THE DETERMINATION OF THE VELOCITIES AFTER IMPACT FOR THE CONSTRAINED BAR PROBLEM

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Abstract. *In this paper, a simple mathematical model for a constrained robotic manipulator is investigated. Besides the fact that this model is relatively simple, all the features present in more complex problems are similar to the ones analyzed here. The fully plastic impact is considered in this paper. Expressions for the velocities of the colliding bodies after impact are developed. These expressions are important in the numerical integration of the governing equations of motion when one must exchange from the set of unconstrained equations for the set of constrained equation. The theory presented in this work can be applied to problems in which robots have to follow some prescribed patterns or trajectories when in contact with the environment. It can also de applied to problems in which robotic manipulators must handle with payloads.*

Keywords: *contact dynamics, coefficient of restitution, fully plastic impact*

1. Introduction

There are several ways to deal with the problem of interaction between bodies. Impact dynamics and continuous contact between bodies can both be included in the mathematical model of the constrained problem, or just one of these effects can be considered. It depends, obviously, on the characteristics of the studied problem.

The investigations about the contact between bodies include (at least) two different kind of analysis (Pfeiffer and Glocker, 1996): one associated with the beginning of contact and one associated with its termination. In the first analysis, the distance between the bodies must be checked in order to know when contact occurs; in the second analysis, once the contact is established, the reaction (normal; compression) force between the bodies must be checked. In the second analysis, contact finishes when the contact force is equal to zero.

One of the hardest parts in the study of contact problems involves the different models that must be developed for contact and non-contact situations and the switching between these models when integrating the equations of motion (Lanczos, 1970; Whittaker, 1965). The unconstrained problem and the constrained problem do not have the same number of degrees of freedom. Dynamic systems when constrained have less degrees of freedom than when unconstrained.

The transition between constrained and unconstrained motion is sometimes called contact (including impact) and sometimes called just impact (mostly when the bodies separates after the collision). When contact occurs, the new velocities of the bodies involved must be known in order to generate the initial conditions to the second part (constrained problem) of the numerical integration. In the constrained problem, the concept of coefficient of restitution is very important (Fufaev, 1972).

2. Geometric model of the system and governing equations of motion

The problem discussed here is depicted in Figure 1. According to this figure, in a part of its trajectory the free end of the bar moves along the constraint represented by the mass named m_w. All the movements occur in the horizontal plane. When contact occurs, impact and bouncing are also allowed to occur.

The mass in which the rigid bar is pivoted (m_s) oscillates when excited by the movement of the bar (free and constrained). In the axis Z, passing through the connection between the bar and m_s (perpendicular to the paper sheet), there is a prescribed moment, M_θ, acting to turn the bar.

The dashed lines represent the position of the masses in which the springs and dampers are free of forces. The dotted line represents the position from which one starts to count the angular displacement, θ.

In physical terms, this system may represent a robot with a translational joint and a rotational joint; m_w can be thought as an obstructing wall on the robot's trajectory (or some object this robot must handle or interact with) and M_θ can be thought as an external torque provided by a dc motor.

