GALERKIN LEAST-SQUARES MULTI-FIELD APPROXIMATIONS FOR NON-NEWTONIAN FLOWS

Flávia Schwarz Franceschini Zinani[1]
LAMAC – Laboratory of Applied and Computational Fluid Mechanics
DEMEC – Mechanical Engineering Department
UFRGS – Universidade Federal do Rio Grande do Sul
R. Sarmento Leite, 425, Porto Alegre, RS, Brazil
fla@mecanica.ufrgs.br

Sérgio Frey[2]
LAMAC – Laboratory of Applied and Computational Fluid Mechanics
DEMEC – Mechanical Engineering Department
UFRGS – Universidade Federal do Rio Grande do Sul
R. Sarmento Leite, 425, Porto Alegre, RS, Brazil
frey@mecanica.ufrgs.br

Abstract. *It is common knowledge that most fluids in engineering applications, such as polymer melts and solutions, food products, cosmetics, paints and others, are non-Newtonian. They present features as shear-thinning, normal stress differences, viscoplasticity, extension thickening and elasticity. Computational fluid dynamics is a powerful tool for solving problems of non-Newtonian flows, although it still faces considerable numerical difficulties, specially when dealing with viscoelastic constitutive models. One is the need to approximate conservation and constitutive equations in a multi-field fashion, adding to the usual mixed velocity-pressure model a tensorial field such as the extra stress, the strain rate or the velocity gradient. In the scope of finite element methods, besides the computation of a higher number of degrees of freedom, multi-field formulations impose additional compatibility conditions for the finite-element subspaces. The aim of this work is to study multi-field formulations for flow problems employing Galerkin least-squares (GLS) techniques in order to circumvent those conditions. We present a GLS three-field formulation and employ it to approximate 2-D steady flows with various combinations of elements. Such formulation showed good stability features in comprehensive numerical results. Approximations for Newtonian flows in a lid-driven cavity are introduced to validate the formulation and its implementation. We also show some preliminary results using this multi-field GLS formulation to approximate flows of pseudoplastic fluids through a sudden planar contraction.*

Keywords*: finite elements, Galerkin least-squares, non-Newtonian fluids, stress-velocity-pressure formulations*

1. Introduction

The investigation of the flow of non-Newtonian fluids is a subject of a great scope of applications in engineering. Examples of non-Newtonian fluids are products involved in the industries of petroleum, food, polymers, cosmetics, etc. Nevertheless, the difficulties in the study of such flows are many, from the modeling of their non-linear material behavior to the mathematical dealing with the constitutive equations coupled to specific flow mechanical models.

The analysis of non-Newtonian flows has always been a challenge for fluid mechanists. Specially since the last decades, an extensive research field, the numerical simulation of non-Newtonian fluid flow, has been found of great interest (Crochet *et al.*, 1984; Owens & Phillips, 2002). Prediction of fluid behavior and detailed flow visualization in complex geometry, mostly not accomplishable in experimental analysis, has stimulated the research in this area.

The mathematical modeling for non-Newtonian flows eventually originates non-explicit constitutive equations for the extra-stress tensor. In the numerical standpoint, this feature is a difficulty, since it becomes unfeasible to deal with only two primal variables (velocity and pressure). It is necessary to compute the extra-stress as an additional variable, which increases the number of functional spaces comprised and also the number of degrees of freedom in the numerical model. In the context of finite element methods, the multi-field discrete model consists of the equations of momentum and mass conservation, plus a general constitutive equation for the extra-stress. In such cases, two compatibility conditions arise between the finite element subspaces for the variables: the classical Babuška-Brezzi condition for velocity and pressure subspaces, and another compatibility condition between the extra-stress and velocity subspaces.

As for the mixed formulations in two variables approach, the classical Galerkin method for incompressible fluids suffers from two major difficulties. First, the need to satisfy Babuška-Brezzi condition (Ciarlet, 1978) in order to achieve a compatible combination of velocity and pressure subspaces in mixed formulations. Further, the inherent instability of central difference schemes in approximating advective dominated equations (Brooks & Hughes, 1982). In the context of the Stokes equations for Newtonian fluids, the Galerkin least-squares method (GLS) (Hughes *et al.*, 1986) was developed to provide stability to the original Galerkin method by adding mesh-dependent terms to Galerkin

[1] This author is a graduate student at the Mechanical Engineering Program PROMEC/UFRGS.
[2] Corresponding author.

formulation, which are functions of the residuals of Euler-Lagrange equations evaluated elementwise. Since the residuals of the Euler-Lagrange equations are satisfied by the exact solutions, consistency is preserved in these methods. This idea was also extended to incompressible Navier-Stokes equations in (Franca & Frey, 1992) employing stability parameters designed to optimize stability and convergence.

