OPTIMIZATION OF LOW-THRUST LIMITED-POWER TRANSFERS BETWEEN COPLANAR CIRCULAR ORBITS

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Abstract. *In this paper a numerical analysis of optimal low-thrust limited power trajectories for simple transfer (no rendezvous) between circular coplanar orbits in a central Newtonian gravity field is presented. This analysis is carried out by means of two different algorithms based on second variation theory. One of them considers a direct approach of the optimization problem and involves gradient techniques, and the other one considers an indirect approach based on the set of necessary conditions of Pontryagin Maximum Principle and involves an iterative solution of the two-point boundary value problem describing the optimal solutions. The fuel consumption is taken as the performance criterion and the analysis is carried out considering various radius ratios and transfer durations. The results are compared to the ones provided by a linear analytical theory.*

Keywords: *Optimal low-thrust limited power trajectories, transfers between circular coplanar orbits.*

1. Introduction

The main purpose of this paper is to present a numerical analysis of optimal low-thrust limited power trajectories for simple transfers (no rendezvous) between circular coplanar orbits in a Newtonian central gravity field. The fuel consumption is taken as the performance criterion and it is calculated for various radius ratios $\rho = r_f / r_0$, where r_0 is the radius of the initial circular orbit O_0 and r_f is the radius of the final circular orbit O_f, and for various transfer durations $t_f - t_0$. Transfers with small and moderate amplitudes are studied and the numerical results are compared to the ones provided by a linear theory (Marec, 1967, 1979; Da Silva Fernandes, 1989). This analysis has been motivated by the renewed interest in the use of low-thrust propulsion systems in space missions verified in the last ten years due to the development and the successes of two space mission powered by ionic propulsion: Deep Space One and SMART 1.

Two different algorithms based on second variation theory are applied in determining the optimal trajectories. The first algorithm (Da Silva Fernandes and Golfetto, 2003, 2004) combines the main positive characteristics of the steepest-descent (first order gradient method) and of a direct method based upon the second variation theory (second order gradient method), and it has two distinct phases. In the first one, the algorithm uses a simplified version of the steepest-descent method developed for a Mayer problem of optimal control with free final state and fixed terminal times, in order to get great improvements of the performance index in few iterates with satisfactory accuracy. In the second phase, the algorithm switches to a direct method based upon the second variation theory developed for a Bolza problem with fixed terminal times and constrained initial and final states, in order to improve the convergence as the optimal solution is approached. The second algorithm is based on the linearization about an extremal solution of nonlinear two-point boundary value problem defined by the set of necessary conditions for a Bolza problem of optimal control with fixed initial and final times, fixed initial state and constrained final state. The resulting linear two-point boundary value problem is solved through Riccati transformation. For completeness, a brief description of these algorithms is presented in the next section.

2. Optimization methods based on second variation theory

2.1 The combined algorithm

As discussed in the book by Bryson and Ho (1975), the main characteristic of the steepest-descent method is the great improvements in the first few iterates, but it has poor convergence as the optimal solution is approached. On the other hand, the second-order gradient method exhibits excellent convergence characteristics as the optimal solution is approached, but it requires the nominal solution to be "convex". Numerical experiments have shown that the accuracy of the solution is very sensitive, if the first approximation of the control is poor (Da Silva Fernandes and Golfetto, 2003). Considering these remarks, an algorithm which combines the best characteristics of each method is implemented: at the first step, the steepest descent method is applied to improve the nominal solution to a point where it is convex; then, at the second step, the second-order gradient method is applied. This algorithm will be simply referred to as the combined algorithm. The algorithms of steepest-descent and second order gradient methods are described in what follows.

2.1.1 Steepest-descent method

The steepest-descent method is an iterative direct method that minimizes a scalar performance index in an optimization problem. The most known version has been proposed by Bryson and Denham (1962). In this section, a simplified version of the steepest descent method is presented for a Mayer problem of optimal control with free final state and fixed terminal times. Problems with constraints on the state variables at the final fixed time are treated by using the penalty function method (O'Doherty and Pierson, 1974), and, problems with free final time are treated by using a transformation approach which introduces a new independent variable and an additional control variable (Hussu, 1972).

Consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n, \tag{1}$$

where x is a n-vector of state variables, u is a m-vector of control variables, $f(.): R^n \times R^m \to R^n$,$f_i(.)$ e $\partial f_i / \partial x_j$, i and j =1,...,n, are defined and continuous on $R^n \times R^m$. It is assumed that there exist no constraints on the state or control variables. The problem consists in determining the control $u^*(t)$ that transfers the system (1) from the initial conditions:

$$x(t_0) = x_0, \tag{2}$$

to the final conditions at t_f :

$$x(t_f) - \text{free}, \tag{3}$$

and minimizes the performance index:

$$J[u] = g(x(t_f)), \tag{4}$$

with g: $R^n \to R$ and $\partial g / \partial x_i$, i =1,...,n, continuous on R^n.

The development of the algorithm is based on the classic Calculus of Variations (Gelfand and Fomin, 1963). The step by step computing procedure to be used in the steepest-descent method is summarized as follows:

1. Integrate the differential equations (1) from the initial point x_0 at t_0 to an unspecified final point $x(t_f)$ at t_f, with the nominal control $u^0(t), t_0 \le t \le t_f$;

2. Integrate the adjoint equations $\frac{d\lambda}{dt} = -H_x^T$ from t_f to t_0, with the terminal conditions $\lambda(t_f) = -g_{x_f}^T$. H is the Hamiltonian function defined as $H(x,\lambda,u) = \lambda^T f(x,u)$;

3. Compute the Lagrangian multiplier $\nu = \frac{1}{2K}\left\{\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt\right\}^{1/2}$;

4. Compute the control "correction" $\delta u = \frac{1}{2\nu} W^{-1} H_u^T$;

5. Obtain a new nominal control by using $u^1(t) = u^0(t) + \delta u(t)$, and repeat the process (1) through (4) until the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ tends to zero or other convergence criterion is satisfied.

It should be noted that as the nominal control $u^n(t)$ approaches the optimal control $u^*(t)$, the integral $\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt$ approaches zero and the Lagrangian multiplier ν tends to zero; thus the control "correction" $\delta u(t)$ can become too large and the process diverges. In order to avoid this drawback, the step size in control space K must be redefined. The following heuristic approach is proposed: suppose that K^m be the step size in control space at the m-th iterate, if

$\int_{t_0}^{t_f} H_u W^{-1} H_u^T dt < L$, where L is a critical value for this integral, then K is redefined as $K^{m+1} = \beta K^m$, with $0 < \beta < 1$ is a reduction factor. Numerical experiments show that this approach gives good results. We note that $W(\tau)$ is an arbitrary, time-varying, positive-definite symmetric $m \times m$ matrix of weighting functions chosen to improve convergence of the steepest-descent method. L, β, $W(\tau)$ and K, must be chosen by the user of the algorithm.

2.1.2 The direct second variation method

The direct method based on the second variation, also known as second order gradient, is an iterative method used for computing a m-vector of control variables $u(t), t_0 \le t \le t_f$, that minimizes a scalar performance index in an optimization problem (Bullock and Franklin, 1967; Longmuir and Bohn, 1969; Bryson and Ho, 1975; Imae, 1998). The direct second variation method is developed for a Bolza problem of optimal control with constrained final state and fixed terminal times. The generalized Riccati transformation (Longmuir and Bohn, 1969) is applied in solving the linear two-point boundary value problem associated to the accessory minimization problem obtained from the second variation of the augmented performance index of the original optimization problem. In this way, the algorithm here presented is different from the one described by Bullock and Franklin (1967), which is developed for a Mayer problem and involves a different set of transformation matrices for solving the linear two-point boundary value problem associated to the accessory minimization problem. The algorithm is also different from the one proposed by Bryson and Ho (1975), since it does not include the stopping condition for time-free problems which are treated by a transformation approach as mentioned before.

