MEASURING THE STRESS AMPLITUDE WITHIN THE CONTEXT OF MULTIAXIAL FATIGUE

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Abstract. *In this paper, we propose a new measure of shear stress amplitude within the setting of multiaxial fatigue, based on the concept of the circumscribing rectangular prism associated with the stress path. Assessment of the resulting fatigue criterion shows that the proposed measure of shear stress amplitude is essentially the same as the one based on the ellipsoid circumscribing the stress path, proposed by a number of authors. Application of both measures to iso-frequency out-of-phase as well as more general multi-axial loadings provides very good results for the resulting fatigue criteria.*

Keywords: *fatigue, multi-axial high-cycle fatigue, stress amplitude*

1. Introduction

In a previous paper (Gonçalves et al. (2005)), the authors reported a fatigue endurance criterion well suited for situations involving synchronous, sinusoidal, multiaxial, in-phase or out-of-phase loadings. As in many classical multiaxial criteria for high cycle fatigue, it was based on measures of the shear stress amplitude and of the normal stress. The main contribution of that model was a new measure for the shear stress amplitude, defined as a function of quantities associated with a circumscribing rectangular prism. A mathematical result presented in that paper shows that this measure does not depend on the orientation of the circumscribing rectangular prism. As a consequence, the calculation of the shear stress amplitude becomes a very simple task. However, the aforementioned result was stated only for proportional, afine or nonproportional synchronous sinusoidal multiaxial stress paths.

The goal of the present study is to propose and to assess a fatigue endurance criterion for metals subjected to very general multiaxial stress paths. Its main feature is a new definition of the shear stress amplitude within the setting of multiaxial, proportional or nonproportional stress paths. The criterion presented here extends the one presented by Gonçalves et al. (2005) by selecting a particular rectangular prism from the set which circumscribes the stress path. Assessment shows that the proposed criterion compares very well with experimental results reported in the literature describing limiting situations of fatigue strengths.

2. The fatigue model

Many multiaxial fatigue endurance criteria can be represented in terms of the inequality:

$$f(\tau) + \kappa\, g(\sigma) \leq \lambda, \tag{1}$$

where κ and λ are material parameters, while f and g are functions of the shear stress τ and of the normal stress σ acting upon the material point under consideration. A number of authors — including Crossland (1956), Deperrois (1991) and Papadopoulos et al. (1997) — adopt, as the function $g(\sigma)$ of the normal stress, the maximum value of the hydrostatic stress acting upon the material point:

$$g(\sigma) = ph_{max} := \max_t \left[\frac{1}{3}\, tr\, \mathbf{T}(t)\right], \tag{2}$$

where $\mathbf{T}(t)$ accounts for the Cauchy stress tensor at time instant t. On the other hand, the measure of the shear stress amplitude, in the setting of multiaxial stress paths has been the subject of several distinct approaches. Crossland considers, as a measure of the shear solicitation to fatigue, the $\sqrt{J_2}$ radius of the sphere circumscribing the stress path (after projection onto the deviatoric space). This criterion leads to very good results when proportional or affine stress histories are taken into account. Nevertheless, proportional and nonproportional paths can be circumscribed by the same sphere (as illustrated in Fig. 1) and hence leading to the same measure of shear stress amplitude, although a more severe solicitation is expected when the nonproportional stress history is considered. Deperrois proposes, as the equivalent shear stress, the expression:

$$f(\tau) = \frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{5} D_i^2}, \tag{3}$$

where $D_i,\ i = 1, \ldots, 5$ are computed as follows: first, the longest chord D_5 between two distinct points of the stress path in the deviatoric space is determined; next, the stress path is projected onto a subspace orthogonal to such chord; a new longest chord D_4 is computed in this subspace, and the process is repeated successively for the remaining dimensions. As remarked by Papadopoulos (1997), in some situations the Deperrois criterion shows a lack of uniqueness of the longest chord, making the definition of the orthogonal subspace an ill posed problem.

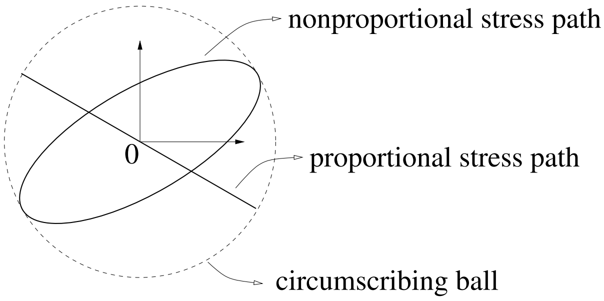


Figure 1. Proportional and nonproportional stress paths associated with the same amplitude $\sqrt{J_{2,a}}$.