Multi-field formulations have been analyzed by several researchers, with the aim of constructing formulations that have the properties of stability and convergence, and also establishing the combinations of finite element subspaces which would be effective for each of those formulations. In the context of the Stokes problem, the three-field formulation is already a challenge, being exploited by authors as Marchal & Crochet (1987), Franca & Stenberg (1991), Baranger & Sandri (1992), Baaijens (1998) and references therein, Carneiro de Araújo & Ruas (1998) and references therein, Bonvin *et al*. (2001), among others. A four field formulation known as *Elastic Viscous Split Stress (*EVSS) is an alternative which employs the rate of strain tensor as an additional variable, in order to appease the compatibility conditions between the tensor and velocity subspaces (Guenette & Fortin, 1995; Sun *et al.*, 1999; Gatica *et al.*, 2004). Stabilized formulations based on the philosophy of the Galerkin lest-squares methods have been employed to multi-field formulations with two main goals. One is the achieving of the necessary stability and convergence for both Newtonian and non-Newtonian models. The other, which may be considered as the main advantage of such methods, is the circumvention of the compatibility conditions (Babuška-Brezzi) which they mostly provide (Franca & Stenberg, 1991; Behr *et al.*, 1993; Fan *et al.*, 1999; Bonvin *et al.*, 2001).

In 1986, Marchal and Crochet proposed finite elements which used Hermite polynomials, and proved that such elements satisfy a sufficiency condition that assure that the solution for the three-field problem is the same as the result for the two-field problem (Marchal & Crochet, 1986; Marchal & Crochet, 1987). Fortin & Pierre (1989) proved that the element proposed by Marchal & Crochet (1987) satisfied the Babuška-Brezzi condition for the stress and velocity fields subspaces. Ruas and co-workers proposed new mixed elements and obtained important results in the numerical analysis of multi-field formulations for the Stokes problem, also extending their methodology for the flow of viscoelastic fluids (Carneiro de Araújo & Ruas, 2005, and references therein).

In this study, we employ a GLS three-field formulation which is based on the stabilized formulations of Behr *et al.* (1993) and Bonvin *et al*. (2001). We have incorporated features of both works to come up with a GLS three-field formulation for isochoric generalized Newtonian flows. We present some preliminary numerical tests for this formulation which showed good stability features and comprehensive results for lid-driven cavity flows and non-Newtonian flows in a sudden planar contraction.

2. Mechanical model

The mechanical modeling presented herein concerns a material body $\mathcal{B}$ for which flow is defined by the triple velocity, mass density and stress tensor fields, ($\mathbf{v}$, ρ, $\mathbf{T}$), and the associated system of contact and body forces, ($\mathbf{t}(\mathbf{n})$, $\mathbf{f}$).

Principle of Mass Conservation: The mass of a mechanical body $\mathcal{B}$ does not change with time: Mathematically, this primer principle may be stated as

$$\frac{d}{dt}\int_{\mathcal{P}} \rho dV = 0 \tag{1}$$

where ρ is the mass density, $\mathcal{P}$ is a part of a configuration $\mathcal{B}_t$ of the body $\mathcal{B}$ at the time t. Applying Reynolds transport theorem (Gurtin, 1981) to Eq.(1), and assuming an incompressible fluid model, i.e., constant ρ, a variational principle for isochoric motion may be derived:

$$\int_{\mathcal{P}} q \operatorname{div} \mathbf{v} dV = 0 \qquad \forall q \in L^2(\mathcal{B}_t) \tag{2}$$

where $\mathbf{v}$ denotes a virtual velocity field of the flow, and $L^2(\mathcal{B}_t)$, accounts for the functional space of the pressure field.

Principle of Power Expended (Gurtin, 1981): This major dynamic principle is equivalent to the laws of conservation of momentum, formulated in a variational sense. It asserts that, for any part $\mathcal{P}$, with $H^1(\mathcal{B}_t)^{nsd}$ denoting the space of virtual velocities associated to $\mathcal{B}_t$, the power expended on $\mathcal{P}$ by external body forces $\mathbf{f}$ and surface forces $\mathbf{t}(\mathbf{n})$ is equal to the stress power plus the rate of change of kinetic energy:

$$\int_{\mathcal{P}} \rho \mathbf{f} \cdot \mathbf{v}\, dV + \int_{\partial\mathcal{P}} \mathbf{t}(\mathbf{n}) \cdot \mathbf{v}\, dA = \int_{\mathcal{P}} \mathbf{T} \cdot \mathbf{D}\, dV + \int_{\mathcal{P}} \rho \dot{\mathbf{v}} \cdot \mathbf{v}\, dV \qquad \forall \mathbf{v} \in H^1(\mathcal{B}_t)^{nsd} \tag{3}$$

where $\mathbf{D}$ is the strain rate tensor, and $\mathbf{T}$ stands for a second-order symmetric tensor, the stress tensor. The first term on the right side of Eq. (3) accounts for the stress power, i.e., the power expended due to the work of internal contact forces, and the second is the rate of change of kinetic energy.

