

ASYMPTOTIC WAVE PROPAGATION ALONG CHAINS OF REPETITIVE SYSTEMS

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Abstract
We consider wave propagation along chains of non-uniform repetitive systems. For harmonic waves, the governing equation for this class of systems is a second order difference equation with non-constant coefficients. When the ratio between the sub-systems inertia force and the sub-systems coupling force is small, we developed an asymptotic theory for harmonic wave propagation along this class of systems. We assume an asymptotic expansion for the dependent variable based on the small parameter defined above. Its zero order term is the solution of a second order differential equation with variable coefficient. To solve it, we use a change of the independent variable, which maps the original equation to a model equation with known solution in terms of special functions. The choice of the model equation is guided by the topology of the turning points of the original equation. By assuming an asymptotic expansion for the change of independent variable, and incorporating it into the asymptotic expansion for the dependent variable, the strategy used to solve the zero order problem is applied to the higher order problems. Each set of turning points is modeled by a transfer matrix. As a result, quantities of interest are given along the system as a product of transfer matrices. We apply the asymptotic theory to case with known solution in terms of Bessel functions. We illustrate how to apply the asymptotic theory to wave propagation problems.
Keywords: *repetitive system, asymptotic method, difference equation, wave reflection, comparison equation method*

1. Introduction.

We consider wave propagation along one-dimensional repetitive systems. These type of systems are chains of interconnected subsystems which can be identical to each other (uniform repetitive system) or they may vary among each other (non-uniform repetitive system). In this work, only single-degree-of-freedom (d.o.f) subsystems are considered. Examples of repetitive systems are chains of coupled oscillators, crystal lattices and any non-uniform one-dimensional continuous medium discretized by pieces where at each piece the medium is assumed to have uniform properties.

A chain of coupled pendula is an example of a chain of coupled oscillators. The subsystem is a pendulum coupled to its two nearest neighbors only through springs. Linear gravity waves propagating along a shallow channel with non-uniform bottom, which is discretized by steps is an example of a continuous one-dimensional medium with non-uniform properties discretized by pieces where at each piece the medium has uniform properties. The subsystem is a shelf between two consecutive steps, and the coupling between the two nearest shelves is given by the continuity of the free surface displacement and of the horizontal velocity (Devillard *et al.*, 1988).

For harmonic wave propagation, the governing equation for such class of repetitive systems is a second order difference equation (2nd order d.e.) which can be reduced to the canonical form

$$y_{x+1} - 2y_x + y_{x-1} + \beta^2 Q(x) y_x = 0 \tag{1}$$

where β^2 represents the ratio between the subsystem inertia and the coupling force between subsystems and $Q(x)$ is the 2nd. order d.e. coefficient function. For the case of a chain of coupled pendula, β^2 is the ratio between the pendulum inertia and the coupling spring force and $Q(x) = 1/(1+\epsilon(x)) - \omega^2$, where $(1+\epsilon(x))$ and ω are, respectively, the non-dimensional pendulum lenght and non-dimensional wave frequency.

The objective of this work is to develop an asymptotic theory for wave propagation along long non-uniform one-dimensional repetitive systems when the parameter β is small. The asymptotic theory reveals to some degree how the non-uniformity in the system parameters affect wave propagation in such class of systems. This could be an useful tool to design the non-uniformity in the system parameters such that the repetitive system behaves, for example, as wave filter, selecting the range of wave frequencies that waves are allowed to pass or reflected back. These design problems are one motivation for this work. The asymptotic theory developed here is outlined in the paragraph below.

We use the concept of pseudo-differential operators to show that 2nd order d.e.s are equivalent to infinite dimensional ordinary differential equations (i.d.o.d.e.s). We adapted the comparison equation method (see chapter 2 of Froman & Froman, 1997) to the infinite dimensional o.d.e. equivalent to 2nd order d.e.s.

In the comparison equation method, the original second order ordinary differential equation (2nd order o.d.e.) with an assumed small parameter is reduced by a change of the independent variable to a model equation with known solution in terms of special functions. The choice of the appropriate model equation is guided by the topology of the turning points of the original equation. Once we decided on the form of the model equation, we obtain a differential equation for the independent variable of the model equation in terms of the independent variable of the original equation. This equation is usually nonlinear and a function of the parameter assumed small. To make further progress, (i) the solution of the original 2nd order o.d.e. is assumed given as an amplitude function times a linear combination of two solution of the model equation, and (ii) expansions in the parameter assumed small are considered for the amplitude function and for the change of the independent variable. Then, the assumed solution is substituted into the original 2nd order o.d.e., which gives a sequence of linear non-homogeneous problems for the terms of the expansion of the amplitude function and change of variables. These problems can be solved and an asymptotic solution in terms of the parameter assumed small is obtained for the original 2nd order o.d.e..

