left(\omega-\frac{\omega_a-\omega_b}{2}\right)-\frac{3\varepsilon^2}{2+\varepsilon^2}\left(\omega-\frac{\omega_a-\omega_b}{2}\right)\cos\phi\right]d\phi}{\left(1+\varepsilon\cos\phi\right)^3} \tag{27}$$

The resulting oil-film forces due to the motion of journal in a squeeze-film damper are usually presented in polar coordinates, since the oil-film forces can be conveniently expressed in this form. The forces in polar coordinates can be easily transformed into Cartesian coordinates by following equations:

$$F_v(t) = \sum F_y = F_t\sin\Phi - F_r\cos\Phi \tag{28}$$

$$F_w(t) = \sum F_z = -F_t\cos\Phi - F_r\sin\Phi \tag{29}$$

4. Finite Element Model Development

4.1 Finite Element Formulation of Rotors

The finite element matrices of a Timoshenko rotor can be obtained by using the standard finite element procedure to obtain

$$[M_s]\{\ddot{x}\} + \omega[C_s]\{\dot{x}(t)\} + [K_s]\{x(t)\} = F(t) \tag{30}$$

in which $[M_s]$, $[C_s]$, and $[K_s]$, are the structural mass, gyroscopic damping, and stiffness matrices, respectively. F is the external excitation force vector. A rotor-bearing system is composed of a uniform shaft of a length L rotating at a constant speed ω and supported by two bearings. Coordinate systems adopted are depicted in "Fig. 3", where $X-Y-Z$ is the fixed frame, and $x-y-z$ the rotating frame. Timoshenko beam includes all effects, i.e., bending deformation, rotary inertia, gyroscopic effect and shear deformation. The strain energy is due to bending and shear. Translation, rotation and gyroscopic effect contribute towards kinetic. In the present model each element has two nodes and each node has four generalized displacements.

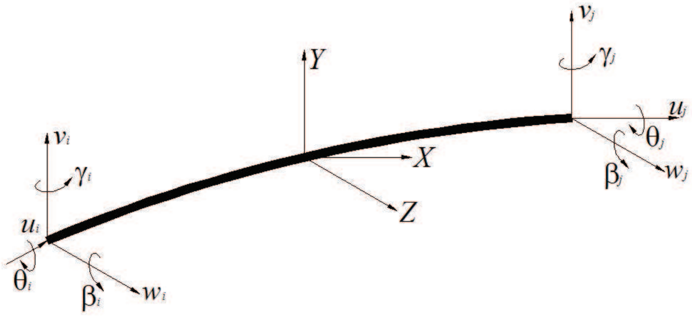

Figure 3. Displacement variables and coordinates system.

$$q_x = \begin{pmatrix} u_i \\ \theta_i \\ u_j \\ \theta_j \end{pmatrix}, \quad q_y = \begin{pmatrix} v_i \\ \beta_i \\ v_j \\ \beta_j \end{pmatrix}, \quad q_z = \begin{pmatrix} w_i \\ \gamma_i \\ w_j \\ \gamma_j \end{pmatrix}, \tag{31}$$

The corresponding element degrees of freedom in which θ_i and θ_j are the rotational degrees of freedom associated with the twisting moment, and γ_i, γ_j, β_i e β_j are slopes associated with bending moments. Employing the Finite Element Method one may obtain the solution for the proposed problem. This method considers the structure, in this case a rotating shaft, as an assembly of finite elements. Any displacement component along the element can be depicted as nodal values interpolations, according to the following expression:

$$q(x,t) = \sum_{i=1}^{n} N_i(x) q_i(t) \tag{32}$$

Here the shape functions "(Reddy, 1997)" (or the interpolation functions) are

$$N_x = \begin{pmatrix} 1-\xi \\ 1-\xi \\ \xi \\ \xi \end{pmatrix}, \quad N_y = \begin{pmatrix} \frac{2\xi^3-3\xi^2+\psi(1-\psi)+1}{\psi+1} \\ \frac{L\left(\xi^3-2\xi^2+\xi+\frac{1}{2}\psi\left(\xi-\xi^2\right)\right)}{\psi+1} \\ \frac{-2\xi^3+3\xi^2+\psi\xi}{\psi+1} \\ \frac{L\left(\xi^3-\xi^2-\frac{1}{2}\psi\left(\xi-\xi^2\right)\right)}{\psi+1} \end{pmatrix}, \quad N_z = \begin{pmatrix} \frac{2\xi^3-3\xi^2+\psi(1-\xi)+1}{\psi+1} \\ -\frac{L\left(\xi^3-2\xi^2+\xi+\frac{1}{2}\psi\left(\xi-\xi^2\right)\right)}{\psi+1} \\ \frac{-2\xi^3+3\xi^2+\psi\xi}{\psi+1} \\ -\frac{L\left(\xi^3-\xi^2-\frac{1}{2}\psi\left(\xi-\xi^2\right)\right)}{\psi+1} \end{pmatrix}, \tag{33}$$

and

$$\theta = \left(\frac{\partial N_x}{\partial x}\right) q_x, \quad \beta = -\left(\frac{\partial N_z}{\partial x}\right) q_y, \quad \gamma = \left(\frac{\partial N_y}{\partial x}\right) q_z \tag{34}$$

where

$$\psi = \frac{12IE}{AGk_sL^2} \quad and \quad \xi = \frac{x}{L} \tag{35}$$

A MATLAB computer code was written to assembly the shaft element matrices into global matrixes, and simulates the shaft response, for several dimensions, materials angular frequency and loads "(Kwon-Bang, 1996)". To obtain the time domain response, the integration procedure of Newmark method was adopted "(Rao, 1995)". For the use of Lagrange's equations, one has to express the kinetic and potential energies of the element. The elemental kinetic energy consists of two translational terms related to y and z motion, and accordingly, the potential energy of the element takes into account the bending in both planes. Replacing "Eqs. (31), (33) and (34)", obtain the total kinetic and strain energies for the element, according as function of the nodal displacements $q_i(x,t)$. According to Lagrange's equations, the motion equations are obtained by the direct integration of the kinetic and strain energy equations, followed by differentiations with respect to each nodal displacement. The set of eight equations for each element can be written in matrix form as:

4.2 Finite Element Formulation Disc

The equation of motion for disc is

$$[M_d]\{\ddot{x}\} + \omega[C_d]\{\dot{x}(t)\} = 0 \tag{36}$$

Where $[M_d]$, $[C_d]$ are the mass and gyroscopic matrices of disc only.

5. Integration of the Equation of Motion

In 1959 Newmark presented a family of single-step integration methods for the solution of structural dynamic problems. During the past 40 years Newmark´s method has been applied to the dynamic analysis of many practical engineering structures. In addition, it has been modified and improved by many other researches. In order to illustrate the use of this family of numerical integration methods one may consider the solution of the linear dynamic equilibrium equations written in the following form:

$$[M_g]\{\ddot{q}_{n+1}\} + \omega\,[C_g]\{\dot{q}_{n+1}\} + [K_g]\{q_{n+1}\} = \{Q_{n+1}\} \tag{37}$$

The most basic self-starting method is simply a Taylor Series Expansion truncated after some arbitrary number of terms. By truncating the series, which is known as Newmark's Method "(Rao, 1995)" one may obtain:

$$\{q_{n+1}\} = \{q_n\} + \Delta t\,\{\dot{q}_n\} + \frac{(\Delta t)^2}{4}\{\ddot{q}_n\} + \frac{(\Delta t)^2}{4}\{\ddot{q}_{n+1}\} \tag{38}$$

$$\{\dot{q}_{n+1}\} = \{\dot{q}_n\} + \frac{\Delta t}{2}\{\ddot{q}_n\} + \frac{\Delta t}{2}\{\ddot{q}_{n+1}\} \tag{39}$$