3. Material Behavior

Although Cauchy theorem (Gurtin, 1981) describes the form of contact forces for any continuous body, the way in which materials deform or flow when submitted to any dynamic condition is not stated by this theorem. Besides, the behavior of continuous bodies submitted to arbitrary conditions differs drastically, due to the material dependent relation between contact forces within the body upon its motion and deformation. This relation is described by the rheological constitutive equations, which are mathematical models for the stress tensor, **T**. These equations are constructed in order to obey certain axiomatic principles (Astarita & Marrucci, 1974): determinism, local action and frame indifference. A functional dependence of **T** with the strain rate tensor, **D**, is acceptable in view that this could represent a frame indifferent model, as **D** is frame indifferent (Gurtin, 1981). The most general linear relation between **T** and **D** tensors may be given as the following isotropic function:

$$\mathbf{T} = (-P + \varpi \,\mathrm{div}\,\mathbf{v})\mathbf{I} + 2\mu\mathbf{D} \tag{4}$$

where μ is the fluid viscosity and the parameter ϖ is related to the scalar function κ, called bulk coefficient of viscosity . For an incompressible fluid, the divergence of the velocity field is null, and Eq. (4) may be written as function of a mean pressure, p, the mean of the normal components of **T,** as:

$$\mathbf{T} = -p\mathbf{I} + 2\mu\mathbf{D} = -p\mathbf{I} + \boldsymbol{\tau} \tag{5}$$

where $\boldsymbol{\tau}$ is the extra stress tensor.

The practically observed phenomena of shear-thinning, viscoplasticity and shear-thickening in pure shear flows give rise to the construction of the generalized Newtonian models (Bird *et al.*, 1987). These models apply the empirical models which fit the behavior stress versus strain rate in viscometric flows for modeling the stress tensor. They maintain the mathematical structure of a Newtonian fluid, in the form that $\mu=\eta(\dot{\gamma})$, being the non-Newtonian viscosity of the fluid, which is a function of the strain rate, $\dot{\gamma}$. For flows in general, $\dot{\gamma}$ is defined as a Frobenius norm of **D**:

$$\dot{\gamma} = (2II_{\mathbf{D}})^{1/2} = (2\,\mathrm{tr}\,\mathbf{D}^2)^{1/2} \tag{6}$$

An example of an empirical model for the viscosity is the Carreau-Yasuda model (Bird *et al.*, 1987), employed to model the behavior of pseudoplastic fluids. This model is given by the following constitutive equation:

$$\frac{\eta(\dot{\gamma}) - \eta_\infty}{\eta_0 - \eta_\infty} = \left[1 + (\lambda\dot{\gamma})^a\right]^{\frac{n-1}{a}} \tag{7}$$

The Reynolds number for a generalized Newtonian fluid may be defined, for a general characteristic viscosity, η_c, which depends on the model employed, as follows,

$$\mathrm{Re} = \frac{\rho L u_0}{\eta_c} \tag{8}$$

in which L and u_0 are the characteristic length and velocity of the flow. Note that for the Newtonian model $\eta_c = \mu$.

For viscoelastic fluids, i.e., fluids with memory, the most common models are the differential constitutive equations (Astarita & Marrucci, 1974; Bird *et al.*, 1987). In general, they are given as functions of the objective derivatives (Astarita & Marrucci) of the extra stress tensor, in order to maintain the necessary feature of frame indifference. Among these viscoelastic models, two are of great importance in numerical simulation of non-Newtonian flows, due to their widespread employment: the Maxwell-B and the Oldroyd-B models, given as:

$$\boldsymbol{\tau} + \lambda \overset{\triangledown}{\boldsymbol{\tau}} = 2(\mu_1\mathbf{D} + \mu_2 \overset{\triangledown}{\mathbf{D}}) \tag{9}$$

where λ and μ_i are the material functions for these models, with μ_2=0 for the Maxwell-B model. The symbol $\triangledown$ represents the upper convected derivative of the respective variable. In numerical approximations, the Oldroyd-B constitutive equation (Eq. (9)) is sometimes decomposed in the following form (Crochet & Keunings, 1982):

$$\begin{aligned} &\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \\ &\boldsymbol{\tau}_1 + \lambda_1 \overset{\triangledown}{\boldsymbol{\tau}_1} = 2\mu_1\mathbf{D} \\ &\boldsymbol{\tau}_2 = 2\mu_2\mathbf{D} \end{aligned} \tag{10}$$