When we apply the comparison equation method to an i.d.o.d.e., new issues arise. 2nd order o.d.e.s have semi-infinite passband, but 2nd order d.e.s or its equivalent i.d.o.d.e. has a finite passband. Each passband edge gives an equation which has part of the the system turning points as their solutions, while for 2nd order o.d.e.s there is only a single equation for the system turning points. The first term in the asymptotic expansion is a solution of a 2nd order o.d.e., which is capable of seeing only the lower edge of the repetitive system passband. To deal with this limitation, we split the real axis in intervals. Each interval is related to a set of turning points which are related only to the upper or lower passband edge. At the intervals with turning points related to the upper passband edge we modify the coefficient function $Q(x)$ in Eq. (1) such that the 2nd order o.d.e. governing the first term in the asymptotic expansion sees these turning points. To continue the asymptotic solution across the end points of each interval, we impose the continuity of the asymptotic solution and of its 1st order derivative. This leads naturally to the transfer matrix method. We can use a transfer matrix to relate the value of the asymptotic solution and the value of its 1st order derivative at one end point of an interval to their values at the other end of same interval, since we know the solution along each interval. As a result, the asymptotic solution along the repetitive system is given as a product of transfer matrices.

In the next section, we developed the asymptotic theory to solve Eq. (1) based on the comparison equation method. In section 3., we illustrate the performance of the asymptotic theory by applying it to Eq. (1) when the coefficient is linear. In this case Eq. (1) has known solution in terms of Bessel functions. In section 3, we also illustrate how to apply the asymptotic theory to wave propagation problems. In section 4., we discuss when to apply the asymptotic theory, its limitations, and the future work to pursue.

2. Asymptotic Theory.

Here we describe the asymptotic theory developed in this work. In the section 2.1 we use the concept of pseudo-differential operator to show that Eq. (1) is equivalent to an infinite-dimensional o.d.e. and we also show that the first term of an appropriate asymptotic expansion for the solution of Eq. (1) is governed by a 2nd order o.d.e.. The differences in behavior between the solution of 2nd order d.e.s and 2nd order o.d.e.s are discussed in section 2.2. The appropriate asymptotic expansion is given in section 2.3 and the application of the transfer matrix method to the present asymptotic theory is outlined in section 2.4

2.1 Pseudo-differential operator approach.

We use the concept of pseudo-differential operator (see Maslov and Nazaikinskii 1998) to show that Eq. (1) is equivalent to an i.d.o.d.e.. We consider the domain of Eq. (1) to be the whole real axis, so we can use the concept of a pseudo-differential operator. Since Eq. (1) is the governing equation for a finite, but long, repetitive system ($1 \leq x \leq N$, where N is the number of subsystems), we need to extend the coefficient function $Q(x)$ to the whole x axis. We assume $Q(x)$ constant for $x < 1$ and for $x > N$, but with $Q(x) = Q(1)$ for $x < 1$ and with $Q(x) = Q(N)$ for $x > N$. We consider the Fourier transform pair

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{y}(k)\exp(ikx)dk \text{ and } \hat{y}(k) = \int_{-\infty}^{\infty} y(x)\exp(-ikx)dx. \tag{2}$$

With Eq. (2) in mind, we can write Eq. (1) in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} L(x,k)\hat{y}(k)\exp(ikx)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty}[(\underbrace{(\exp(ik) - 2 - \exp(-ik)}_{2\cosh(ik)-2}) + \beta^2 Q(x)]\hat{y}(k)\exp(ikx)dk = 0. \tag{3}$$

We do a Taylor series expansion of the symbol $L(x,k)$ of the difference operator in Eq. (1) in terms of the variable k. Then, we apply the inverse Fourier transform to the resulting equation, and we obtain the i.d.o.d.e.

$$\left[\left(\sum_{n=0}^{\infty} \frac{2}{(2n)!}\frac{d^{2n}}{dx^{2n}}\right) + \beta^2 Q(x)\right] y(x) = 0. \tag{4}$$

Next, we consider the change of the independent variable. We write $x = \beta^{-2}z$ and substitute it into Eq. (4). In the limit $\beta \to 0$, the resulting infinite-dimensional o.d.e. in the z independent variable reduces to the 2nd order o.d.e.

$$\left(\frac{d^2}{dz^2} + Q(z)\right) y(z) = 0 \tag{5}$$

This fact motivate us to consider an asymptotic expansion in the parameter β to obtain an asymptotic solution for Eq. (1). The first term of the asymptotic expansion will be the solution of a 2nd order o.d.e. in the form given by Eq. (5). Before we proceed to construct the above mentioned asymptotic expansion in the parameter β, we need to be aware of the differences in behavior between 2nd order o.d.e.s and 2nd order d.e.s, otherwise the asymptotic theory we are trying to build are not going to be successful.