"Equation (38)" can be used to express $\{\ddot{q}_{n+1}\}$ in terms of $\{q_{n+1}\}$, and the resulting expression can be substituted into "Eq. (39)" to express $\{\dot{q}_{n+1}\}$ in terms of $\{q_{n+1}\}$. By substituting these expressions for $\{\dot{q}_{n+1}\}$ and $\{\ddot{q}_{n+1}\}$ into "Eq. (37)", we can obtain an expression for obtaining $\{q_{n+1}\}$:

$$\left(\frac{4}{(\Delta t)^2}[M_g] + \frac{2}{\Delta t}[C_g] + [K]\right)\{q_{n+1}\} = \{Q_{n+1}\} + [M_g]\left(\frac{4}{(\Delta t)^2}\{q_n\} + \frac{4}{\Delta t}\{\dot{q}_n\} + \{\ddot{q}_n\}\right) + [C_g]\left(\frac{2}{\Delta t}\{q_n\} + \{\dot{q}_n\}\right) \tag{40}$$

The previous equation can be solved for $\{q_{n+1}\}$, as a linear system of equations, and the nodal displacements of the vibrating problem can be obtained for any time. Employing "Eqs. (38) and (39)", nodal velocities and accelerations are obtained for the present time as a function of the present displacement, and previous velocities and accelerations.

6. Results

A computer code, based on Newmark approach, according to "Eq. (40)", has been written. An interactive routine had to be created to get convergence at each time step. For numerical analysis purposes, the following parameters of a simply supported rotating shaft.

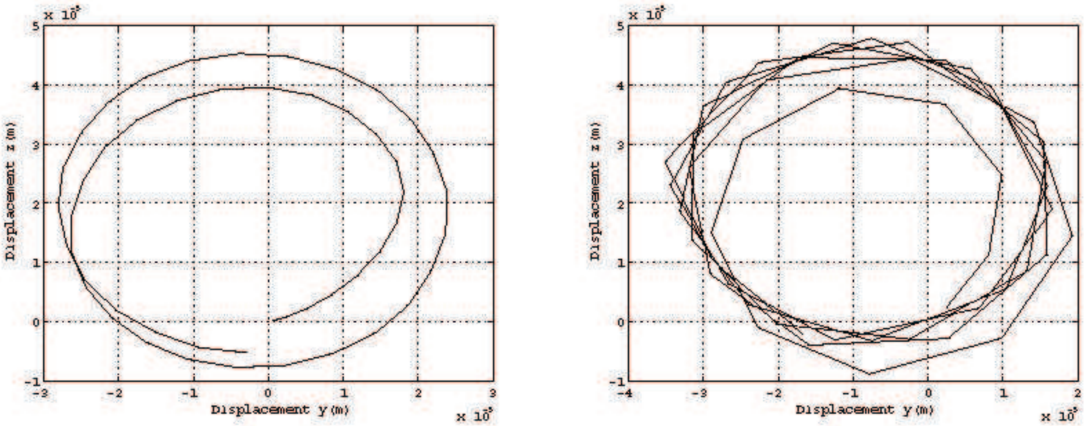

Figure 4. Journal orbit supported by Squeeze-film; (a) $\Delta t = 1e - 3s$ and (b) $\Delta t = 3e - 3s$

7. Comments and Justification

A computer code written in MATLAB language allows the study of the design parameters of bearings in a rotating shaft submitted to an unbalance force. Preliminary results show a considerable sensitivity to the support parameters in a rotating shaft vibration problem. The proposed program also allows the inclusion of a stiff disk to the problem, not considering the rotating inertia effect. Such study is introductory to a more complex model including the actual hydrodynamic bearing forces, gyroscopic effects of the rotating shaft and rotating inertia of stiff and heavy disk coupled to the shaft.

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10. Responsibility notice

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