Thus, the model is viewed as the sum of an elastic ($\boldsymbol{\tau}_1$) portion of $\boldsymbol{\tau}$ and a viscous and one ($\boldsymbol{\tau}_2$).

4. Multi-field finite element formulations

The problems considered herein are defined on a bounded domain $\Omega \subset \Re^{nsd=2,3}$, with a polygonal or polyhedral boundary Γ, formed by the union of Γ_g, where Dirichlet conditions are imposed, and Γ_h, subjected to Neumann boundary conditions. As usual, $L^2(\Omega)$, $L_0^2(\Omega)$, $H^1(\Omega)$, $H_0^1(\Omega)$ stand for Hilbert and Sobolev functional spaces (Ciarlet, 1978). Finally, $\|\ ,\ \|_0$ denotes the $L^2(\Omega)$ norm and $\|\ ,\ \|_{0,K}$ the $L^2(\Omega_K)$ one.

Multi-field formulations are those that employ as primal variables, besides the usual velocity (**u**) and pressure (p), any of the fields of extra stress ($\boldsymbol{\tau}$), strain rate (**D**) or velocity gradient (grad **u**). In this section we make some basic comments on mixed multi-field formulations for isochoric flows and also present the stabilized formulation employed in this study. We use the following notation: C_h is a partition of the closed domain $\bar{\Omega}$ into elements, R_k denote the polynomial spaces of degree k , and $(\cdot\,,\cdot)$ represents the L^2 inner product (Ciarlet, 1978).

The Galerkin formulation for the isochoric flow of a Newtonian fluid may be stated as: given **f**, find the triple $(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as:

$$B_1(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_1(\mathbf{S}, q, \mathbf{v}), \quad (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times \mathbf{V}^h \times P^h \tag{11}$$

where

$$\begin{aligned} &B_1(\boldsymbol{\tau}, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = \frac{1}{2\mu}(\boldsymbol{\tau}, \mathbf{S}) - (\mathbf{D}(\mathbf{u}), \mathbf{S}) + \rho([\operatorname{grad}\mathbf{u}]\mathbf{u}, \mathbf{v}) + (p, \operatorname{div}\mathbf{v}) - (\boldsymbol{\tau}, \mathbf{D}(\mathbf{v})) + (\operatorname{div}\mathbf{u}, q) \\ &F_1(\mathbf{S}, q, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \end{aligned} \tag{12}$$

introducing the finite element subspaces:

$$\begin{aligned} P^h &= \{q \in C^0(\Omega) \cap L_0^2(\Omega) \,\big|\, q_{|K} \in R_l(K), K \in C_h\} \\ \mathbf{V}^h &= \{\mathbf{v} \in H_0^1(\Omega)^{nsd} \,\big|\, \mathbf{v}_{|K} \in R_k(K)^{nsd}, K \in C_h\} \\ \mathbf{W}^h &= \{\mathbf{S} \in \mathbf{W} \,\big|\, \mathbf{S}_{|K} \in R_j(K)^{nsd\times nsd}, K \in C_h\} \end{aligned} \tag{13}$$

with

$$\mathbf{W} = \{\mathbf{S} \in C^0(\Omega)^{nsd\times nsd} \cap L^2(\Omega)^{nsd\times nsd} \,\big|\, \mathbf{S}_{ij} = \mathbf{S}_{ji}, i = 1, ..., nsd\} \tag{14}$$