2.2 Uniform repetitive system.

We would like to illustrate aspect of the behavior of a uniform repetitive system governed by Eq. (1) that would be relevant to the development of the asymptotic theory. For a uniform repetitive system the coefficient function in Eq. (1) is constant, and in this case we write $Q(x) = Q$. We assume a plane wave solution of Eq. (1) in the form

$$y(x) = A\exp(ikx), \tag{6}$$

where A is the wave amplitude and k is the wavenumber. If we substitute Eq. (6) into Eq. (1), we obtain the dispersion relation

$$\cos(k) - 1 + \frac{\beta^2}{2}Q = 0 \text{ or } \; k = -i\ln\left\{1 - \frac{\beta^2}{2}Q \pm \left[-\beta^2 Q + \frac{\beta^4}{4}Q^2\right]^{1/2}\right\}. \tag{7}$$

According to Eq. (7), the wavenumber k assumes real values ($-\pi \le k \le \pi$) only for $-1 \le cos(k) \le 1$ which implies that $0 \le \beta^2 Q \le 4$. For real values of the coefficient function $\beta^2 Q$ outside this interval, the wavenumber assumes pure imaginary values and no wave propagation happens. The system has a passband behavior. Wave propagation is observed only for a finite interval of the system parameter. At the edges of the passband ($\beta^2 Q = 0$ and $\beta^2 Q = 4$), the solution behavior changes from a wave like behavior to an exponentially decaying or growing behavior. This behavior is shared by a non-uniform repetitive system, and in the presence of non-uniformity the lower and upper passband edges condition became equations for the non-uniform system turning points. These are solutions of

$$\beta^2 Q(x) = 0, \tag{8}$$
$$\beta^2 Q(x) = 4, \tag{9}$$

The 2nd order o.d.e. of the form

$$\frac{d^2y}{dx^2} + \beta^2 Q(x)y(x) = 0, \tag{10}$$

with $Q(x) = Q$ constant admits solution of the form (6). In this case, wave propagation happens for $\beta^2 Q > 0$. For 2nd order o.d.e.s with constant coefficients, the passband is semi-infinite. The turning points of Eq. (10) are solutions of Eq. (8). The turning points given by Eq. (9) are irrelevant for 2nd order o.d.e.s. This fact is the difference in behavior between 2nd order o.d.e.s and 2nd order d.e.s we are concerned.

As we will show below, the leading order problem of the asymptotic expansion in the parameter β to be considered has a non-uniform 2nd order o.d.e. as governing equation. As discussed above, this type of equation does not have the correct passband behavior and its turning points are only the solutions of Eq. (8). For the leading term of the asymptotic expansion in β to have the correct passband behavior, we proceed as follows. First, we split the x axis in intervals, where each of them contains only turning points solutions of Eq. (8) or Eq. (9). Second, in the intervals with turning points solutions of Eq. (9) we apply the asymptotic expansion in the parameter β to Eq. (1) with coefficient function $\tilde{Q}(x) = 4/(\beta^2) - Q(x)$ instead of $Q(x)$, and in the intervals with turning points solutions of Eq. (8) we apply the asymptotic expansion in the parameter β to Eq. (1) with the usual coefficient function $Q(x)$. The coefficient function $\tilde{Q}(x)$ results from the change of the dependent variable in Eq. (1). We write $y(x) = \exp(i\pi x)w(x)$ and substitute into Eq. (1). This results in Eq. (1) with w as dependent variable, but with the coefficient function $\tilde{Q}(x)$ instead of $Q(x)$. Third, to continue the asymptotic solution across the intervals endpoints, we assume the continuity of the asymptotic solution and its derivative across both interval endpoints. For the interval (x_j, x_{j+1}) we write

$$y_j(x_{j+1}) = y_{j+1}(x_{j+1}) \qquad \text{and} \qquad y_{j-1}(x_j) = y_j(x_j), \tag{11}$$

$$\frac{dy_j}{dx}(x_{j+1}) = \frac{dy_{j+1}}{dx}(x_{j+1}) \qquad \text{and} \qquad \frac{dy_{j-1}}{dx}(x_j) = \frac{dy_j}{dx}(x_j). \tag{12}$$

where $y_j(x)$ represents the asymptotic solution of Eq. (1) in the interval (x_j, x_{j+1}). The linearity of Eq. (1) and the conditions (11) and (12) allow us to relate the values $y_{j-1}(x_j)$ and $(dy_{j-1}/dx)(x_j)$ to the values $y_j(x_{j+1})$ and $(dy_j/dx)(x_{j+1})$ through a transfer matrix. As a result, the asymptotic solution of Eq. (1) and its first order derivative at both ends of the non-uniform part of the chain can be related by the product of transfer matrices.

2.3 Asymptotic expansion.

We assume the asymptotic expansion of the form

$$y(x) = \eta(z)\psi(\beta^{\theta}\phi(z)) \tag{13}$$

where $z = \beta^{\gamma}x$. For $\eta(z)$ and $\phi(z)$ we assume the expansions below in terms of the parameter β.

$$\eta(z) = \eta_0(z)\left\{1 + \sum_{n=1}^{\infty}\beta^{n\omega}\eta_n(z)\right\}, \qquad \text{and} \qquad \phi(z)= \phi_0(z)\left\{1 + \sum_{t=1}^{\infty}\beta^{t\delta}\phi_t(z)\right\}. \tag{14}$$

The value of the coefficients θ, ω and δ are assumed unknown a priori. Their values are chosen later such that the governing equation for the first term of the asymptotic expansion is a 2nd order o.d.e.. Next, we substitute Eq. (13), and the expansions for $\eta(z)$ and $\phi(z)$ in Eq. (14) into Eq. (4). As a result we obtain the equation