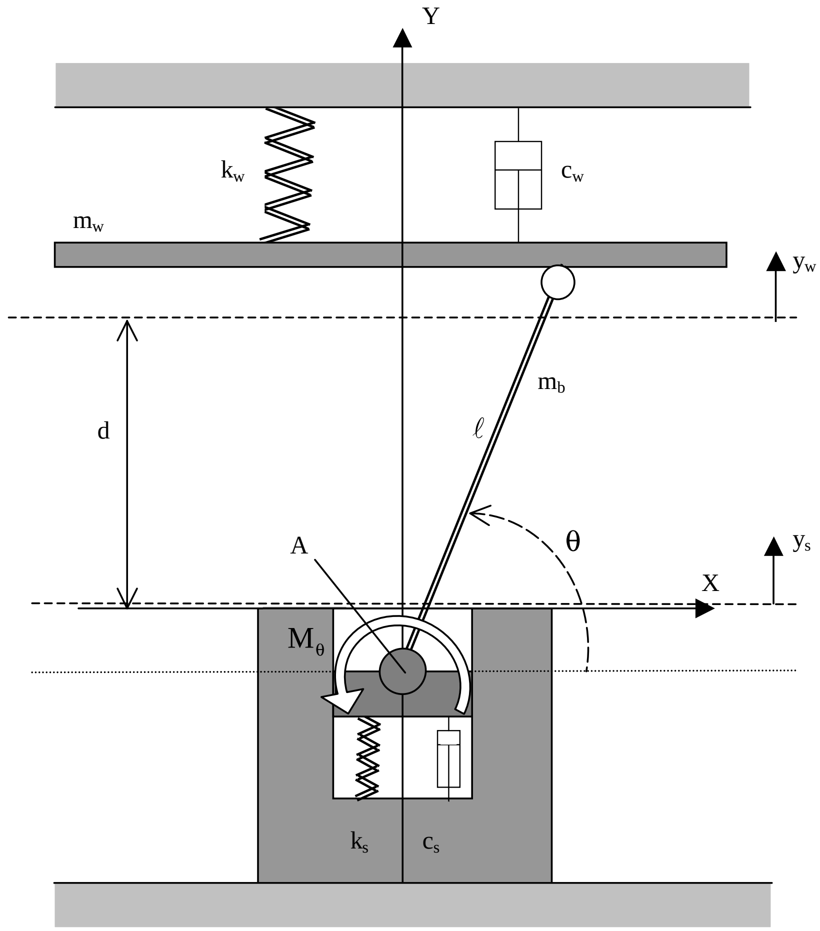

Figure 1 – Oscillating constrained bar.

According to (Fenili, 2005), the constrained governing equations of motion for this system are given by:

$$(m_b + m_s)\ddot{y}_s + c_s\dot{y}_s + k_s y_s - m_b d_{Acmb}\dot{\theta}^2 \sin\theta + m_b d_{Acmb}\ddot{\theta}\cos\theta + F_N = 0 \qquad (1)$$

$$m_w\ddot{y}_w + c_w\dot{y}_w + k_w y_w - F_N = 0 \qquad (2)$$

$$(I_{b,cm} + m_b d^2_{Acmb})\ddot{\theta} + m_b d_{Acmb}\ddot{y}_s \cos\theta + F_N \ell \cos\theta = M_\theta \qquad (3)$$

and the constraint condition is given by:

$$d - y_s + y_w - \ell \sin\theta = 0 \qquad (4)$$

where $I_{b,cm}$ represents the bar moment of inertia around its center of mass, m_b represents the mass of the bar, d_{Acmb} represents the distance from A to the cm of the bar, c_w represents the damping coefficient of m_w, c_s represents the damping coefficient associated with mass m_s, k_w represents the stiffness coefficient of mass m_w, k_s represents the stiffness coefficient associated with m_s, F_N represents the amplitude of the normal force. It is assumed the there are no friction forces involved and ℓ represents the total length of the bar.

Equation (1), Eq. (2) and Eq. (3) are the equations of motion for y_s, y_w, and θ. Equation (4) is an additional relationship between the generalized coordinates y_s, θ and y_w when contact occurs. The Eqs. (1) to (4) provides four equations and four unknowns (y_s, θ, y_w and F_N) considering the constrained problem and three equations and three unknowns (y_s, θ and y_w) considering the unconstrained problem. In the unconstrained case, Eq. (4) does not apply and $F_N = 0$.

3. The contact case

In contact, for this problem, there is the loss of one degree of freedom. In other words, one of the variables is dependent of all the others. The best choice is the elimination of the generalized coordinate y_w, which is not always present into the system represented by the oscillating bar (Schäfer et all, 2004). The new set of equations (Fenili, 2005) is given by:

$$\begin{aligned}\ddot{y}_s + \frac{1}{a_1 m_t + a_3\cos^2\theta}\Big(& a_1(c_s + c_w)\dot{y}_s + a_1(k_s + k_w)y_s + a_1 c_w \ell\dot{\theta}\cos\theta + a_1 k_w \ell\sin\theta - a_1 a_2\dot{\theta}^2\sin\theta \\ & - m_b c_w \ell^2 d_{Acmb}\dot{\theta}\cos^3\theta - m_b k_w \ell^2 d_{Acmb}\sin\theta\cos^2\theta + m_b k_w d\ell d_{Acmb}\cos^2\theta + \ell(m_w \ell c_s - m_b d_{Acmb} c_w)\dot{y}_s\cos^2\theta \\ & + \ell(m_w \ell k_s - m_b d_{Acmb} k_w) y_s\cos^2\theta - a_1 k_w d\Big) = -\frac{a_2\cos\theta}{a_1 m_t + a_3\cos^2\theta} M_\theta\end{aligned} \tag{5}$$