Alternatively, one could consider a quantity associated with the minimum ellipsoid circumscribing the stress path, as suggested by Bin Li et al. (2000): the equivalent shear stress is defined as the square root of the sum of the squared semi-axes $\lambda_i,\ i = 1, \ldots, 5$ of the minimum cricumscribing ellipsoid (in the deviatoric space):

$$f(\tau) = \sqrt{\sum_{i=1}^{5} \lambda_i^2}. \tag{4}$$

Although this measure provides very good results when compared to experimental results associated with iso-frequency in-phase and out-of-phase sinusoidal loadings, it demands quite elaborate algorithms to compute the semi-axesof the ellipsoid.

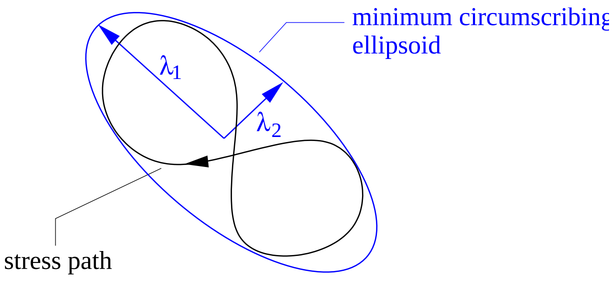


Figure 2. Non-proportional stress path and its corresponding minimum circumscribing ellipsoid.

Seeking for an easer to compute measure of shear stress amplitude, Mamiya & Araújo (2002) showed that the following equality holds whenever iso-frequency in-phase or out-of-phase sinusoidal multiaxial loadings are considered:

$$\sum_{i=1}^{5} \lambda_i^2 = \sum_{i=1}^{5} a_i^2, \tag{5}$$

where a_i are the distances of the centre of the ellipsoid to the faces of any arbitrarily oriented rectangular prism circumscribing the stress path in the deviatoric space, as illustrated in Fig. 3. As a consequence, the shear stress amplitude can be described by the quantity:

$$f(\tau) = \sqrt{\sum_{i=1}^{5} a_i^2}, \quad (6)$$

for any arbitrarily oriented rectangular prism. This result enables us to derive $f(\tau)$ easily whenever stress paths are described by elliptical curves, without the need to search for the minimum circumscribing the elliptic hull.

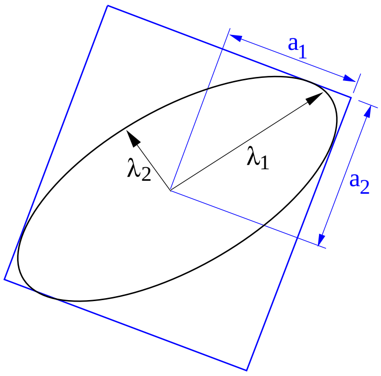


Figure 3. Rectangular prism circumscribing an elliptic stress path.

In this paper, we show that, when more general stress paths are taken into account (distinct frequencies or non-sinusoidal loadings), then: (i) the minimum circumscribing ellipsoid still provides very good fatigue endurance estimates, (ii) a measure of the shear stress amplitude associated with circumscribing rectangular prisms also assesses very well when compared to experimental data reported in the literature (provided a properly oriented prism is chosen) and (iii) both measures give essentially the same results, indicating that an equivalence between them could exist.

2.1 A new measure of shear stress amplitude

We claim that, for any proportional or nonproportional general multiaxial cyclic loadings, the shear stress amplitude, within the setting of fatigue endurance, can be expressed as:

$$f(\tau) = \max_{\theta} \sqrt{\sum_{i=1}^{5} a_i^2(\theta)}, \quad (7)$$

where θ is the orientation of the rectangular prism circumscribing the stress path, while $a_i(\theta),\ i = 1, \ldots, 5$ are the distances from the centre of the prism to its faces, in the deviatoric space. The definition of $f(\tau)$ in Eq. 7 is an extension of the one proposed by Mamiya & Araújo (2002) to situations where the stress path components have not necessarily the same frequency or are not sinusoidal. Since in these cases the invariance property stated in Eq. 5 does no longer hold, the rectangular prism with orientation θ leading to the maximum value of the quantity $\sqrt{\sum_{i=1}^{5} a_i^2(\theta)}$ has to be chosen.