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{2n}\frac{2}{k!(2n-k)!}\left(\frac{d^{2n-k}\eta_0}{dz^{2n-k}} + \sum_{m=1}^{\infty}\beta^{m\omega}\sum_{p=0}^{2n-k}\frac{(2n-k)!}{p!(2n-k-p)!}\frac{d^{2n-k-p}\eta_0}{dz^{2n-k-p}}\frac{d^p\eta_m}{dz^p}\right)\right.\\
&\left[\sum_{l=1}^{k}\sum_{j_1+\ldots+j_l=k}\beta^{l\theta+2n\gamma}\frac{k!}{l!j_1!\ldots j_l!}\frac{d^l\psi}{d\phi^l}\left\{\frac{d^{j_1}\phi_0}{dz^{j_1}}\cdots\frac{d^{j_l}\phi_0}{dz^{j_l}} + \sum_{t=1}^{\infty}\beta^{t\delta}\left(\sum_{k_1=0}^{t}\sum_{k_2=0}^{t-k_1}\sum_{k_3=0}^{t-k_1-k_2}\cdots\sum_{k_{l-1}=0}^{t-k_1-\ldots-k_{l-2}}\right.\right.\right.\\
&\left.\left.\left.\left(\frac{d^{j_1}\phi}{dz^{j_1}}\right)_{k_1}\cdots\left(\frac{d^{j_{l-1}}\phi}{dz^{j_{l-1}}}\right)_{k_{l-2}}\left(\frac{d^{j_l}\phi}{dz^{j_l}}\right)_{t-k_1-\ldots-k_{l-1}}\right)\right\}\right] + \beta^{2n\gamma}\left[\frac{2}{(2n)!}\frac{d^{2n}\eta_0}{dz^{2n}} + \sum_{m=1}^{\infty}\beta^{m\omega}\right.\\
&\left.\left.\left(\sum_{p=0}^{2n}\frac{2}{p!(2n-p)!}\frac{d^{2n-p}\eta_0}{dz^{2n-p}}\frac{d^p\eta_m}{dz^p}\right)\right]\psi(\phi)\right\} + \left(\beta^2\eta_0 + \sum_{m=1}^{\infty}\beta^{2+m\omega}\eta_0\eta_m\right)Q(x)\psi(\phi(z)) = 0
\end{aligned} \tag{15}$$

where $Q(x)$ is the usual coefficient function of Eq. (1) if $x \in (x_j, x_{j+1})$ which has only turning points solutions of Eq. (8), or it is the coefficient function $\tilde{Q}(x) = 4/\beta^2 - Q(x)$ mentioned above if $x \in (x_j, x_{j+1})$ which has only turning points solutions of Eq. (9). We also define

$$\left(\frac{d^{j_l}\phi}{dz^{j_l}}\right)_t = \frac{d^{j_l}\phi_0}{dz^{j_l}}\phi_t(z) + \sum_{p=1}^{j_l}\left[\frac{(j_l)!}{p!(j-l-p)!}\frac{d^{j_l-p}\phi_0}{dz^{j_l-p}}\frac{d^p\phi_t}{dz^p}\right]. \tag{16}$$

There is no unique choice of the values of the coefficients γ, ω, θ and δ such that the leading term of the expansion in terms of the parameter β in Eq. (15) is a 2nd order o.d.e. for the function $\psi(\phi)$. We chose $\gamma = 3/2, \theta = -1/2, \omega = 1/2$ and $\delta = 1/2$. With this choice, the leading term in Eq. (15) is of the order $O(\beta^2)$, which is

$$\frac{d^2\psi}{d\phi^2}\left(\frac{d\phi_0}{dz}\right)^2 + Q(x)\psi(\phi) = 0. \tag{17}$$

The change of independent variable $\phi(z)$, given in Eq. (14) as an expansion in the parameter β, is not known yet. We specify $\phi_0(z)$ (term of order $O(\beta^0)$) such that Eq. (17) is reduced to the form

$$\frac{d^2\psi}{d\phi^2} + F(\phi)\psi(\phi) = 0, \tag{18}$$

with known solution in terms of special functions of mathematical physics. This implies that $\phi_0(z)$ satisfies

$$Q(x) = F(\phi)\left(\frac{d\phi_0}{dz}\right)^2. \tag{19}$$

Notice that in the limit $\beta \to 0$, $\phi(z) \to \phi_0(z)$ according to Eq. (14). Therefore, we can integrate Eq. (19), which results in the implicit equation

$$\int_{\phi_0(A)}^{\phi_0(z)} \sqrt{F(s)}ds = \int_A^z \sqrt{Q(t)}dt \tag{20}$$

for $\phi_0(z)$. A is a reference point which is chosen according to the problem the asymptotic theory is applied to. The next term of the expansion (term of $O(\beta^{5/2})$) gives an equation involving $\eta_0(z)$ and $\phi_1(z)$. The satisfaction of this equation implies that

$$\eta_0(z) = \left(\frac{d\phi_0}{dz}\right)^{-1/2} \qquad \text{and} \qquad \phi_1(z) = \frac{1}{\phi_0(z)} \text{ or } \phi_1(z) = 0. \tag{21}$$