Consider the system of differential equations:

$$\frac{dx_i}{dt} = f_i(x,u), \quad i = 1,...,n, \tag{5}$$

where x is a n-vector of state variables, u is a m-vector of control variables, $f(.): R^n \times R^m \to R^n$, $f_i(.)$ e $\partial f_i / \partial x_j$, i and j =1,...,n, are defined and continuous on $R^n \times R^m$. It is assumed that there exist no constraints on the state or control variables. The problem consists in determining the control $u^*(t)$, that transfers the system (5) from the initial conditions:

$$x(t_0) = x_0, \tag{6}$$

to the final conditions at t_f:

$$\psi(x(t_f)) = 0, \tag{7}$$

and minimizes the performance index:

$$J[u] = g(x(t_f)) + \int_{t_0}^{t_f} F(x,u)\, dt, \tag{8}$$

with $g: R^n \to R$ and $\partial g / \partial x_i$, i =1,...,n, continuous on R^n. $F(.)$ and $\partial F / \partial x_i$, i =1,...,n, are also defined and continuous on $R^n \times R^m$, and $\psi: R^n \to R^q$, $q < n$, $\psi_i(.)$ and $\partial \psi_i / \partial x_j$, i = 1,...,q, and j = 1,...,n, are defined and continuous on R^n. Furthermore, it is assumed that the matrix $[\partial \psi / \partial x]$ has maximum rank.

The direct second variation method is an extension of the steepest-descent method presented in the previous section and is also based on the classic Calculus of Variations. The main difference is the inclusion of the second-order terms in the expansion of the augmented performance index about a nominal solution. The step by step computing procedure to be used in the direct second variation method is summarized as follows:

1. Integrate the state equations (5) from the initial point x_0 at t_0 to an unspecified final point $x(t_f)$ at t_f, with the starting nominal control $u^0(t), t_0 \le t \le t_f$.
2. Choose a starting Lagrange multiplier μ^0;
3. Integrate the adjoint equations $\frac{d\lambda}{dt} = -H_x^T$ from t_f to t_0, with the boundary conditions $\lambda(t_f) = -(g_x + \mu^T \psi_x)^T$. H is the Hamiltonian function defined as $H(x,\lambda,u) = -F(x,u) + \lambda^T f(x,u)$;
4. Compute and store the partial derivatives of the Hamiltonian function H - $H_u, H_{uu}, H_{xu}, H_{x\lambda}, H_{xx}$ and $H_{\lambda u}$;

5. Integrate the system of differential equations $-\dot{R} = RA + A^T R + RBR - C$, $-\dot{L} = (A^T + RB)L$, $-\dot{Q} = L^T BL$, $-\dot{s} = (A^T + RB)s + RD - E$, $-\dot{r} = L^T(D + Bs)$, backward from t_f to t_0, with the boundary conditions $R(t_f) = -\Phi_{xx}$, $L(t_f) = -\psi_x^T$, $Q(t_f) = 0$, $s(t_f) = 0$, $r(t_f) = -k\,\psi$, where A, B, C, D and E are defined as $A = H_{\lambda x} - H_{\lambda u} H_{uu}^{-1} H_{ux}$, $B = -H_{\lambda u} H_{uu}^{-1} H_{u\lambda}$, $C = H_{xu} H_{uu}^{-1} H_{ux} - H_{xx}$, $D = -H_{\lambda u} H_{uu}^{-1} H_u$, $E = H_{xu} H_{uu}^{-1} H_u$;
6. Compute $\delta\mu = -Q(t_0)^{-1}\left[L^T(t_0)\delta x(t_0) + r(t_0)\right]$;
7. Integrate the linear perturbation equation $\delta\dot{x} = (A + BR)\delta x + BL\delta\mu + Bs + D$;
8. Compute $\delta\lambda(t) = R(t)\delta x(t) + L(t)\delta\mu + s(t)$;
9. Compute $\delta u^* = -H_{uu}^{-1}[H_u^{\;T} + H_{ux}\delta x + H_{u\lambda}\delta\lambda]$.
10. Compute the new control $u^1(t) = u^0(t) + \delta u(t), t_0 \le t \le t_f$, and the Lagrange multiplier $\mu^1 = \mu^0 + \delta\mu$;
11. Test the convergence. Repeat the process until it converges.