3. Assessment

In this section, the following fatigue criteria:

$$\sqrt{\sum_{i=1}^{5} \lambda_i^2} + \kappa\, ph_{max} \leq \lambda, \quad (8)$$

$$\max_{\theta} \sqrt{\sum_{i=1}^{5} a_i^2(\theta)} + \kappa\, ph_{max} \leq \lambda, \quad (9)$$

are assessed for a number of situations reported in the literature, describing nonproportional sinusoidal loadings with distinct frequencies and nonsinusoidal loadings. The criterion described by Eq. 8 is essentially the one proposed by Bin

Li et al. (2000), but with a better understanding of the concept of the minimum circumscribing ellipsoid. The criterion presented by Eq. 9 is new, and considers a quantity associated with circumscribing rectangular prisms as a measure of the shear stress amplitude.

If f_1 and t_1 are fatigue endurance limits under alternate bending and alternate torsion solicitations, respectively, then the parameters κ and λ can be computed in both cases as:

$$\kappa = \sqrt{2}\Big(3\,\frac{t_{-1}}{f_{-1}} - \sqrt{3}\Big) \quad \text{and} \quad \lambda = \sqrt{2}\,t_{-1}. \tag{10}$$

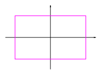
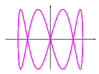
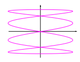
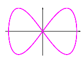
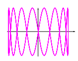
Experimental data presented in Table 1 — reported by Heidenreich et al. (1983) and Kaniut, C. (1983) for hard metals $(1.3 \leq f_{-1}/t_{-1} \leq \sqrt{3})$ — describe biaxial normal/shear trapezoidal iso-frequency stress history and sinusoidal biaxial normal/shear stress histories of the form:

$$\sigma(t) = \sigma_a \sin(\omega t) + \sigma_m \tag{11}$$

$$\tau(t) = \tau_a \sin(\eta\omega t - \beta) + \tau_m \tag{12}$$

Parameters σ_a, σ_m, τ_a,τ_m, η and β describe limiting situations of loadings that the specimen can stand without failing before 10^6 cycles. σ and τ describe respectively the normal and shear histories, the subscripts a and m stand respectively for the amplitude and the mean value of stresses, β is the phase angle and η is a frequency factor.

Table 1. Multiaxial loadings: (1) trapezoidal and (2 to 5) sinusoidal with distinct frequencies

	σ_a (MPa)	σ_m (MPa)	τ_a (MPa)	τ_m (MPa)	β (^{o})	η	σ x τ
1	240	0	120	0	90	1	
2	263	0	132	0	0	4	
3	210	0	105	0	0	1/4	
4	220	0	110	0	0	2	
5	196	0	98	0	0	8	

1 and 2: 34Cr4 ($f_{-1} = 415$ MPa, $t_{-1} = 256$ MPa) Heidenreich et al. (1983)
3, 4 and 5: 25CrMo4 ($f_{-1} = 340$ MPa, $t_{-1} = 228$ MPa) Kaniut, C. (1983)

3.1 Trapezoidal stress path

First, let us consider the trapezoidal traction/torsion stress path studied on 34Cr4 alloy by Heidenreich et al. (1983) (Table 1, line 1). The loading history can be fully characterized by the corner point $(\sigma_a, \tau_a) = (240\,MPa,\,120\,MPa)$. Projection of this stress state into the deviatoric space define the stress components:

$$(s_1, s_3) = \left(\sqrt{\frac{2}{3}}\sigma_a,\,\sqrt{2}\,\tau_a\right), \tag{13}$$

provided the following orthonormal basis of the deviatoric stress space is considered:

$$N_1 = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{6}} \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix},$$
$$N_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, N_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}, N_5 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}, \tag{14}$$