In the context of the Stokes problem, i.e., when the inertial term is negligible, Franca & Stenberg (1991) proposed a GLS formulation which is stable for any combinations of finite elements. That formulation is given by: given **f**, find the triple $(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as:

$$B_{GLS1}(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_{GLS1}(\mathbf{S}, q, \mathbf{v}), \quad (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h \tag{15}$$

where

$$\begin{aligned} &B_{GLS1}(\boldsymbol{\tau}, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = \frac{1}{2\mu}(\boldsymbol{\tau}, \mathbf{S}) - (\mathbf{D}(\mathbf{u}), \boldsymbol{\tau}) + (p, \operatorname{div}\mathbf{v}) - (\mathbf{D}(\mathbf{v}), \mathbf{S}) + (q, \operatorname{div}\mathbf{u}) - \alpha(\boldsymbol{\tau} - 2\mu\mathbf{D}(\mathbf{u}), \mathbf{S} - 2\mu\mathbf{D}(\mathbf{v})) + \\ &+\beta \sum_{K\in C_h} h_K^2 (\operatorname{div}\boldsymbol{\tau} - \operatorname{grad} p, \operatorname{div}\mathbf{S} - \operatorname{grad} q)_K \\ &F_{GLS1}(\mathbf{S}, q, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) - \beta \sum_{K\in C_h} h_K^2 (\mathbf{f}, \operatorname{div}\mathbf{S} - \operatorname{grad} q)_K \end{aligned} \tag{16}$$

Convergence and stability for this formulation were established by Franca & Stenberg (1991). Behr *et al.* (1993) improve these results presenting a stabilized formulation very similar to this former, but also incorporating the inertia terms which had been neglected by Franca & Stenberg (1991) in their formulation for the three-field Stokes problem. Behr *et al.* (1993) also use a design of the stability parameter which incorporates the local Reynolds number and the mesh size parameter h, as in Franca & Frey (1992).

A mixed formulation which is largely employed (Baaijens, 1998) is the one based on a linear version of the Oldroyd-B model, using the split of Eq. (10). The model is linear in view that the parameter λ_1 in Eq. (10) is null. The Galerkin finite element formulation for such model is given as: find the triple $(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as

$$B_2(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_2(\mathbf{S}, q, \mathbf{v}), \quad (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h \tag{17}$$

with

$$\begin{aligned} &B_2(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = \frac{1}{2\eta_p}(\boldsymbol{\tau}_1, \mathbf{S}) - (\mathbf{D}(\mathbf{u}), \mathbf{S}) + \rho([\operatorname{grad}\mathbf{u}]\mathbf{u}, \mathbf{v}) + 2\eta_s(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{v})) + (\boldsymbol{\tau}_1, \mathbf{D}(\mathbf{v})) - (p, \operatorname{div}\mathbf{v}) + (\operatorname{div}\mathbf{u}, q) \\ &F_2(\mathbf{S}, q, \mathbf{v}) = -(\mathbf{f}, \mathbf{v}) \end{aligned} \tag{18}$$

Bonvin *et al.* (2001) prove uniqueness and existence of the problem of Eq. (17), neglecting the inertial term ($\rho = 0$). They also establish the stability lemma for the finite element formulation, which depends on two compatibility conditions, one between the finite element subspaces of stress and velocity, and other between the subspaces of velocity and pressure. The former is the classical Babuška-Brezzi condition for the Stokes problem. Bonvin *et al.* (2001) derive Galerkin least-squares methods that circumvent those compatibility conditions, adding terms to the forms B_2 and F_2, which correspond to the least squares forms of the residual equations.

In this study, we employed a GLS formulation based on the Galerkin scheme of Eq. (17), based on the stabilized formulations of Behr *et al.* (1993) and Bonvin *et al.* (2001): Find the triple $(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h$ such as:

$$B_{GLS2}(\boldsymbol{\tau}_1^h, p^h, \mathbf{u}^h; \mathbf{S}, q, \mathbf{v}) = F_{GLS2}(\mathbf{S}, q, \mathbf{v}) \ \ \forall\ (\mathbf{S}, q, \mathbf{v}) \in \mathbf{W}^h \times P^h \times \mathbf{V}^h \tag{19}$$

where

$$\begin{aligned} &B_{GLS2}(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) = B_2(\boldsymbol{\tau}_1, p, \mathbf{u}; \mathbf{S}, q, \mathbf{v}) + \sum_{K \in C_h} \tau(\mathrm{Re}_K)([\operatorname{grad}\mathbf{u}]\mathbf{u} - 2\eta_s \operatorname{div}\mathbf{D}(\mathbf{u}) + \operatorname{grad} p - \operatorname{div}\boldsymbol{\tau}_1, \\ &[\operatorname{grad}\mathbf{v}]\mathbf{u} - 2\eta_s \operatorname{div}\mathbf{D}(\mathbf{v}) + \operatorname{grad} q - \operatorname{div}\mathbf{S})_K + 2\eta_p \beta \left(\frac{1}{2\eta_p}\boldsymbol{\tau}_1 - \mathbf{D}(\mathbf{u}), \frac{1}{2\eta_p}\mathbf{S} - \mathbf{D}(\mathbf{v}) \right) \\ &F(\mathbf{S}, q, \mathbf{v})_{GLS2} = F_2(\mathbf{S}, q, \mathbf{v}) + \sum_{K \in C_h} \tau(\mathrm{Re}_K)(\mathbf{f}, [\operatorname{grad}\mathbf{v}]\mathbf{u} - 2\eta_s \operatorname{div}\mathbf{D}(\mathbf{v}) + \operatorname{grad} q - \operatorname{div}\mathbf{S})_K \end{aligned} \tag{20}$$

