DEFLECTIONS OF THIN PLATES: INFLUENCE OF THE SLOPE OF THE PLATE IN THE APLICATION OF LINEAR AND NONLINEAR THEORIES

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Abstract. *In thin plates, always that the deflection and the slope not do exceed, respectively, 20% of thickness and 10^{-3} rd, the Linear Theory (LT) can be apllied. If these limits of deflection and slope are exceeded, the Nonlinear-Theory (NLT) of the plates, which takes into account the displacement of the midplane of the plate, has to be used. A lot of authors have checked the application limits from the LT and the NLT through the numerical and analytical methods. However, the analysis are carried out by checking only the relation between load-deflection and never the relation plate load-slope. The objective of this work is to study the relation load-slope of the plate using the LT and the NLT and to stablish the application limits of these theories. The work is developed to square plates, clamped and simply supported, sumitted to uniformly distributed load. The finite element method with isoparametric quadrilateral elements is applied in the numerical solutions.*

Keywords: *Thin Plates, Large Deflections, Slope, Finite Element Method*

1. Introduction

Many applications of engineering structures use plates as structural elements, in mechanics, in civil or aviation, for instance in metal platforms, in commercial floor or industries, engine discs, tank bases, containers and tanks of different sizes and that have to support internal and external efforts. These important applications led to many researchers, in the beginning of 1800, studying this problem in order to develop a plate fundamental theory, to get analytic solutions for simple cases. From 1930, some numeric methods were developed for solving complex problems. The results from those analytic works, from that time, are still used as base of comparison with new numeric models.

Taking Kirchhoff hypothesis for linear theory – LT (or classic theory, or small displacement theory) for thin plates, isotropic plates, homogeneous and elastic plates, results

$$w < \frac{t}{5} \tag{1}$$

$$\theta < 10^{-3} \text{ rd} \tag{2}$$

The displacement limit w from Eq. (1) is checked in many numeric and analytic development (Chia, 1980; Duan and Mahendran, 2003; Zhang and Cheung, 2003), while the slope limit θ from Eq. (2) did not receive analysis before.

2. Fundamental equations for thin plates subjected to small deflection

Taking an infinitesimal element *dxdy* from a plate that is under load pressure across all surfaces, distributed in a uniform way in the area, *p*, the application of equilibrium equations and Hooke Law provides the different equation below

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \tag{3}$$

where

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{4}$$

The Eq. (3) is known by Sophie-Germain-Lagrange equation, presented in 1811. The Eq. (4) represents the plate flexural rigidities In this equation, E is the material modulus of elasticity, ν is the Poisson's ratio and t is the thickness of the plate.

3. Formulation of the finite element method for the problem of thin plates subjected to small deflections

3.1. Element properties

For solving by numbers the problem of thin plates, the rectangular plate is discretized in quadrilateral elements. The plate element, placed in xy plan, is shown in Fig. 1. Each nodal point has three degree of freedom: the vertical displacement, w, the rotation around x axis, θ_x , and the rotation around y axis, θ_y . The rotations are related with the slopes through

$$\theta_x = \frac{\partial w}{\partial y} \quad , \qquad \theta_y = \frac{\partial w}{\partial x} \tag{5}$$

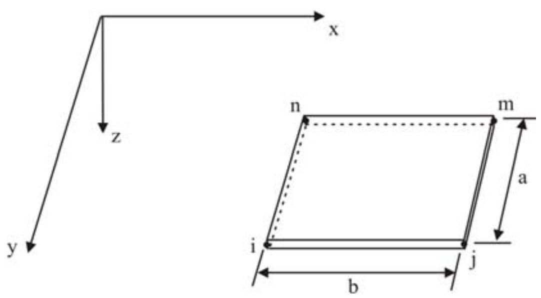

Figure 1 – Quadrilateral Element

The vector of nodal displacements of the element is represented by

$$\{\delta\}_e = \{w_i, \theta_{xi}, \theta_{yi}, w_j, \theta_{xj}, \theta_{yj}, w_m, \theta_{xm}, \theta_{ym}, w_n, \theta_{xn}, \theta_{yn}\} \tag{6}$$

The displacement function, w_e ,define the displacement of any point of the element. The nodal displacements link to the displacement function through the expression

$$\{w\}_e = [P]\{\delta\}_e \tag{7}$$

The matrix $[P]$ is a function of the point position considered and it is called the shape matrix. For each element, the linear theory for thin plates provides

(a) The vector of deformation-displacement:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_x \\ \gamma_{xy} \end{Bmatrix}_e = \left\{ -\frac{\partial^2 w}{\partial x^2}, -\frac{\partial^2 w}{\partial y^2}, -2\frac{\partial^2 w}{\partial x \partial y} \right\} \tag{8}$$

which results in

$$\{\varepsilon\}_e = [B]\{\delta\}_e \tag{9}$$

(b) The relation of stress-displacement:

$$\begin{Bmatrix} \sigma_x \\ \sigma_x \\ \tau_{xy} \end{Bmatrix}_e = \frac{Ez}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}\{\varepsilon\}_e \tag{10}$$

(c) The moments according the stresses:

$$\{M\}_e = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix}_e = \int_{-t/2}^{t/2} z\{\sigma\}_e dz \tag{11}$$

Using the Eq. (10), the solution of Eq. (11) provides

$$\{M\}_e = [D]\{\varepsilon\}_e \tag{12}$$

where

$$[D] = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \tag{13}$$

is the elasticity matrix of the plate element.