$$\begin{aligned}\ddot{\theta} + \frac{1}{a_1 m_t + a_3\cos^2\theta}\Big(& c_w \ell(m_t\ell - a_2)\dot{\theta}\cos^2\theta + k_w \ell(m_t\ell - a_2)\sin\theta\cos\theta - k_w d(m_t\ell - a_2)\cos\theta \\ & + \big(a_2 m_b d_{Acmb} - m_w \ell(m_t\ell - a_2)\big)\dot{\theta}^2\sin\theta\cos\theta + \big(k_w(m_t\ell - a_2) - a_2 k_s\big)y_s\cos\theta \\ & + \big(c_w(m_t\ell - a_2) - a_2 c_s\big)\dot{y}_s\cos\theta\Big) = \frac{m_t}{a_1 m_t + a_3\cos^2\theta} M_\theta\end{aligned} \tag{6}$$

It is considered here the fully plastic impact case for the calculation of the velocities immediately after contact. Separation will take place when the normal force is zero.

As soon as these two variables are known, the remaining variable, y_w, is also known through Eq. (4). Equations (5) and (6) represent, respectively, the time behavior of the generalized coordinates y_s and θ during the contact condition. In (Fenili, 2005), an analytical expression to the reaction force, F_N, is also presented.

4. The determination of the velocities after contact (impact)

To better distinguish between velocities right before and right after impact, they are denoted with superscripts '+' (after) and '-' (before). Their two components in X- and Y-direction are indicated by corresponding subscripts 'x' and 'y'. The subscript 'b' is for the bar.

During impact, one has the additional equation, which relates the velocities before and after impact, in the direction normal to the contact surface, i.e. in Y-direction:

$$\varepsilon_y = -\frac{v_{1y}^+ - v_{2y}^+}{v_{1y}^- - v_{2y}^-} \tag{7}$$

with

$$\begin{aligned} v_{1y}^- &= \dot{y}_s^- + \dot{\theta}^-\ell\cos\theta \\ v_{1y}^+ &= \dot{y}_s^+ + \dot{\theta}^+\ell\cos\theta \\ v_{2y}^- &= \dot{y}_w^- \\ v_{2y}^+ &= \dot{y}_w^+ \end{aligned} \tag{8}$$

where ε_y represents the coefficient of restitution in Y direction. In this work, it is assumed that there is fully plastic impact, i.e. the impacting bodies maintain steady contact as far as the contact force is not zero (otherwise, they will separate). This leaves $\varepsilon_y = 0$, and hence

$$v_{1y}^{+} = v_{2y}^{+} \tag{9}$$

or

$$\dot{y}_w^{+} = \dot{y}_s^{+} + \dot{\theta}^{+}\ell\cos\theta = v_{by}^{+} + \dot{\theta}^{+}\left(\ell - d_{Acmb}\right)\cos\theta \tag{10}$$

With these equations it is possible to calculate all the velocities right after impact, given the velocities before impact. Additionally, but not needed here, it is also possible to calculate the appropriate linear impulses. $\hat{A}_x$ and $\hat{A}_y$ are impulses calculated at point A and $\hat{P}_y$ is the impulse calculated at the point of contact.

To summarize, one has the following eight equations to determine all the five velocities right after impact ($\dot{y}_s^{+}$, v_{bx}^{+}, v_{by}^{+}, $\dot{y}_w^{+}$, $\dot{\theta}^{+}$), as well as the impulses ($\hat{A}_x$, $\hat{A}_y$, $\hat{P}_y$):

$$m_w\dot{y}_w^{+} - m_w\dot{y}_w^{-} = \hat{P}_y \tag{11}$$

$$m_b v_{by}^{+} - m_b v_{by}^{-} = \hat{A}_y - \hat{P}_y \tag{12}$$

$$m_b v_{bx}^{+} - m_b v_{bx}^{-} = \hat{A}_x \tag{13}$$

$$I_{b,cm}\dot{\theta}^{+} - I_{b,cm}\dot{\theta}^{-} = \hat{A}_x d_{Acmb}\sin\theta - \hat{A}_y d_{Acmb}\cos\theta - \hat{P}_y\left(\ell - d_{Acmb}\right)\cos\theta \tag{14}$$

