ANALYSIS OF PURELY KINEMATIC CRITERIA FOR FLOW CLASSIFICATION

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Abstract. *Among the presently available flow classification criteria, the ones proposed by Tanner and Huilgol (1975) and Astarita (1979) are of particular interest. While the former is restricted to MWCRPSH (motions with constant relative principal stretch history), the latter was shown by Huilgol (1980) to fail for certain 3-D kinematics. In the present work we propose a modification of Tanner-Huilgol's classification criterion to encompass motions other than MWCRPSH. We also establish both the set of flows that constitute the range of applicability of Astarita's criterion and the relationship between the two criteria. An interesting subset of flows within this range was identified, the so-called quasi-plane flows. The Generalized Maxwell Rheometer Flow and the Doubly Superposed Viscometric Flow are analyzed from the standpoint of these two criteria and also from the standpoint of the criterion recently proposed by Thompson and Souza Mendes (2005). The persistence-of-straining parameter presented in this last reference is shown to perform advantageously as a flow classification parameter.*
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1. Introduction

In several flows found in industrial processes the material experiences a variety of types of flow as it moves along the process line. Therefore, it is of practical importance from the rheological point of view to map as accurately as possible the regions of shear, extension, rigid body motion, etc. of a given flow.

The issue of flow classification can be directly related to a key concept for the present paper, namely, the persistence-of-straining concept (Lumley (1969), Frank and Mackley (1976), Astarita (1979), Schunk and Scriven (1990), Thompson and Souza Mendes (2005). To briefly introduce the persistence-of-straining concept, for simplicity let us employ the particular case of plane flows. We start by considering the eigenvalues and principal directions of the rate-of-strain tensor for these flows. In the absence of relative rotation, a material filament which is aligned, say, with the eigenvector corresponding to the largest positive eigenvalue, will be persistently stretched. On the other hand, it may happen that the fluid rotates in such a way that a different material filament is aligned with this eigenvector at each instant of time. In this case, a filament which is aligned with this eigenvector at some instant of time will subsequently rotate towards directions of less stretching. Thus, this material rotation relative to the eigendirections decreases the persistence of straining.

1.1 Previous work

Tanner and Huilgol (1975) proposed a classification criterion for steady homogeneous flows based on the eigenvalues of the velocity gradient, $\mathbf{L}$. According to this criterion, if there is at least one eigenvalue of $\mathbf{L}$ with a positive real part, then the flow is classified as *strong*. Otherwise, it is called *weak*. The reason for this terminology is based on the fact that, if a material filament is aligned with the principal direction corresponding to an eigenvalue with a positive real part, then it will undergo an exponential growth in its length. This type of classification is not general, because it is still restricted to MWCRPSH. Nevertheless, this idea has been used in the literature quite frequently to discuss the response of materials to certain flows.

Astarita (1979) proposed a persistence-of-straining parameter that is objective and not restricted to MWCRPSH. Objectivity was attained by introducing the tensor that measures the rate of rotation of the material as measured by an observer who is fixed to the principal directions of the rate of strain. This tensor, $\overline{\mathbf{W}}$, is called the *relative-rate-of-rotation tensor*, and is defined by

$$\overline{\mathbf{W}} = \mathbf{W} - \mathbf{\Omega} \qquad (1)$$

where $\mathbf{\Omega}$ is the tensor that gives the rate of rotation of the eigenvectors of $\mathbf{D}$:

$$\dot{\mathbf{e}}_i = \mathbf{\Omega}\mathbf{e}_i = \omega \times \mathbf{e_i}, \quad \mathbf{i} = \mathbf{1}, \mathbf{2} \text{ or } \mathbf{3} \qquad (2)$$

where $\mathbf{e}_i$ is any of the three eigenvectors of $\mathbf{D}$, ω is the angular velocity of these eigenvectors, and the dot indicates the (material) time derivative.

To complete the definition of $\overline{\mathbf{W}}$, the flows for which two of the eigenvalues of $\mathbf{D}$ are equal (but distinct from the third) $\mathbf{\Omega}$ is such that the component of ω normal to the plane defined by the two multiply defined eigenvectors coincides with the vorticity component normal to the same plane.