where $\tau(\mathrm{Re}_K)$ is the stability parameter, given as suggested by Franca & Frey (1992) and Behr *et al.* (1993):

$$\begin{aligned} &\tau(\mathrm{Re}_K) = \frac{h_K}{2|\mathbf{u}|_\infty}\xi(\mathrm{Re}_K) \\ &\xi(\mathrm{Re}_K) = \begin{cases} \mathrm{Re}_K, 0 \le \mathrm{Re}_K < 1 \\ 1\ , \mathrm{Re}_K \ge 1 \end{cases} \\ &\mathrm{Re}_K = \frac{m_k |\mathbf{u}|_\infty h_K}{4\eta_p(\dot{\gamma})/\rho} \\ &m_k = \min\{1/3, 2C_k\} \\ &\sum_K h_K^2 \|\operatorname{div}\mathbf{T}\|_{0,K} \le C_k \|\mathbf{T}\|_0^2 \end{aligned} \tag{21}$$

REMARK: The differences between our formulation (Eq. (19)) and the formulation of Behr *et al.* (1993) are that ours represents a truly Galerkin least-squares formulation, in the sense that the stabilizing terms are added as the least squares forms of the residual equations; and that the term containing the solvent viscosity η_s is also present. The differences between the formulation of Bonvin et. al. (2001) and ours are the design of the stability parameter and that our formulation also accounts for inertia effects.

5. Numerical results

We implemented the formulation of Eq. (19) in the finite element code in development by LAMAC group. We present some results for 2-D isochoric flows which were obtained using meshes of quadrilateral bilinear elements for all variables ($Q_1/Q_1/Q_1$ - $\boldsymbol{\tau}_1$-p-$\mathbf{u}$). We have also obtained similar results for the combinations of elements $Q_2/Q_1/Q_2$ and

$Q_1/Q_1/Q_2$, which are not shown here. To solve the resulting algebraic system of equation, we have implemented a Newton-based method (Dalquist & Bjorck, 1969).

5.1. Newtonian flow in a square lid-driven cavity

We present the results obtained with formulation of Eq. (19) for Newtonian flow in a lid-driven cavity flow, using a 40x40 mesh, which the problem statement is given as in Fig. 1(a), with L=1m, u_0=1m/s. The Newtonian viscosity was split in two parts as follows: for the Stokes flow $\eta_p=\eta_s=0.5$; for flows with inertia $\eta_p>\eta_s$. Figures 1(b) and 1(c) depict the streamlines for the Stokes flow and for Re=400 (η_p=0.002, η_s=0.0005), respectively. Streamlines were calculated using the finite element formulation of Carneiro de Araújo *et al.* (1999). In Fig. 2 we depict pressure elevation and components τ_{11} and τ_{12} for the Stokes flow. In Fig. 3 we depict pressure contours and components τ_{11} and τ_{12} for the inertial flow. One may notice that the method is stable even when inertial effects are accounted for. Those results agree qualitatively with the ones of Behr *et al.* (1993).

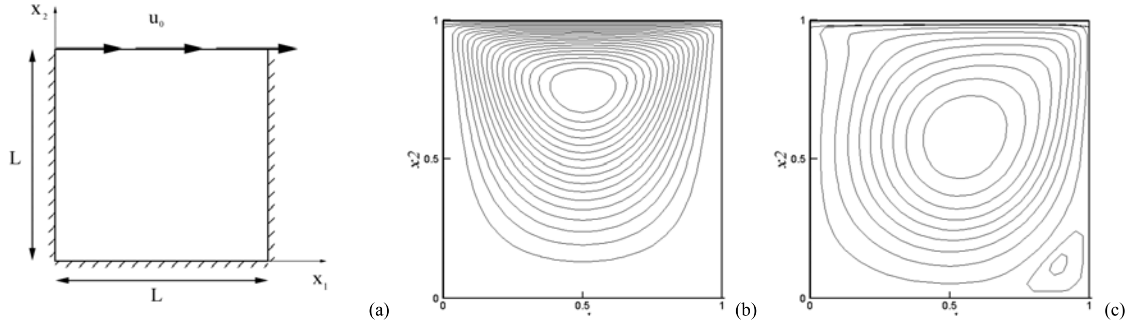

Figure 1: (a) problem statement for the lid-driven cavity, (b) streamlines for Stokes flow, (c) streamlines for Re=400.