$$m_s\dot{y}_s^{+} - m_s\dot{y}_s^{-} = -\hat{A}_y \tag{15}$$

$$v_{bx}^{+} = -\dot{\theta}^{+} d_{Acmb}\sin\theta \tag{16}$$

$$v_{by}^{+} = \dot{y}_s^{+} + \dot{\theta}^{+} d_{Acmb}\cos\theta \tag{17}$$

$$\dot{y}_w^{+} = \dot{y}_s^{+} + \dot{\theta}^{+}\ell\cos\theta = v_{by}^{+} + \dot{\theta}^{+}\left(\ell - d_{Acmb}\right)\cos\theta \tag{18}$$

First, it is eliminated the impulses. $\hat{A}_y$ is simply obtained from Eq. (15) or by adding the two equations (11) and (12), giving

$$\begin{aligned}\hat{A}_y &= -\left(m_s\dot{y}_s^{+} - m_s\dot{y}_s^{-}\right) \\ &= m_b v_{by}^{+} - m_b v_{by}^{-} + m_w\dot{y}_w^{+} - m_w\dot{y}_w^{-}\end{aligned} \tag{19}$$

$\hat{A}_x$ also goes simply with Eq. (13),

$$\hat{A}_x = m_b v_{bx}^{+} - m_b v_{bx}^{-} \tag{20}$$

and $\hat{P}_y$ is simply obtained directly from Eq. (11) or by adding the two equations (12) and (15),

$$\begin{aligned}\hat{P}_y &= m_w \dot{y}_w^+ - m_w \dot{y}_w^- \\ &= -\left(m_b v_{by}^+ - m_b v_{by}^-\right) - \left(m_s \dot{y}_s^+ - m_s \dot{y}_s^-\right)\end{aligned} \tag{21}$$

Comparing Eq. (19) with (21), it is observed that both equations yield the same result for the linear momenta before and after impact. To determine now the velocities right after impact, one can rely on the Eqs. (14), (16), (17), (18) and (19) (or (21), which is the same). Replacing v_{bx}^+, v_{by}^+ and $\dot{y}_w^+$, one arrives at the two equations for the unknown velocities $\dot{y}_s^+$ and $\dot{\theta}^+$:

$$\left(m_s + m_b + m_w\right)\dot{y}_s^+ = m_s \dot{y}_s^- + m_b v_{by}^- + m_w \dot{y}_w^- - \dot{\theta}^+\left(m_b d_{Acmb} + m_w \ell\right)\cos\theta \tag{22}$$

$$\begin{aligned}&\left[I_{b,cm} + m_b d_{Acmb}^2 \sin^2\theta + m_w \ell\left(\ell - d_{Acmb}\right)\cos^2\theta\right]\dot{\theta}^+ = \dot{y}_s^+\left[m_s d_{Acmb} - m_w\left(\ell - d_{Acmb}\right)\right]\cos\theta \\ &+ I_{b,cm}\dot{\theta}^- - m_s \dot{y}_s^- d_{Acmb}\cos\theta - m_b v_{bx}^- d_{Acmb}\sin\theta + m_w \dot{y}_w^-\left(\ell - d_{Acmb}\right)\cos\theta\end{aligned} \tag{23}$$

And with

$$v_{by}^- = \dot{y}_s^- + \dot{\theta}^- d_{Acmb}\cos\theta$$

and

$$v_{bx}^- = -\dot{\theta}^- d_{Acmb}\sin\theta$$

these equations can finally be express by means of the independent velocities, $\dot{y}_s^-$, $\dot{y}_w^-$ and $\dot{\theta}^-$, right before impact:

$$\left(m_s + m_b + m_w\right)\dot{y}_s^+ = \left(m_s + m_b\right)\dot{y}_s^- + m_w \dot{y}_w^- + m_b \dot{\theta}^- d_{Acmb}\cos\theta - \dot{\theta}^+\left(m_b d_{Acmb} + m_w \ell\right)\cos\theta \tag{24}$$

$$\begin{aligned}&\left[I_{b,cm} + m_b d_{Acmb}^2 \sin^2\theta + m_w \ell\left(\ell - d_{Acmb}\right)\cos^2\theta\right]\dot{\theta}^+ = \dot{y}_s^+\left[m_s d_{Acmb} - m_w\left(\ell - d_{Acmb}\right)\right]\cos\theta \\ &+ \left(I_{b,cm} + m_b d_{Acmb}^2 \sin^2\theta\right)\dot{\theta}^- - m_s \dot{y}_s^- d_{Acmb}\cos\theta + m_w \dot{y}_w^-\left(\ell - d_{Acmb}\right)\cos\theta\end{aligned} \tag{25}$$