The effects of higher order derivatives in Eq. (4) ($n > 2$) appear only for terms of order $O(\beta^4)$ and higher. Therefore, to have a glimpse of their effect in the asymptotic theory we solve the perturbation problems up to the order $O(\beta^{9/2})$. The term of order $O(\beta^{7/2})$ of the expansion (15) gives an equation where $d^2\psi/d\phi^2$, $d\psi/d\phi$ and $\psi(\phi)$ appear. In the equations for the previous order problems, only $d\psi/d\phi$ and $\psi(\phi)$ appeared. We were able to solve these equations by setting the coefficients multiplying $d\psi/d\phi$ and $\psi(\phi)$ equal to zero. But, to be able to solve the equation of the problem of order $O(\beta^{7/2})$, we use Eq. (17) to write $d^2\psi/d\phi^2$ in terms of $\psi(\phi)$, so we can use the same strategy used for the previous order equations. For terms of the expansion (15) of order higher than $O(\beta^{7/2})$, derivatives of $\psi(\phi)$ of the third order and higher appears. To express this higher order derivatives of $\psi(\phi)$ in terms of $d\psi/d\phi$ and $\psi(\phi)$, we derivate Eq. (17) with respect to the z variable the appropriate number of times. Then we substitute the equation for the lower order derivatives in terms of $d\psi/d\phi$ and $\psi(\phi)$ into the equation for the higher order derivatives such that these are written in terms of only $d\psi/d\phi$ and $\psi(\phi)$. Finally, we substitute these equations into the equation we want to solve and use the strategy applied to the previous order problems. We summarize the solution of each perturbation problem below.

- Solution of the problem of order $O(\beta^3)$:

$$\eta_1(z) = 1 \qquad \text{and} \qquad \phi_2(z) = \frac{1}{\phi_0(z)}\int_A^z \frac{1}{2Q(t)}\left[\frac{3}{4}\left(\frac{\phi_0}{dt}\right)^{-1}\left(\frac{d^2\phi_0}{dt^2}\right)^2 - \frac{1}{2}\frac{d^3\phi_0}{dt^3}\right]dt \tag{22}$$

- Solution of the problem of order $O(\beta^{7/2})$.

$$\eta_2(z) = \frac{1}{2}\int_A^z \left(\frac{d\phi_0}{dt}\right)^{-1}\left[\left(\frac{d\phi_0}{dt}\right)^{-1}\frac{d^2\phi_0}{dt^2}\left(\frac{d\phi}{dt}\right)_2 - \left(\frac{d^2\phi}{dt^2}\right)_2\right]dt, \text{ and } \phi_3(z) = \frac{1}{\phi_0(z)} \text{ or } \phi_3(z) = 0, \tag{23}$$

where $(d\phi/dt)_2$ and $(d^2\phi/dt^2)_2$ is defined according to Eq. (16) with $t = 2$.

- Solution of the problem of order $O(\beta^4)$.

$$\eta_3(z) = \frac{1}{2}\int_A^z \left(\frac{d\phi_0}{dt}\right)^{-1}\left[\left(\frac{d\phi_0}{dt}\right)^{-1}\frac{d^2\phi_0}{dt^2}\left(\frac{d\phi}{dt}\right)_2 - \left(\frac{d^2\phi}{dt^2}\right)_2 - \frac{\beta^{-\theta}}{3}Q(t)\frac{d\phi_0}{dt}\frac{d^2\phi_0}{dt^2} + \frac{\beta^{-\gamma-\theta}}{6}\frac{dQ}{dx}\left(\frac{d\phi_0}{dt}\right)^2\right]dt,$$

and

$$\phi_4(z) = -\frac{1}{\phi_0(z)}\int_A^z \left\{\frac{1}{2}\left(\frac{d\phi_0}{dt}\right)^{-1}\left(\frac{d\phi}{dt}\right)_2^2 + \eta_2\left(\frac{d\phi}{dt}\right)_2 - \frac{1}{Q(x)}\frac{d\phi_0}{dt}\left[\frac{d^2\eta_0}{dt^2}\frac{\eta_2}{\eta_0} + \frac{2}{\eta_0}\frac{d\eta_0}{dt}\frac{d\eta_2}{dt} + \frac{d^2\eta_2}{dt^2}\right] + \frac{\beta^{-2\theta}}{24}\left[\frac{5\beta^{-\gamma}}{Q(x)}\frac{dQ}{dx}\frac{d^2\phi_0}{dt^2} - 8\left(\frac{d\phi_0}{dt}\right)^{-1}\left(\frac{d^2\phi_0}{dt^2}\right)^2\right.\right.$$

$$+2\frac{d^3\phi_0}{dt^3} - \frac{\beta^{-2\gamma}}{Q(x)}\frac{d^2Q}{dt^2}\frac{d\phi_0}{dt}\bigg] - \frac{Q(t)}{24}\frac{d\phi_0}{dt}\bigg\} dt. \tag{24}$$

- Solution of order $O(\beta^{9/2})$.

$$\begin{aligned}\eta_4(z) =& \frac{1}{2}\int_A^z \left\{-2\frac{d}{dz}(\eta_0\eta_2)\left(\frac{d\phi}{dz}\right)_2 - \eta_0\eta_2\left(\frac{d^2\phi}{dz^2}\right)_2 - 2\frac{d\eta_0}{dz}\left(\frac{d\phi}{dz}\right)_4 - \eta_0\left(\frac{d^2\phi}{dz^2}\right)_4\right.\\ &\left.+\frac{1}{12}\left[4\frac{d\eta_0}{dz}\left(\frac{d\phi_0}{dz}\right)^3 + 6\eta_0\frac{d^2\phi_0}{dz^2}\left(\frac{d\phi_0}{dz}\right)^2\right]Q(t)\left(\frac{d^2\phi_0}{dz^2}\right)\right\} dt\end{aligned} \tag{25}$$

We do not present the expression for $\phi_5(z)$ since it is not used in the section 3where we present the results.