It should be noted that the algorithm described above diverges if the Legendre condition $H_{uu} < 0$, computed for the nominal solution over the whole time interval $t_0 \le t \le t_f$, is not satisfied. However, by adding a term

$$\frac{1}{2}\|\delta u\|^2 = \frac{1}{2}\int_{t_0}^{t_f} \delta u^T W_2 \delta u dt,$$

which further constraints the control effort, the resulting functional $\Delta\bar{J}$ can be made satisfy the Legendre condition if the $m \times m$ matrix W_2 is chosen large enough. Therefore, H_{uu} must be replaced by $H_{uu} + W_2$ in the algorithm described above (Bullock and Franklin, 1967).

2.2 The indirect second variation method

The indirect method based on the second variation is an iterative method used for solving the two-point boundary value problem resulting from the necessary conditions, which characterize the optimal control and the corresponding optimal trajectory in an optimization problem (Breakwell et al, 1963). Consider the Bolza problem with constrained final state and fixed terminal described by Eqns. (5) – (8). From the application of the Pontryagin Maximum Principle (Pontryagin et al, 1962), the following two-point boundary values problem is obtained

$$\frac{dx}{dt} = H_\lambda^T, \quad \frac{d\lambda}{dt} = -H_x^T, \quad H_u = 0, \quad x(t_0) = x_0, \quad \lambda(t_f) = -\left(\phi_x^T + \mu\psi_x\right), \quad \psi(x(t_f)) = 0,$$

where H is the Hamiltonian function, $H(x,\lambda,u) = -F(x,u) + \lambda^T f(x,u)$. The indirect second variation method consists in determining iteratively the unknown Lagrange multipliers $\lambda(t_0)$ and μ_f. The step-by-step computing procedure to be used in the indirect second variation method is summarized as follows:

1. Choose starting Lagrange multipliers $\lambda(t_0)$ e μ.
2. The control $u = u(x,\lambda)$ is obtained from equation $H_u = 0$;
3. Integrate state and adjoint equations $\frac{dx}{dt} = H_\lambda^T$, $\frac{d\lambda}{dt} = -H_x^T$ from t_0 to t_f to obtain $x(t_f)$ and $\lambda(t_f)$;
4. The values of matrix *A, B, C* are obtained from Eqns. (75);
5. Integrate the system of differential equations $-\dot{R} = RA + A^T R + RBR - C$, $-\dot{L} = (A^T + RB)L$, $-\dot{Q} = L^T BL$, with the boundary conditions $R(t_f) = -\Phi_{xx}$, $L(t_f) = -\psi_x^T$, $Q(t_f) = 0$, along with the state and adjoint equations, backward from t_f a t_0, where $A = H_{\lambda x} - H_{\lambda u} H_{uu}^{-1} H_{ux}$, $B = -H_{\lambda u} H_{uu}^{-1} H_{u\lambda}$, $C = H_{xu} H_{uu}^{-1} H_{ux} - H_{xx}$;
6. Compute $\delta\mu = -Q(t_0)^{-1}\left[k\psi(x(t_f))\right]$, $0 < k < 1$ indicates the correction is partial;
7. Compute $\delta\lambda(t_0) = L(t_0)\delta\mu$;
8. Compute the new value of the initial adjoint variable $\lambda^{i+1} = \lambda^i + \delta\lambda$;
9. Compute the new value of the Lagrange multiplier $\mu^{i+1} = \mu^i + \delta\mu$;
10. Test the convergence. Repeat the process until it converges;

As it has been done in the second order gradient method, if the Legendre condition $H_{uu} < 0$, computed for the nominal solution over the whole time interval $t_0 \le t \le t_f$ is not satisfied, we take $H_{uu} + W_2$.