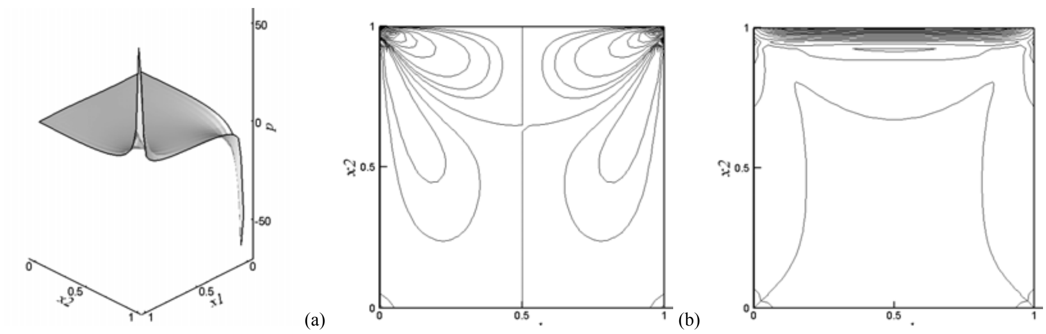

Figure 2: (a) pressure elevation for Stokes flow, (b) τ_{11} and (c) τ_{12} for Stokes flow.

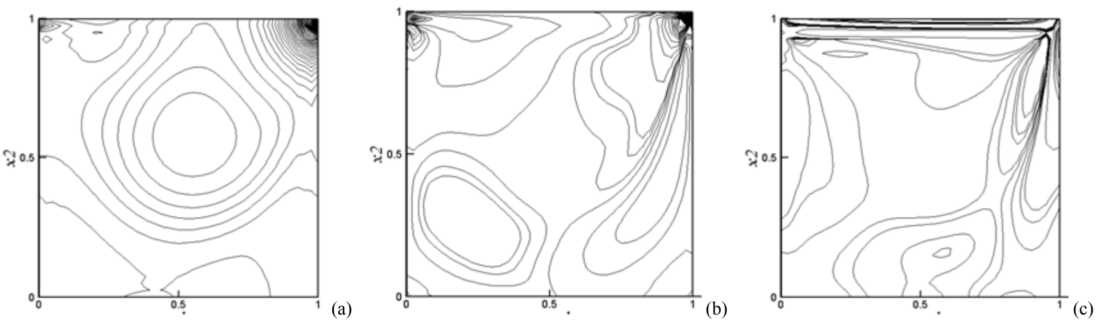

Figure 3: (a) pressure contours for Stokes flow, (b) τ_{11} and (c) τ_{12} for flow with Re = 400.

5.2. Pseudoplastic flow through a sudden contraction

In this section we present some preliminary results for non-Newtonian flows trough a planar 4:1 sudden contraction. The problem statement is given as in Fig. 5 (a), with L=1m, u_0=1m/s. We use only half of problem's actual domain to build the geometric model. A mesh dependency analysis was performed over three meshes, comprised of 3264 (m1), 5275 (m2) and 7290 (m3) elements. The results were found to be mesh independent for mesh m2, as it may be seen in Fig. 4, which shows some Newtonian results for Re=100 (this was one case in which numerical results were most affected by mesh refinement). The velocity $U=u_1/u_0$ in the plane of contraction is plotted against $h=(8x_2)/L$ in Fig. 4 (a) and the pressure $p^*=(pL)/\rho u_0$ is plotted versus x_1 in Fig. 4 (b).