2.4 Transfer Matrix.

For each cluster of turning points (solutions of Eq. (8) or Eq. (9)) we have a different model equation (18), which have different solutions. To be able to construct a solution along the whole x axis, we divide it in intervals as mentioned in the previous section. Each interval is related to a cluster of turning points with a specific model equation (specific functional form of $F(\phi)$ in Eq. (18)). To connect the solution through the end points of each interval, we have the conditions (11) and (12). We assume that at the interval (x_j, x_{j+1}), Eq. (18) has a pair of linearly independent solutions $\psi_{1,j}(\beta^\theta\phi)$ and $\psi_{2,j}(\beta^\beta\phi)$. Then we can write

$$y_j(x) = f(x)\eta^j(z)\{A_{1,j}\psi_{1,j}(\beta^\theta\phi^j(z)) + A_{2,j}\psi_{2,j}(\beta^\theta\phi^j(z))\} \tag{26}$$

for $x_j \le x \le x_{j+1}$. In the eq. (26) we set $f(x) = 1$ (or $f(x) = \exp(i\pi x)$) if in the interval (x_j, x_{j+1}) we have only turning points given by condition (8) (or condition (9)). The functions $\eta^j(z)$ and $\phi^j(z)$ are defined, respectively, by the left and right expansions in Eq. (14) restricted to the interval (x_j, x_{j+1}). To apply conditions (11) and (12), we need the derivative of $y_j(x)$ with respect to x. To obtain it, we differentiate Eq. (26) with respect to x. If we substitute Eq. (26) and its derivative with respect to the x variable into the conditions (11) and (12), we obtain the matrix relation

$$\left\{\begin{array}{c} y_j(x_{j+1}) \\ \frac{dy_j}{dx}(x_{j+1}) \end{array}\right\} = [N_{j+1}][M_j]^{-1}\left\{\begin{array}{c} y_{j-1}(x_j) \\ \frac{dy_{j-1}}{dx}(x_j) \end{array}\right\} \tag{27}$$

where the matrices $[N_{j+1}]$ and $[M_j]$ are given by

$$[N_{j+1}] = \left[\begin{array}{c} f(x)\eta^j(z)\psi_{1,j}(\beta^\theta\phi^j(z))|_{x=x_{j+1}} \\ \left(\left[\frac{df}{dx}\eta^j(z) + f(x)\frac{d\eta^j}{dz}\right]\psi_{1,j}(\beta^\theta\phi^j(z)) + \beta^{\theta-\gamma}f(x)\eta^j(z)\frac{d\phi^j}{dz}\frac{\psi_{1,j}}{d\phi}(\beta^\theta\phi^j(z))\right)_{x=x_{j+1}} \\ f(x)\eta^j(z)\psi_{2,j}(\beta^\theta\phi^j(z))|_{x=x_{j+1}} \\ \left(\left[\frac{df}{dx}\eta^j(z) + f(x)\frac{d\eta^j}{dz}\right]\psi_{2,j}(\beta^\theta\phi^j(z)) + \beta^{\theta-\gamma}f(x)\eta^j(z)\frac{d\phi^j}{dz}\frac{\psi_{2,j}}{d\phi}(\beta^\theta\phi^j(z))\right)_{x=x_{j+1}} \end{array}\right], \tag{28}$$

$$[M_j] = \left[\begin{array}{c} f(x)\eta^j(z)\psi_{1,j}(\beta^\theta\phi^j(z))|_{x=x_j} \\ \left(\left[\frac{df}{dx}\eta^j(z) + f(x)\frac{d\eta^j}{dz}\right]\psi_{1,j}(\beta^\theta\phi^j(z)) + \beta^{\theta-\gamma}f(x)\eta^j(z)\frac{d\phi^j}{dz}\frac{\psi_{1,j}}{d\phi}(\beta^\theta\phi^j(z))\right)_{x=x_j} \\ f(x)\eta^j(z)\psi_{2,j}(\beta^\theta\phi^j(z))|_{x=x_j} \\ \left(\left[\frac{df}{dx}\eta^j(z) + f(x)\frac{d\eta^j}{dz}\right]\psi_{2,j}(\beta^\theta\phi^j(z)) + \beta^{\theta-\gamma}f(x)\eta^j(z)\frac{d\phi^j}{dz}\frac{\psi_{2,j}}{d\phi}(\beta^\theta\phi^j(z))\right)_{x=x_j} \end{array}\right]. \tag{29}$$