3.2. Potential energy principle

For determining the equations which are used in the finite element method of the thin plates, it is applied the principle of the minimum potential energy (Pilkey and Wunderlich, 1994). The variation of the potential energy, $\Delta\Pi$, of a plate is

$$\Delta\Pi = \sum_1^n \iint_A \left(M_x\Delta\varepsilon_x + M_y\Delta\varepsilon_y + M_{xy}\Delta\varepsilon_{xy}\right)dxdy - \sum_1^n \iint_A (p\Delta w)dxdy = 0 \tag{14}$$

where n is the number of elements which consists the plate and A is the surface area of the element. The Eq. (14) can be rewritten in this form

$$\sum_1^n \iint_A \left(\{\Delta\varepsilon\}_e^T\{M\}_e - p\Delta w\right)dxdy = 0 \tag{15}$$

With the replacements of the Eq. (7), (9) and (12) in the Eq. (15), it is obtained

$$\sum_1^n \iint_A \{\Delta\delta\}_e^T\left([k]_e\{\delta\}_e - \{Q\}_e\right) = 0 \tag{16}$$

In the Eq. (16), the stiffness matrix of the element, $[k]_e$ and the vector of nodal forces of the element, $\{Q\}_e$, are, given, respectively, by

$$[k]_e = \iint_A [B]^T [D][B] dxdy \tag{17}$$

$$\{Q\}_e = \iint_A [P]^T p dxdy \tag{18}$$

As the variations in $\{\delta\}_e$ are independent and arbitrary, the Eq. (16) leads to the following expression to the equilibrium of nodal force of the element

$$[k]_e \{\delta\}_e = \{Q\}_e \tag{19}$$

Considering, now, the whole plate, the Eq. (16) provides the following equation that is valid for any variation of displacement $\{\Delta\delta\}$

$$\{\Delta\delta\}^T ([K]\{\delta\} - \{Q\}) = 0 \tag{20}$$

whose development results in

$$[K]\{\delta\} = \{Q\} \tag{21}$$

The stiffness matrix of the plate, $[K]$ and the plate nodal vector forces, $\{Q\}$ are obtained by superposition of the stiffness matrix and nodal forces of the elements, like

$$[K] = \sum_1^n [k]_e \tag{22}$$

and

$$\{Q\} = \sum_1^n \{Q\}_e \tag{23}$$

The general procedure for solving problems of thin plates subjected to small deformations can be summarized into the following steps (Ugural, 1981):

1. Determine $[k]_e$ through Eq. (17) in terms of element property. Generate $[K]$ through Eq. (22).
2. Determine $\{Q\}_e$ through Eq. (18) in terms of the load applicated. Generate $\{Q\}$ through Eq. (23).
3. Determine the nodal displacement through Eq. (21), satisfying the boundary conditions.
4. Determine the stresses and the moments in the element through the Eqs. (10) and (12), respectively.

3.3. Isoparametric quadrilateral element

The displacement function w_e is expressed by a third order polynomial, like

$$w_e = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^3 + a_{10} y^2 + a_{11} x^3 y + a_{12} xy^3 \tag{24}$$

that defines the displacement of any point of the element *ijmn*. The number of the terms of this function is equal to the number of degrees of freedom of the element.

The replacement of the Eqs. (5) and (24) in the Eq. (7) results in

$$\{\delta\}_e = [C]\{a\} \tag{25}$$

Even so,

$$\{a\} = [C]^{-1}\{\delta\}_e \tag{26}$$

In the Eq. (26), the matrix [C] only depends of the nodal coordinate element. However, the Eq. (7) is restricted like

$$\{w\}_e = [L]\{a\} \tag{27}$$

where

$$[L] = [1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3]$$

Using the Eqs. (9), (25), (26) and (27), the vector displacement can be expressed like

$$\{\varepsilon\}_e = [B][C]\{a\} = [H]\{a\} \tag{28}$$

The deformation-displacement matrix is obtained replacing the Eq. (26) in the Eq. (28). The resulting equation supplies that [B]=[H][C]$^{-1}$. The replacement of this equation in Eq. (17) results in the equation of the stiffness matrix of the element like

$$[k]_e = \left[[C]^{-1}\right]^T \left(\iint_A [H]^T [D][H] dxdy \right) [C]^{-1} \tag{29}$$

4. Fundamental equation to deflection of thin plates subjected to large deflections

When the deformations are large, the plate element must be considered in its deformed condition. The application of Hooke´s Law and the equations of equilibrium of plates, in this case, results in

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] \tag{30}$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{t}{D}\left(\frac{p}{t} + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right) \tag{31}$$

The Eqs. (30) and (31) are the equations developed and presented in 1910 by Von Kármán. In these equations, $\phi=\phi(x,y)$ is the stress function.