3. Computation of Optimal Low Thrust Trajectories

In the two-dimension optimization problem, the state equations are given by

$$\frac{du}{dt} = \frac{v^2}{r} - \frac{\mu}{r^2} + R \qquad \frac{dv}{dt} = -\frac{uv}{r} + S \qquad \frac{dr}{dt} = u \qquad \frac{dJ}{dt} = \frac{1}{2}\left(R^2 + S^2\right),$$

where μ is the gravitational parameter, R and S are the radial and the circumferential components of the thrust acceleration vector, respectively. The optimization problem is stated as: it is proposed to transfer a space vehicle M from the initial conditions at t_0, $u(0) = 0$, $v(0) = 1$, $r(0) = 1$, $J(0) = 0$, to the final state at the prescribed final time at t_f $u(t_f) = 0$, $v(t_f) = \sqrt{\mu / r_f}$, $r(t_f) = r_f$, such that J_f is a minimum.

The combined algorithm and the indirect second variation method are applied to solve the transfer problem defined above for several ratios $\rho = r_f / r_0$, $\rho =$ 0,727; 0,8; 0,9; 0,95; 0,975; 1,025; 1,05; 1,1; 1,2; 1,5236, and non-dimension time duration of 2, 3, 4, 5. The Earth-Mars transfer corresponds to $\rho = 1{,}5236$ and Earth-Venus to $\rho = 0{,}727$. The terminal constraints are obtained with an error less than 5.0 × 10^{-6}, that means $\left\|\Psi\left(x(t_f)\right)\right\| \le 5{,}0 \times 10^{-6}$ and the performance index are calculated with an error of $e = \left|J^{n+1} - J^n\right| < 5{,}0x10^{-10}$, where n is the number of iterates.

The values of the consumption variable J computed through the algorithms are plotted in the Fig. 1 as function of the radius of the final orbital for various transfer durations, showing the good agreement between the methods. Note that the linear theory provides a good approximation for the solution of the low-thrust limited power transfer between close circular coplanar orbits in a Newtonian central gravity field. Figure 1 also show that the fuel consumption can be greatly reduced if the duration of the transfer is increased. The fuel consumption for transfers with duration $t_f - t_0 = 2$ is approximately ten times the fuel consumption for a transfer with duration $t_f - t_0 = 4$.