The parameter proposed by Astarita is

$$R_D = -\frac{tr\overline{\mathbf{W}}^2}{tr\mathbf{D}^2} \tag{3}$$

One important advantage of the classification criterion based on this parameter is that it associates a numerical value of R_D with each type of flow. Huilgol (1980) analyzed Astarita's criterion and gave three examples which illustrate weaknesses of R_D as a flow classifier.

2. Persistence of straining at a material filament

Following Thompson and Souza Mendes (2005), we now define the scalar quantity I_n as the intensity of the rate of strain of a material filament whose orientation at a given instant of time is $\mathbf{e}_n$, an eigenvector of $\mathbf{D}$ ($n = 1, 2$, or 3). It is given by:

$$I_n = \mathbf{e}_n \cdot \mathbf{D}\mathbf{e}_n \tag{4}$$

The material derivative of I_n (Thompson and Souza Mendes (2005)) can be written as

$$\dot{I}_n = \mathbf{e}_n \cdot \mathbf{D}'\mathbf{e}_n + \mathbf{e}_n \cdot (\mathbf{D}\overline{\mathbf{W}} - \overline{\mathbf{W}}\mathbf{D})\mathbf{e}_n \tag{5}$$

where $\mathbf{D}'$ is given by

$$\mathbf{D}' = \sum_{i=1}^{3} \dot{\lambda}_i \, \mathbf{e}_i \otimes \mathbf{e}_i \tag{6}$$

where λ_i are the eigenvalues of $\mathbf{D}$. The term $\mathbf{e}_n \cdot \mathbf{D}'\mathbf{e}_n$ gives the rate of change of I_n due to changes with time of the eigenvalues λ_i (following the motion), and $\dot{I}_n^* = \mathbf{e}_n \cdot (\mathbf{D}\overline{\mathbf{W}} - \overline{\mathbf{W}}\mathbf{D})\mathbf{e}_n$ gives the rate of change of I_n due to the relative rate of rotation and so is the part of $\dot{I}_n$ which is related to the concept of persistence of straining.

The persistence-of-straining parameter is defined as

$$\mathcal{R} = \frac{\sqrt{\frac{1}{2} tr\left[(\mathbf{D}\overline{\mathbf{W}} - \overline{\mathbf{W}}\mathbf{D})^2\right]}}{tr\left[\mathbf{D}^2\right]} \tag{7}$$

3. Relationship between the criteria of Tanner-Huilgol and Astarita

The analysis conducted in this section cannot be applied to flows other than MWCRPSH when $\mathbf{D}$ has two equal eigenvalues, but distinct from the third, because in this case, the tensor $\bar{\mathbf{L}}$ is not defined. In order to generalize the criterion created by Tanner and Huilgol and allow a comparison among the different criteria, we replace $\mathbf{L}$ with $\bar{\mathbf{L}} = \mathbf{L} - \mathbf{\Omega}$ in the analysis of Tanner and Huilgol (1975). Defining l_i as the eigenvalues of $\bar{\mathbf{L}}$, the characteristic equation for this tensor can be written as

$$\begin{aligned} -l^3 \quad & + I_D l^2 - \tfrac{1}{2}(I_D^2 - tr\mathbf{D}^2 - tr\overline{\mathbf{W}}^2)l \\ & + III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} - \tfrac{1}{2} I_D tr\overline{\mathbf{W}}^2 = 0 \end{aligned} \tag{8}$$

where $I_D \equiv tr\mathbf{D}$ and $III_D \equiv \det \mathbf{D}$ are the first and third invariants of the rate-of-strain tensor respectively. Therefore, for isochoric flows ($I_D = \mathbf{0}$) the three flow types given by this criterion are respectively related to the following conditions:

- The flow is marginally weak ($l_1 = 0,\ l_2 = 0,\ l_3 = 0$) when the two conditions below are simultaneously met:

  $$III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} = 0$$

  and

  $$-tr\mathbf{D}^2 - tr\overline{\mathbf{W}}^2 = 0 \iff R_D = 1$$

- The flow is strictly weak(($l_1 = 0$, l_2 and l_3 are imaginary and conjugate) when the two conditions below are simultaneously met:

  $$III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} = 0$$

  and

  $$-tr\mathbf{D}^2 - tr\overline{\mathbf{W}}^2 > 0 \iff R_D < 1$$

- The flow is strong (at least one among l_1, l_2 and l_3 has a positive real part) when one of the two conditions below is met:

$$III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} \neq 0$$

or

$$-tr\mathbf{D}^2 - tr\overline{\mathbf{W}}^2 > 0 \iff R_D > 1$$

Therefore, as indicated above, there is an equivalence between the modified Tanner-Huilgol criterion and Astarita's criterion provided the latter is restricted to motions that obey the following restriction:

$$III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} = 0 \tag{9}$$

The class of flows that obey Eq. (9) includes (but is not restricted to) plane flows. In summary, when Eq. (9) is satisfied, the correspondence between the R_D values and the modified Tanner-Huilgol classification is

- marginally weak flows $\iff R_D = 1$
- strictly weak flows $\iff R_D < 1$
- strong flows $\iff R_D > 1$

with an advantage for Astarita's criterion due to its quantitative nature.

In fact, in the first of the ingenious examples given by Huilgol (1980), namely,

$$\mathbf{v} = (a_1 x - wy)\mathbf{e_x} + (wx + a_2 y)\mathbf{e}_y + a_3 z\mathbf{e}_z \tag{10}$$

the expression for R_D considering isochoric ($a_3 = -a_1 - a_2$) flow is

$$R_D = \frac{w^2}{a_1^2 + a_2^2 + a_1 a_2} \tag{11}$$

The condition given by Eq. (9) for isochoric flows when applied to Eq. (10) yields

$$(a_1 + a_2)(a_1 a_2 + w^2) = 0 \tag{12}$$

Combining Eqs. (11) and (12) we obtain

$$(a_1 + a_2)[a_1 a_2 + R_D(a_1^2 + a_2^2 + a_1 a_2)] = 0 \tag{13}$$

Therefore, for $R_D = 1$ the condition above can be written as

$$\begin{aligned} &(a_1 + a_2)^3 = 0 \iff a_1 = -a_2 \iff a_3 = 0, \\ &w^2 = a_1^2 = a_2^2 \iff \bar{\mathbf{L}}^2 = \mathbf{0} \end{aligned} \tag{14}$$

In summary, if we impose that Eq. (10) obeys Eq. (9), then $R_D = 1$ implies viscometric flow and, conversely, when the flow is viscometric ($\bar{\mathbf{L}}^2 = \mathbf{0}$) then $R_D = 1$.

The second example flow given by Huilgol (1980), namely

$$\mathbf{v} = [ax - wy]\mathbf{e}_x + [wx + ay]\mathbf{e}_y - 2az\mathbf{e}_z \tag{15}$$

does not satisfy Eq. (9) for $a \neq 0$, and hence it is outside the scope of applicability of Astarita's criterion.

The third example given by [9] is a comparison between two doubly superposed viscometric flows. Huilgol showed that the parameter R_D is not continuous. Due to the fact that doubly superposed viscometric flows always obey Eq.(9), R_D becomes equal to unity in this case. This flow is analysed on the next section.

4. Doubly superposed viscometric flows

We now analyze the doubly superposed viscometric flow, defined by

$$\mathbf{v} = (ly + mz)\mathbf{e_x} + nz\mathbf{e_y} \tag{16}$$

The eigenvectors of $\mathbf{D}$ do not rotate in this case, and hence $\overline{\mathbf{W}} = \mathbf{W}$. Moreover, $\bar{w} = \frac{1}{2}\sqrt{l^2 + m^2 + n^2}$. An interesting feature of this flow is that neither Tanner-Huilgol's nor Astarita's criterion can distinguish it from the viscometric flows. Tanner-Huilgol's criterion classifies it as (marginally) weak, and the evaluation of Astarita's parameter yields $R_D = 1$ as for viscometric flows. However, doubly superposed viscometric flows are stronger than viscometric flows, in the sense that they possess filaments whose length grows quadratically with time, in contrast to the filament length in viscometric flows, which grow linearly with time.