5. Formulation of the finite element method to the problem of the thin plates subjected to large deflections

5.1. Element properties

The formulations of plates subjected to small deflections is used in the case of large deflections, that includes the effect of deformations in the midplane of the plate and its respected stress. For this, it is considered a plate subjected at first to the applicated forces in the midplane, which stand constant during the bending. In this case, the deformations and the stress into a midplane of the element can be represented by (Ugural, 1981),

$$\{\varepsilon\}_e = \left\{ \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \right\} \tag{32}$$

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix}_e = t \begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_y \\ \overline{\tau}_{xy} \end{Bmatrix}_e \quad \text{or} \quad \{N\}_e = t\{\overline{\sigma}\}_e \tag{33}$$

The deformations to the bending and moments are given according to the Eqs. (8) and (11). The resulting stresses include the stress provocated by directed forces and by flexural moments. The stress and deformations due to bendings mix through the Eq. (10). The stress and deformations in the plan are related by

$$\begin{Bmatrix} \overline{\sigma}_x \\ \overline{\sigma}_x \\ \overline{\tau}_{xy} \end{Bmatrix}_e = \frac{Ez}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \{\overline{\varepsilon}\}_e \tag{34}$$

5.2. Potential energy principle

Considering that the deformations due to directed force and due to the bending are independent, the expression to the potential energy of the plate is given by

$$\Pi = \frac{1}{2}\iint_A \{\varepsilon\}_e^T \{M\}_e dxdy + \frac{1}{2}\iint_A \begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix}^T [\overline{\sigma}]_e \begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} dxdy - \iint_A (pw) dxdy \tag{35}$$

although

$$[\overline{\sigma}]_e = \begin{bmatrix} \overline{\sigma}_x & \overline{\tau}_{xy} \\ \overline{\tau}_{xy} & \overline{\sigma}_y \end{bmatrix}_e \tag{36}$$

it is the membrane stresses matrix in the midplane of the plate.

With the displacement function given by Eq. (24), the slopes in the element, θ_x e θ_y , are represented by

$$\begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix}_e = \{\theta\}_e = [S]\{a\} \tag{37}$$

Replacing the Eq. (26) in the Eq. (37), it is obtained

$$\{\theta\}_e = [S][C]^{-1}\{\delta\}_e = [G]\{\delta\}_e \tag{38}$$

where $[G]$ is only function of the coordinate of the nodal points of the element.

Replacing the Eq. (37) in the Eq. (35), it has

$$\Pi = \frac{1}{2}\iint_A \{\varepsilon\}_e^T \{M\}_e dxdy + \frac{1}{2}\{\delta\}_e^T \left(\iint_A [G]^T [\overline{\sigma}][G] dxdy \right) \{\delta\}_e - \iint_A (pw) dxdy \tag{39}$$

The application the minimum potential energy principle supplies a modified expression to the equilibrium of the nodal forces of the element, like

$$\{Q\}_e = [k]_e \{\delta\}_e + [k_G]_e \{\delta\}_e = [k_T]_e \{\delta\}_e \tag{40}$$

In the Eq. (40), the new term $[k_G]_e$ it is called initial stress matrix or geometric stress matrix, which can be obtained through

$$[k_G]_e = t\left[[C]^{-1}\right]^T \left(\iint_A [S]^T [\overline{\sigma}][S] dxdy \right) [C]^{-1} \tag{41}$$

The matrix $[k_T]_e$ of Eq. (40) is called total stiffness matrix of the element.

The general procedure for solving plate problems of thin plates subjected to large deflections can be summarized in the following steps:

1. Consider the stress of Eq. (34) due to forces in the plan, at first equal to zero. Apply the procedures (steps 1 to 3) of the section 3.2 to obtain the solution of nodal displacement to small deflections.
2. Determine the slope into the centroid of each element through Eq. (38).
3. Determine the strains of the midplane through Eq. (32).
4. Determine the stresses on the midplane through Eq. (34).
5. Determine the geometric stiffness matrix through Eq. (41).
6. Determine the total stiffness matrix of the element.
7. Repeat the steps 1 to 4, until there is a satisfied convergence to the stress in the midplane of the elements of the plate.