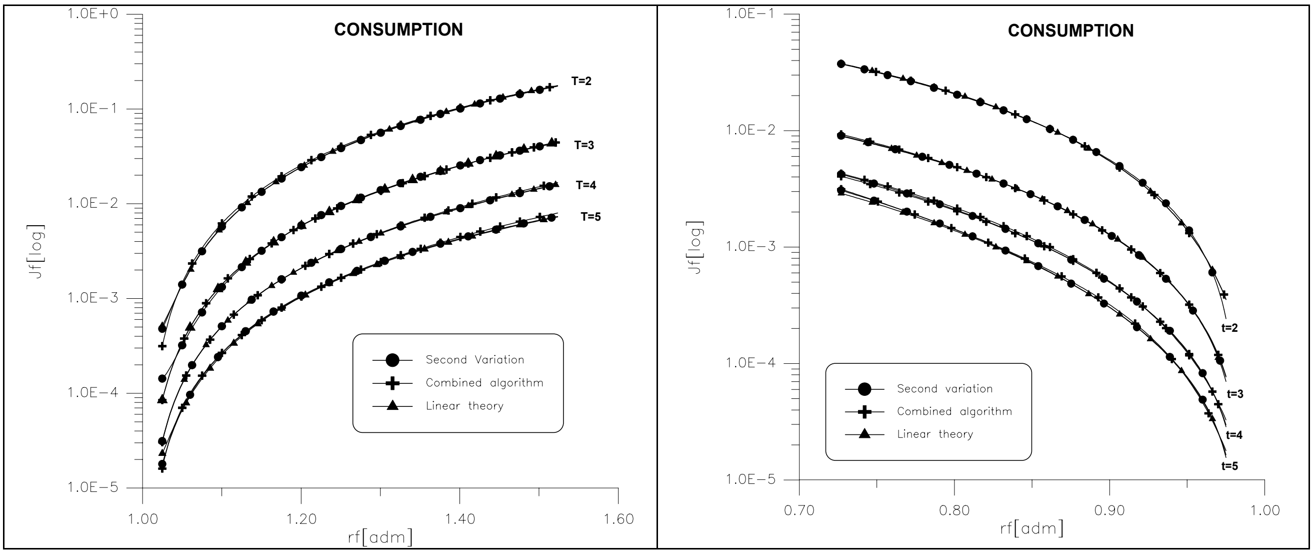

Figure 1 – Comparison of consumption variable J

Figures 2 to 9 present the locus related to the control variables. The magnitude of the optimal thrust acceleration decreases as the duration of the transfer acceleration increases. It can be also noted that when the difference between the initial and final orbits radius increases the acceleration values diminish. The results obtained from the proposed methods are in good agreement with the ones provided by a linear theory.

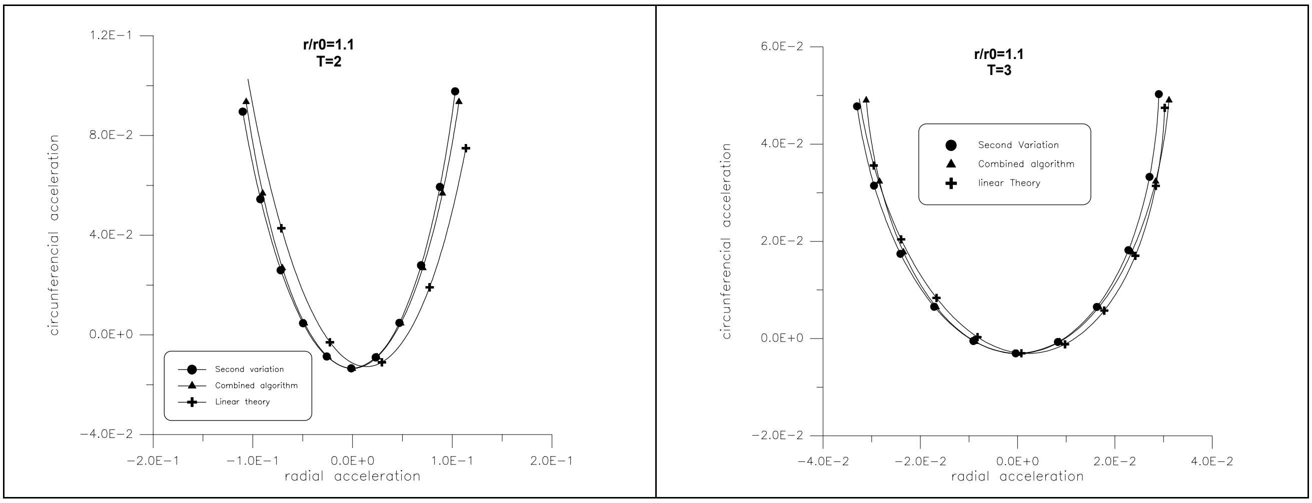

Figure 2 – Acceleration for $\rho = 1{,}1,\ t_f = 2$ e 3.