With the abbreviations:

$$m_{tot} = m_s + m_b + m_w$$

$$I_{tot} = I_{b,cm} + m_b d_{Acmb}^2 \sin^2\theta + m_w \ell\left(\ell - d_{Acmb}\right)\cos^2\theta$$

$$r_1 = \left(m_s + m_b\right)\dot{y}_s^- + m_w \dot{y}_w^- + m_b \dot{\theta}^- d_{Acmb}\cos\theta$$

$$r_2 = -m_s \dot{y}_s^- d_{Acmb}\cos\theta + m_w \dot{y}_w^-\left(\ell - d_{Acmb}\right)\cos\theta + \left(I_{b,cm} + m_b d_{Acmb}^2 \sin^2\theta\right)\dot{\theta}^-$$

$$\alpha_1 = \left(m_b d_{Acmb} + m_w \ell\right)\cos\theta$$

$$\alpha_2 = \left[m_s d_{Acmb} - m_w\left(\ell - d_{Acmb}\right)\right]\cos\theta$$

one finally obtains

$$\dot{y}_s^+ = \frac{r_1 I_{tot} - r_2 \alpha_1}{\alpha_1 \alpha_2 + m_{tot} I_{tot}} \tag{26}$$

$$\dot{\theta}^+ = \frac{r_1 \alpha_2 + r_2 m_{tot}}{\alpha_1 \alpha_2 + m_{tot} I_{tot}} \tag{27}$$

The denominator of these two equations then writes

$$\alpha_1 \alpha_2 + m_{tot} I_{tot} =$$
$$= m_s m_b d_{Acmb}^2 + m_{tot} I_{b,cm} + m_b m_w \left[\ell \left(\ell - 2 d_{Acmb} \right) \cos^2 \theta + d_{Acmb}^2 \right] + m_s m_w \ell^2 \cos^2 \theta + m_b^2 d_{Acmb}^2 \sin^2 \theta$$

In order to check Eqs. (26) and (27), one case is investigated; i.e. for $\theta = 90^o$ we should maintain the simple translational impact between the combined rigid body consisting of the two masses m_s and m_b and the wall with mass m_w.

For the fully plastic impact one obtains from Eqs. (26) and (27) with $\alpha_1 = 0$ and $\alpha_2 = 0$:

$$\dot{y}_s^+ \left(\theta = 90^o \right) = \frac{r_1}{m_{tot}} = \frac{\left(m_s + m_b \right) \dot{y}_s^- + m_w \dot{y}_w^-}{m_{tot}} \tag{28}$$

$$\dot{\theta}^+ \left(\theta = 90^o \right) = \frac{r_2}{I_{tot}} = \frac{I_{b,cm} + m_b d_{Acmb}^2}{I_{tot}} \dot{\theta}^- \tag{29}$$

where the first equation (for the translational motion) coincides with the result obtained for simple impact of two rigid bodies

5. Conclusions

The set of governing equations of motion for the constrained condition is quite different from the one that governs the unconstrained movement of the system. One of these sets is always generating the initial states for the other. The number of degrees of freedom involved changes from one set of equations to the other. In this context, the determination of the velocities after contact (impact) is very important. The velocity expressions presented in Eqs. (26) and (27) are the necessary corrections one must do when considering the fully plastic impact case. If this correction is not taken into consideration in the numerical integration of the governing equations, the system will gain energy after impact, which is not true.

The problem presented in this paper and the procedures developed for its analysis can be extended to many other systems and situations (including more complex ones). The theory presented here can be applied to problems in which robots have to follow some prescribed patterns or trajectories when in contact with the environment (like in painting activities, for instance, or the ROKVISS experiment at DLR).

The next step is the development of the analytical expressions for the velocities after impact considering any value for the coefficient of restitution.

6. References

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