If the interval (x_j, x_{j+1}) has turning points given by the condition (8), we set $f(x) = 1$, but if the turning points in this interval are given by the condition (9), we set $f(x) = \exp(i\pi x)$. Once we have the transfer matrix for each interval (x_j, x_{j+1}), we can relate the quantities $\{y(x_j)(dy/dx)(x_j)\}^T$ to the quantities $\{y(x_{j+N})(dy/dx)(x_{j+N})\}^T$ by a product of transfer matrices, as follows

$$\left\{\begin{array}{c} y(x_{j+N}) \\ \frac{dy}{dx}(x_{j+N}) \end{array}\right\} = \left\{\begin{array}{c} y_{j+N-1}(x_{j+N}) \\ \frac{dy_{j+N-1}}{dx}(x_{j+N}) \end{array}\right\} = \prod_{l=j}^{j+N}[N_{l+1}][M_l]^{-1}\left\{\begin{array}{c} y(x_j) \\ \frac{dy_j}{dx}(x_j) \end{array}\right\}. \tag{30}$$

Notice that $\{y(x_j)\ \ (dy/dx)(x_j)\}^T = \{y_{j-1}(x_j)\ \ (dy_{j-1}/dx)(x_j)\}^T$.

Table 1. Terms of the expansions for $\eta^1(z)$ and $\phi^1(z)$ up to $O(\beta^2)$.

Form of $Q(x)$	$-(1+\alpha x-\omega^2)$	
Function	$\eta^1_l(z)$	$\phi^1_l(z)$
$O(1)(l=0)$	$a_1\beta^{\gamma/6}/\alpha^{1/6}$	$a_2\beta^{2\gamma/3}\alpha^{-2/3}(\omega^2-1-\alpha\beta^{-\gamma}z)$
$O(\beta^{1/2})(l=1)$	1	$1/\phi^1_0(z)$ or 0
$O(\beta)(l=2)$	0	0
$O(\beta^{3/2})(l=3)$	$a_2^{-1}\beta^{-\theta-4\gamma/3}z/6$	$1/\phi^1_0(z)$ or 0
$O(\beta^2)(l=4)$	$\alpha\beta^{-\gamma}z/48$	$-\alpha z(8+\beta^2(2+\beta^{-\gamma}\alpha z-2\omega))/48\beta^{\gamma}(4+\beta^2(1+\alpha\beta^{-\gamma}z-\omega^2))$

3. Results.

The purpose of this section is to illustrate the performance of the asymptotic theory and to discuss how it is applied to the problem of wave interaction with a non-uniform repetitive system.

3.1 Wave Scattering by Linear Non-uniformity.

To illustrate the performance of the asymptotic theory, we aplly it to the problem of wave scattering by a finite repetitive systems with linear non-uniformity embedded in an infinite uniform repetitive system. The coefficient function of Eq. (1) in this case is given by

$$Q(x) = \begin{cases} -(1-\omega^2) & \text{for } x<0 \\ -(1+\alpha x-\omega^2) & \text{for } 0\le x\le 4/\beta^2 \\ -(1+4\alpha/\beta^2-\omega^2) & \text{for } x>4/\beta^2 \end{cases} \tag{31}$$

were α is the slope of the non-uniformity and ω is the non-dimensional wave frequency of the incident wave. Equation (1) with the coefficient function above describes, for example, nono-chromatic wave propagation along a chain of coupled pendula with the variation of the non-dimensional pendula length given as $\epsilon(x)=1/(1+\alpha x)-1$ for $0<x<4/\beta^2$. We consider the case of only left wave incidence. In this case, the system passband for $x<0$ is the interval $1<\omega<\sqrt{1+4/\beta^2}$. For this range of wave frequencies, we have only a single turning point (given by Eq. (8)) in the non-uniform part of the chain. Therefore, for the interval $(0,4/\beta^2)$ we chose functions $\psi_{1,j}(\phi)(j=1,2)$ which are solutions of the model equation equation (18) with $F(\phi)=-\phi$. This is the Airy equation (see Abramowitz & Stegun, 1967). To apply the asymptotic theory, we need also to construct the functions $\phi^1_l(z)$ and $\eta^1_l(z)$ up to order $O(\beta^2)$ ($\delta=\omega=1/2$ and $l=0,1,2,3,4$). We display these funtion in table 1, where we have $a_1=\exp(-i2\pi/3)$, $a_2=\exp(i\pi/3)$.