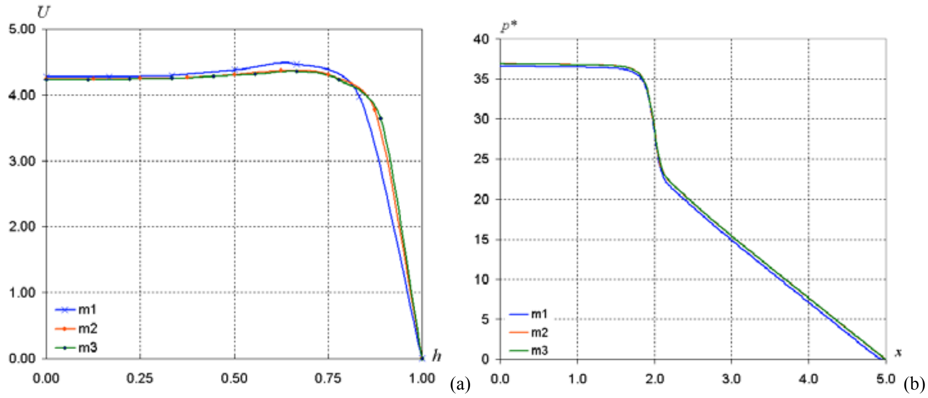

Figure 4: Mesh dependency tests: (a) $U=u_1/u_0$ versus $h=(8x_2)/L$, (b) $p^*=(pL)/\rho u_0$ versus x_1.

The pseudoplastic model of Carreau (Eq. (7)) was employed with parameters as suggested by Bao (2002): $\eta_\infty=0.5$, $\eta_0=1.0$ and $\lambda=1$; $n=0.5$ and $n=0.1$. We assumed a negligible Reynolds number. We present the results obtained with formulation of Eq. (19) and mesh m2, in which we made $\eta_s=\eta_\infty$, $\eta_p=\eta-\eta_\infty$, splitting the viscosity in the Carreau model naturally. Figure 5(b) shows the result for the streamlines for $n=0.5$ in a detail of the domain near the contraction. Figure 6 shows the viscosity field for the two values of n, (a) $n=0.5$ and (b) $n=0.1$, which shows the expected characteristics of lower values in regions of high shear rates and vice versa. It also depicts the differences in the viscosity fields for the two fluids considered, with lower viscosity values for the higher n. Figure 7 depicts the contours for the (a) τ_{12} and (b) u_2, for the Carreau fluid with $n=0.5$.

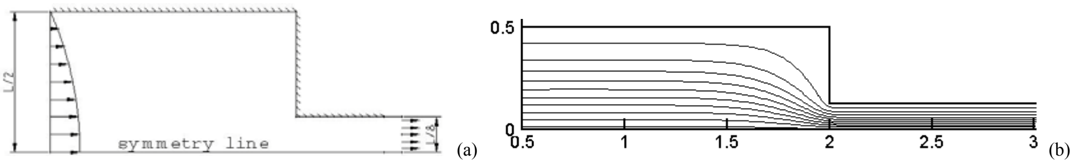

Figure 5: (a) problem statement for the Carreau flow through a sudden 4:1 contraction, (b) streamlines.

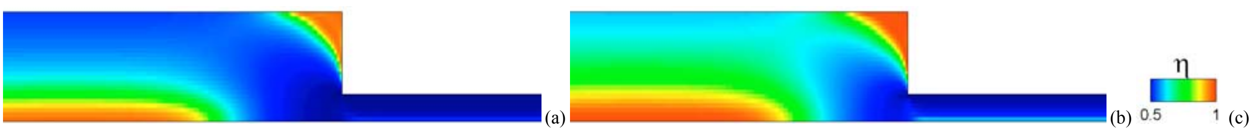

Figure 6: viscosity field for (a) n = 0.1 and (b) n = 0.5 in the Carreau model, (c) legend for viscosity contours.

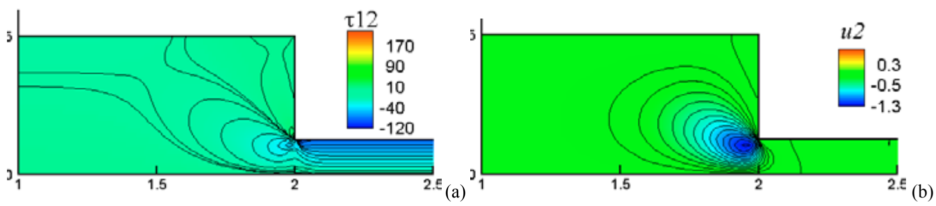

Figure 7: pseudoplastic flow, Carreau model with n = 0.5, (a) contours for τ_{12} (b) contours for u_2.

6. Final remarks

We have presented some comprehensive results for Newtonian and generalized Newtonian flows using a three-field finite element formulation stabilized by a GLS scheme. Such formulation allows the use of simple combinations of finite elements, circumventing the Babuska-Brezzi conditions for all subspaces. This represents a good achievement in view that simple meshes generated by usual software may be employed with such formulation. This study is still in progress and we intend to proceed on investigations in the mathematical analysis of the formulation presented, in the generation and validation of more numerical results and in the extension of this GLS formulation to non-linear viscoelastic models as Maxwell-B and Oldroyd-B.

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