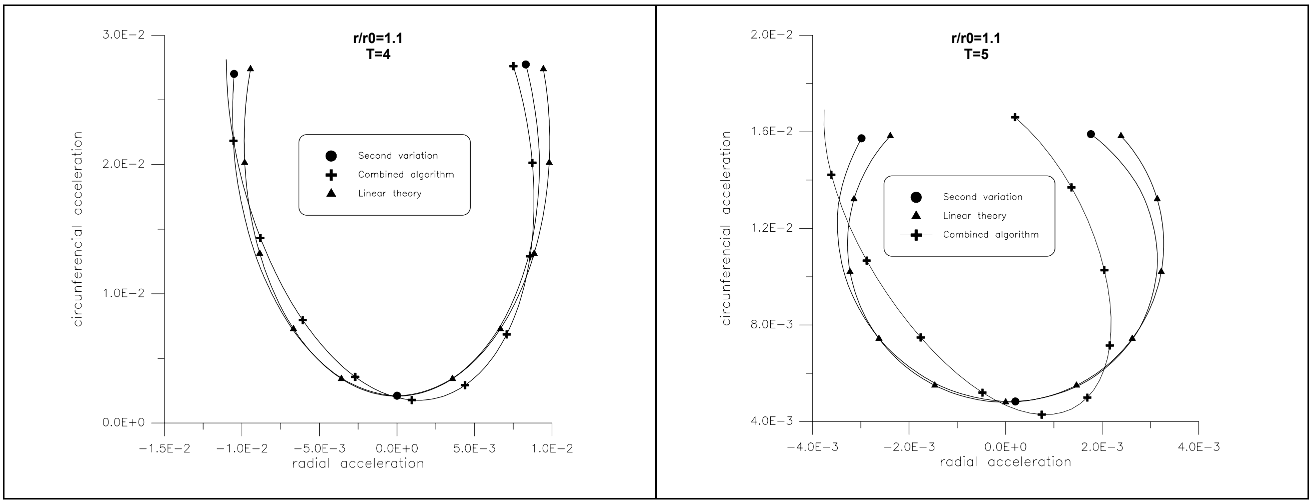

Figure 3 – Acceleration for $\rho = 1{,}1,\ t_f = 4$ e 5.

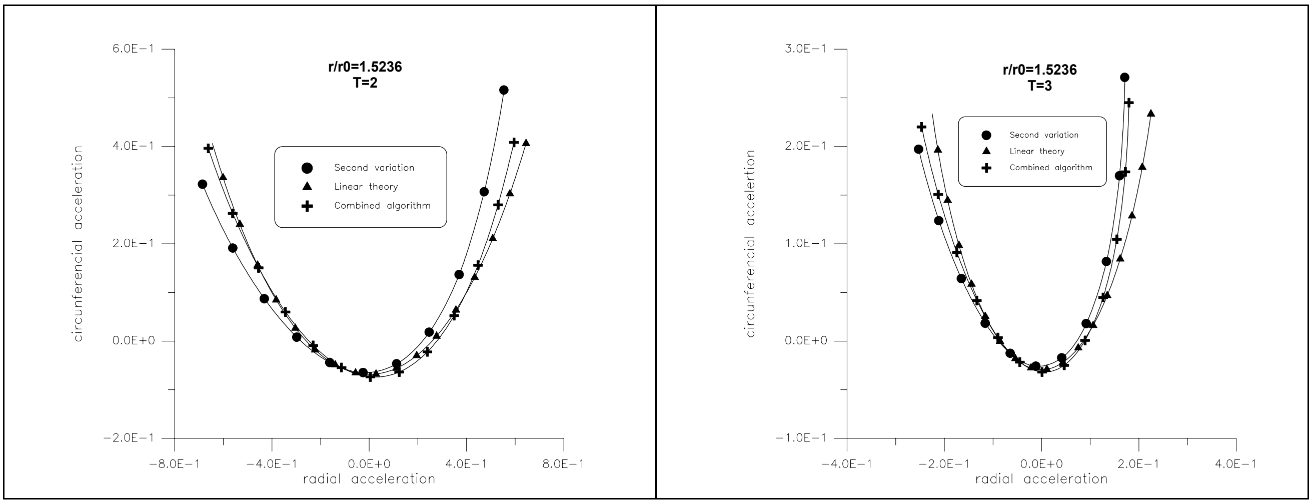

Figure 4 – Acceleration for $\rho = 1{,}5236,\ t_f = 2$ e 3.