Notice that the functions $\phi^1(z)$ assume values along the line with argument $\pi/3$ on the complex ϕ plane for real values of z. According to Eq (31), for $\omega^2\ge 1+\alpha\beta^{-\gamma}$, $Q(x)$ is positive and the solution of Eq. (1) decays or grows exponentially. For $\omega^2<1+\alpha\beta^{-\gamma}$, $Q(x)$ is negative and Eq. (1) has oscillatory behaviour. Therefore, along the axis with argument $\pi/3$ $(-2\pi/3)$ in the complex ϕ plane, we need functions $\psi_{1,j}(\phi)$ which gorws or decays exponentially (oscillates). We chose the following linear combinations of Airy functions

$$y(x)=\eta^1(z)\left\{A_1A_0(\beta^{-1/2}\phi^1(z))+A_2A_{-1}(\beta^{-1/2}\phi^1(z))\right\}, \tag{32}$$

where $A_0(\phi)$ is the Airy's function $Ai(\phi)$ itself, and $A_j(\phi)=Ai(\exp(i2\pi/3)^{-j}\phi)$ with $j=1,-1$ (see page 413 of Olver, 1997). To obtain the coefficients A_j in Eq. (32) and the left reflection R^- and transmission T^- coefficients, we enforce the continuity of the function $y(x)$ and of its derivative dy/dx at $x=0$ and at $x=4/\beta^2$. At the uniform part of the chain the solution is given by

$$y(x)=\begin{cases} \exp(ik_1x)+R^-\exp(-ik_1(x)) & \text{for } x<0 \\ \exp(ik_2x) & \text{for } x>4/\beta^2 \end{cases} \tag{33}$$

where $\cos(k_j)=1-(\beta^2/2)(1+4\alpha\beta^2\delta_{2j}-\omega^2)$. $\delta_{ij}=0$ for $i\ne j$ and it is 1 for $i=j$. The continuity condition of the solution and of its first order derivative at the point $x=0$ and $x=4/\beta^2$ gives a system of equations for the reflection R^- and transmission T^- coefficients. The asymptotic approximation for the coefficients R^- and T^- up to order $O(\beta^2)$ are compared with the numerical solution of this scattering problem in Fig. 1.

According to curves (v) and (vi) in Fig. 1, the right choice for $\phi^1_1(z)$ and $\phi^1_3(z)$ is to set them equal to zero. The results illustrated by curves (vii) and (viii) represents this choice, which is the correct one, since the approximation for

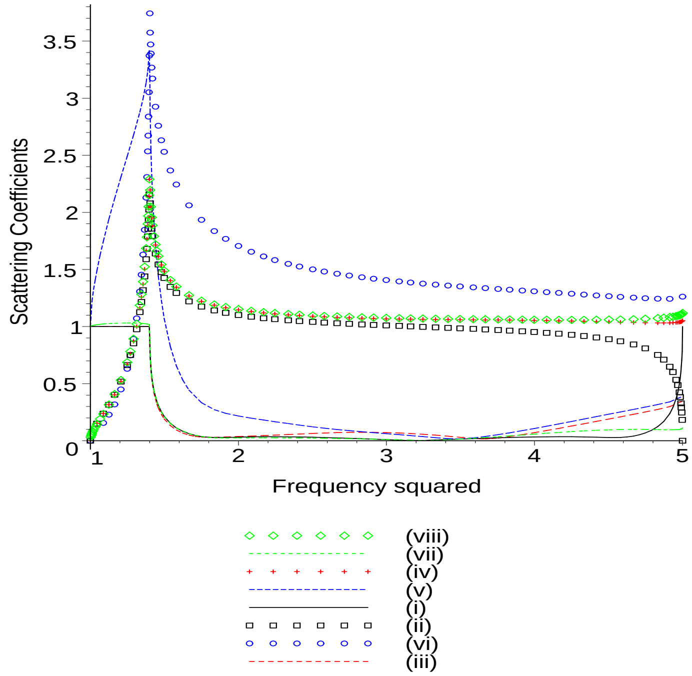

Figure 1. $\beta = 1$, $\alpha = 0.1$ and $1 < \omega^2 < 1 + 4/\beta^2$. Curves (i) and (ii) display, respectively, R^- and T^- obtained numerically. Curves (iii), (v) and (vii) display asymptotic approximations for R^-, respectively, of order $O(1)$, $O(\beta^{1/2})$ and of $O(\beta^2)$. Curves (iv), (vi) and (viii) display asymptotic approximations for T^-, respectively, of order $O(1)$, $O(\beta^{1/2})$ and of $O(\beta^2)$.

the coefficients R^- and T^- are closer to the numerical results. This suggests that we should have chosen $\delta = 2$ instead of $\delta = 1/2$ in the expansion (14) for the function $\phi^j(z)$.

4. Conclusions.

The asymptotic theory presented in this work recovers the right qualitative behavour even for values of the parameter $\beta = 1$, as we can see by Fig. 1. In the application example above, only a single turning point given by Eq. (8) was present. This is an exception. Usually we would have turning points given by both Eqs. (8) and (9). Actually, for $\omega^2 = 1 + 4/\beta^2$ a turning point given by Eq. (9) appear at $x = 0$. If we have taken the effect of this turning point into account, the approximations for the coefficients R^- and T^- would have been better for $\omega^2 \sim 1 + 4/\beta^2$. There is no point in applying this asymptotic theory to wave propagation along non-uniform repetitive system if the non-uniformity variation length scale is close to 1. In this case, each subsystem could be modeled by a transfer matrix, and the solution obtained as a product of the necessary transfer matrices. On the other hand, when the non-uniformity length scale is $>> 1$, the asymptotic theory obtains the solution of the wave propagation problem (see section 2.4) as a product of a very few transfer matrices (see eq. (30)), which represents a huge economy with respect to the product of a large number of transfer matrices.

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