An infinite number of ellipses:

$$\left(\frac{s_1}{\lambda_1}\right)^2 + \left(\frac{s_3}{\lambda_3}\right)^2 = 1. \tag{15}$$

with axis sizes λ_1 and λ_3 passes through the corner point (s_1, s_3). From Eq. 15, we can write λ_1 as a function of λ_3:

$$\lambda_1 = \frac{s_1\,\lambda_3}{\sqrt{\lambda_3^2 - s_3^2}}. \tag{16}$$

Thus, the minimum crcumscribing ellipsoid can be characterized as the one which minimizes the quantity:

$$f(\lambda_3;\, s_1, s_3) = \sqrt{\frac{s_1^2\,\lambda_3^2}{\lambda_3^2 - s_3^2} + \lambda_3^2}. \tag{17}$$

in λ_3. In the present case, we have:

$$f = 365.60\,MPa, \quad g = 80\,MPa \tag{18}$$

and hence the error index — which measures the relative difference between predictio nand experimental data characterizing the fatigue limit — is given by:

$$I := \frac{f + \kappa\, g - \lambda}{\lambda} \times 100 = 4.96\% \tag{19}$$

as indicated in Table 2, line 1. The error index I was expected to be equal to zero, in order to describe a limiting situation of fatigue endurance. Since the computed value of I is positive, it means that the fatigue endurance estimate provided by Eq. 8 is conservative in this example.

In the case of the shear stress amplitude based on the rectangular prism, the goal is to determine the prism circumscribing the deviatoric stress path which gives the maximum value of the quantity $\sqrt{\sum_{i=1}^{5} a_i^2(\theta)}$, as illustrated in Fig. 4. For each $\theta \in [0, \pi/2]$, the quantities $a_i(\theta),\ i = 1, \ldots, 5$ can be computed as:

$$a_i(\theta) = \frac{1}{2}\left(\max_t s_i^*(t) - \min_t s_i^*(t)\right), \tag{20}$$

where:

$$\mathbf{s}^* = \mathbf{Q}\,\mathbf{s}, \tag{21}$$

$$\mathbf{Q} = \begin{pmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{pmatrix}. \tag{22}$$

Due to the simplicity of the stress path, the shear stress amplitude based on the rectangular prism can be determined analytically. As illustrated in Fig. 4, the measures $\|\ \mathbf{u}\ \|$, a_1, and a_3 can be computed as:

$$a_1 = u\,\cos(\alpha - \theta) \tag{23}$$

$$a_3 = u\,\cos(\beta - \theta). \tag{24}$$

where:

$$u = \sqrt{(s_1)^2 + (s_3)^2} \tag{25}$$

The norm $\sqrt{\sum_{i=1}^{5} a_i^2(\theta)}$ attains its maximum value when $\alpha = \beta = \theta = \pi/4$. The resulting quantities are:

$$f = 366.61\,MPa, \quad g = 80\,MPa \tag{26}$$

and hence the error index is given by:

$$I = 4.97\% \tag{27}$$

as indicated in Table 2, line 1. We should notice that results are essentially the same ones obtained when considering the minimum circumscribing ellipsoid.

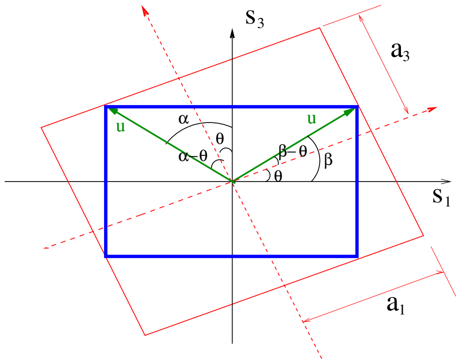
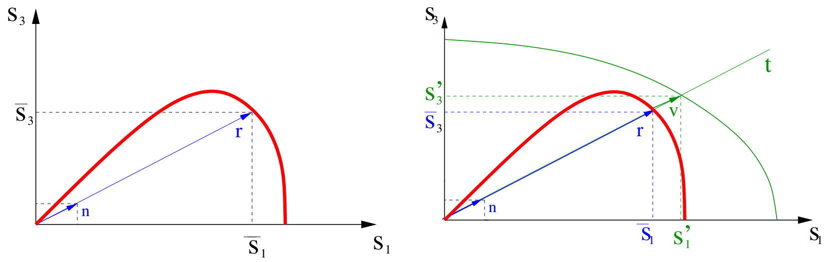


Figure 4. Rectangular prism circumscribing the trapezoidal deviatoric stress history



(a) (b)

Figure 5. Determination of the minimum ellipsoid circumscribing the deviatoric stress path for sinusoidal loadings with distinct frequencies.