For doubly superposed viscometric flows, $\mathcal{R}$ becomes:

$$\mathcal{R} = \frac{\sqrt{(l^2+m^2+n^2)^2 - 3l^2n^2}}{l^2+m^2+n^2} \tag{17}$$

For the sake of simplicity but without loss of generality, we can take $l^2+m^2+n^2=1$ Huilgol (1970), and hence $\mathcal{R}$ is reduced to

$$\mathcal{R} = \sqrt{1-3l^2n^2} \tag{18}$$

The minimum value of $\mathcal{R}$ is reached for the special case when $m=0$. The maximum one happens when $l=0$ or $n=0$ and in this case the flow is viscometric. Because $\frac{1}{2} \leq \mathcal{R} \leq 1$, the persistence-of-strainning parameter $\mathcal{R}$ is able to indicate that the doubly superposed viscometric flows are stronger than viscometric flows and weaker than extensional flows, as expected.

5. The generalized eccentric-disk rheometer flow

We now analyze the eccentric-disk rheometer flow. In this flow the fluid sample is placed between two parallel disks, both of which rotate with the constant angular velocity $\alpha > 0$. The axes of rotation of the disks are separated by a distance a and there is a gap b between the disks. The velocity field is given by

$$\mathbf{v} = -\alpha(y - \frac{a}{b}z)\hat{\imath} + \alpha x\hat{\jmath} \tag{19}$$

This flow can be generalized to the following form Rajagopal(1982):

$$\mathbf{v} = -\alpha(y - g(z))\hat{\imath} + \alpha(x - f(z))\hat{\jmath} \tag{20}$$

Because the eigenvectors of $\mathbf{D}$ do not rotate, $\mathbf{\Omega} = \mathbf{0}$, and hence $\overline{\mathbf{W}} = \mathbf{W}$. Defining

$$\Psi(z) = \sqrt{f'(z)^2 + g'(z)^2}, \tag{21}$$

then the intensity of the relative-rate-of-rotation vector can be written as $\bar{w} = \frac{\alpha}{2}\sqrt{\Psi^2+4}$. Thus, the expression for $\mathcal{R}$ for this flow is

$$\mathcal{R}(z) = \frac{\sqrt{\Psi^2+1}}{\Psi} \tag{22}$$

while the expression for R_D is

$$R_D(z) = \frac{\Psi^2+4}{\Psi^2} \tag{23}$$

Therefore, $\mathcal{R} > 1$ and $R_D > 1$ for the generalized eccentric disk rheometer flow. This is a desired result, since this flow is an elliptical flow in the sense discussed by Astarita and Marrucci(1971) and strictly weak in Tanner-Huilgol's classification. Another interesting result is that, as the flow approaches rigid body motion, i.e. as $\Psi(z) \rightarrow 0$, then $\mathcal{R}(z) \rightarrow \infty$, as it should, the same behavior being presented by the parameter R_D.

6. Final Remarks

In the present work we have proposed a generalization of Tanner and Huilgol (1975) flow classification criterion for flows other than MWCRPSH. This was done by employing the velocity gradient tensor as measured from a reference frame attached to the principal axis of $\mathbf{D}$. We also established the range of applicability of Astarita's criterion, by comparing it with the modified Tanner-Huilgol criterion. We showed that Astarita's criterion is applicable for flows that obey the condition $III_D + tr\overline{\mathbf{W}}\mathbf{D}\overline{\mathbf{W}} = 0$. This class of flows encompasses the doubly superposed viscometric flows and a subset of quasi-plane flows, and therefore Astarita's criterion is not restricted to plane flows as it might have been previously believed Brunn and Ryssel (1997). We have also shown that the persistence-of-strainning parameter proposed by Thompson and Souza Mendes (2005) performs equivalently to other criteria when is applied to the Generalized Maxwell Rheometer and has a better performance as a flow classifying parameter for the doubly superposed viscometric flow, in the sense that it is capable of distinguishing this flow from viscometric flows. According to this parameter, the doubly superposed viscometric flow is stronger than viscometric flows and weaker than extensional flows.

The modified Tanner-Huilgol criterion cannot be applied to flows other than MWCRPSH when two eigenvalues of $\mathbf{D}$ are equal. In addition, it is a qualitative criterion. Astarita's parameter R_D has another restriction in addition to the one just mentioned, but has the advantage of being quantitative. On the other hand, the measure $\mathcal{R}$ of persistence-of-strainning is a local, quantitative and generally applicable parameter, and hence it can be used on general flow classification.

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8. References

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