6. Numerical examples

With the presented formulation above, a computational program was elaborated to determine deflections and slopes of thin plates with nonlinear behavior. The obtained results were compared to the numerical and analytic results obtained by other authors (Chia, 1980; Pica and Wood, 1979; Singh and Elaghabash, 2003). The results are expressed into the nondimensional form to the load, $Q = pa^4/Et^4$, and to the deflection, $W = w/t$. It was analyzed a square plate which side a = 2 m, thickness t = 8 cm, E = 210 GPa, ν = 0.316. The Fig. 2 and Fig. 3 present the deflection in the middle of the plate completely clamped and simply supported, respectively.

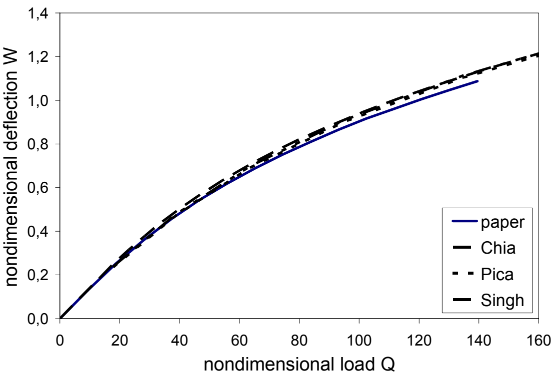

Figure 2. Deflection in the center clamped plate

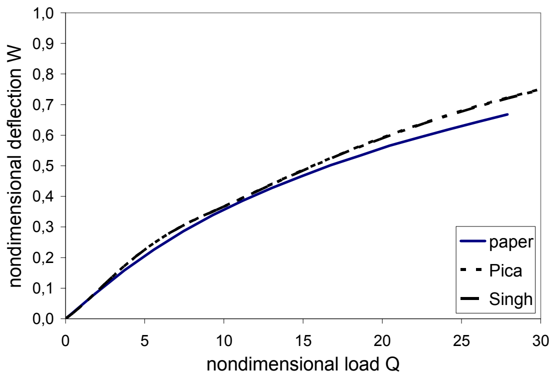

Figure 3. Deflection in the center simply supported plate

The graphics of maximum slope due to the loading for plates with with clamped and simply supported boundaries are shown, respectively, in the Figs. 4 e 5.

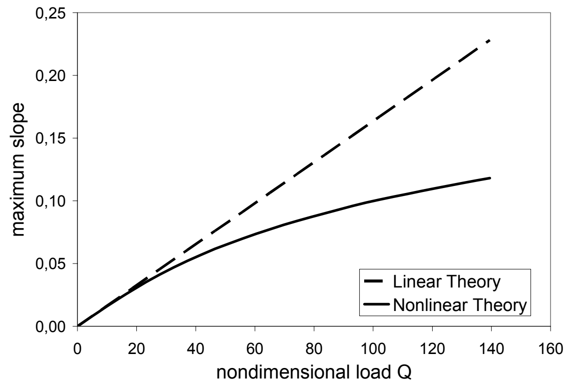

Figure 4. Maximum slope for clamped plate

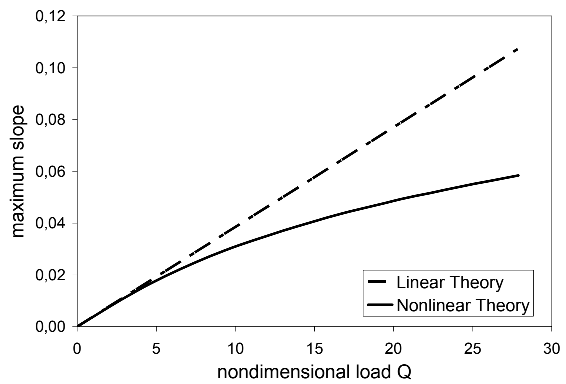

Figure 5. Maximum slope for simply supported plate

7. Conclusions

According to the Fig. 2 and Fig. 3 showed, to the the square clamped plate in all the sides, the maximum difference between the results in this work and the obtained results by Chia, Pica and Sing is smaller than 5%. For square simply supported plates in all the sides, the maximum difference is smaller than 7%.

In the case of clamped plates, the Fig. 4 shows that even the angle of 0.025 rd the LT shows satisfied results, or either, with a 150% limit over the normal used of 0.010 rd. For simply supported plates, the Fig. 5 shows that this limit is of 0.016 rd, or either, 60% over the used limit.

The developed model do not converge to values of nondimensional loads over 150 in the case of clamped plates, and of 30 to supported plates. These values are lower to the ones obtained by other authors, who use nondimensional loads equal to 400. Nevertheless, in the case of thin plates, isotropic, homogeneous and elastic, normally used in the structural applications in the different areas of engineering, nondimensional loads over 25 lead the stress values already, in the plate, over the elastic limit.

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