3.2 Sinusoidal stress paths with distinct frequencies

For the sinusoidal loadings with distinct frequencies, a methodology must be created to determine the point of the stress path which is tangent to the minimum ellipsoid. In this context, let us consider the load history described in Table 1 line 4.
Each point of the stress path can be represented by a vector $\mathbf{r}$ and its corresponding unity vector $\mathbf{n}$:

$$\mathbf{r} = \begin{pmatrix} s_1 \\ s_3 \end{pmatrix}, \quad \mathbf{n} = \frac{\mathbf{r}}{||\mathbf{r}||}. \tag{28}$$

Arbitrary values of λ_1 and λ_3 define an ellipsis which may eventually cross or circumscribe the stress path. Given $\mathbf{r}$, let us define the vector $\mathbf{v}$ on the ellipsis, parallel to $\mathbf{r}$ (see Fig. 5):

$$\mathbf{v} = \alpha\, \mathbf{r} \tag{29}$$

such that:

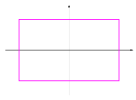
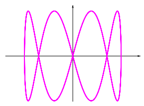
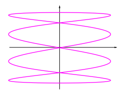
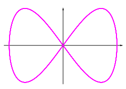
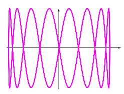
$$\left(\frac{\alpha\, s_1}{\lambda_1}\right)^2 + \left(\frac{\alpha\, s_3}{\lambda_3}\right)^2 = 1 \tag{30}$$

If α is lesser or equal to one, then the ellipsis is secant or tangent to the stress path at the stress state defined by $\mathbf{r}$. If the largest value of α, computed by considering all the points from the stress path, is equal to one, then the ellipsis defined by radii λ_1 and λ_3 is tangent to the loading history. This procedure can be adopted to build a set of ellipses tangent to the stress path. We select from this set the one with the smallest value of the quantity $\sqrt{\lambda_1^2 + \lambda_3^2}$.

Experimental data reported by Heidenreich et al. (1983) for the 34Cr4 alloy and by Kaniut (1983) for 25CrMo4, representing fatigue endurance situations, are listed in Table 1, lines 2 to 5. Frequencies of stress components (traction/torsion) differ from a factor which varies from $1/4$ to 8.

The corresponding fatigue endurance estimates are listed in Table 2, lines 2 to 5: when considering the criterion (8), based on the quantities associated with the minimum circumscribing ellipsoid, the calculated index error varies from -0.33% to 10.67%. The predictions are essentially the same when considering the criterion (9), based on the measure associated with the rectangular prism.

Table 2. Results for the shear stress amplitude $f(\tau)$, contribution of the normal stresses $g(\sigma)$ and error indexes I

		circumscribing ellipsoid	rectangular prism	
1	$f(\tau)$ (MPa)	365.60	366.61	
	$g(\sigma)$ (MPa)	80.0	80.0	
	I (%)	4.96	4.97	
2	$f(\tau)$ (MPa)	385.97	385.97	
	$g(\sigma)$ (MPa)	87.67	87.67	
	I (%)	10.67	10.67	
3	$f(\tau)$ (MPa)	309.19	309.19	
	$g(\sigma)$ (MPa)	70.0	70.0	
	I (%)	4.47	4.47	
4	$f(\tau)$ (MPa)	293.28	293.33	
	$g(\sigma)$ (MPa)	73.33	73.33	
	I (%)	-0.05	-0,03	
5	$f(\tau)$ (MPa)	295.52	295.52	
	$g(\sigma)$ (MPa)	65.33	65.33	
	I (%)	-0,33	-0,33	

1 and 2: 34Cr4 ($f_{-1} = 415$ MPa, $t_{-1} = 256$ MPa) Heidenreich et al. (1983)
3, 4 and 5: 25CrMo4 ($f_{-1} = 340$ MPa, $t_{-1} = 228$ MPa) Kaniut, C. (1983)

4. Conclusions

The assessment showed very good agreement between predictions provided by both fatigue criteria and experimental data reported in the literature. Since the predictions were essentially the same when considering the measure of the shear stress amplitude based on the minimum circuscribing ellipsoid and on the maximum circumscribing rectangular prism, the equivalence between these measures appears as a possibility, although further assessment is needed, by considering a larger number of experimental data. The advantage of the criterion based on the rectangular prism is its simplicity, which makes it well suited for applications within the context of finite element analysis.

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