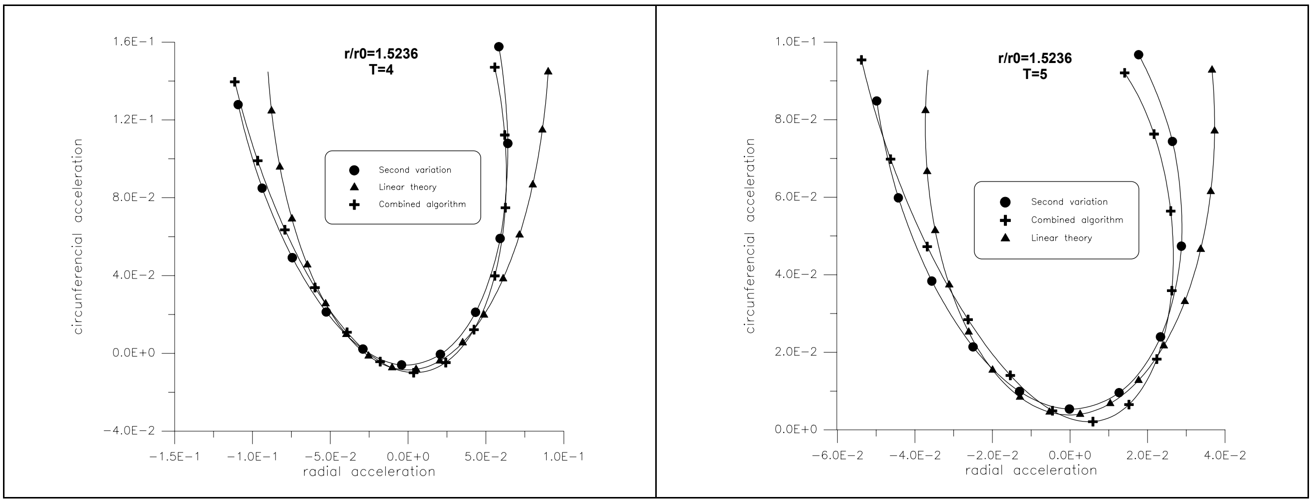

Figure 5 – Acceleration for $\rho = 1{,}5236, t_f = 4$ e 5 .

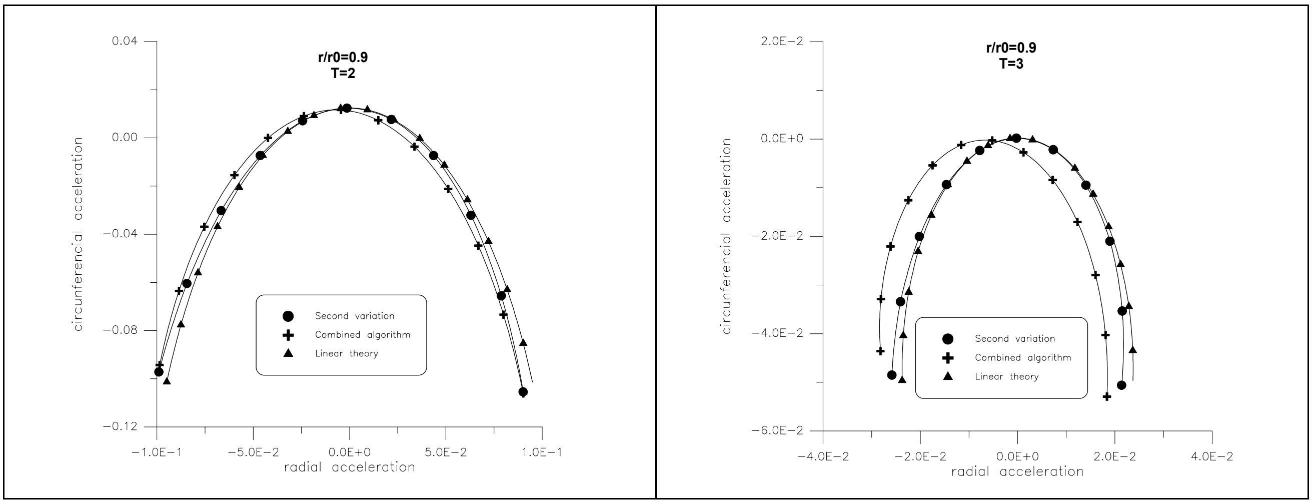

Figure 6 – Acceleration for $\rho = 0.9,\ t_f = 2$ e $t_f = 3$.

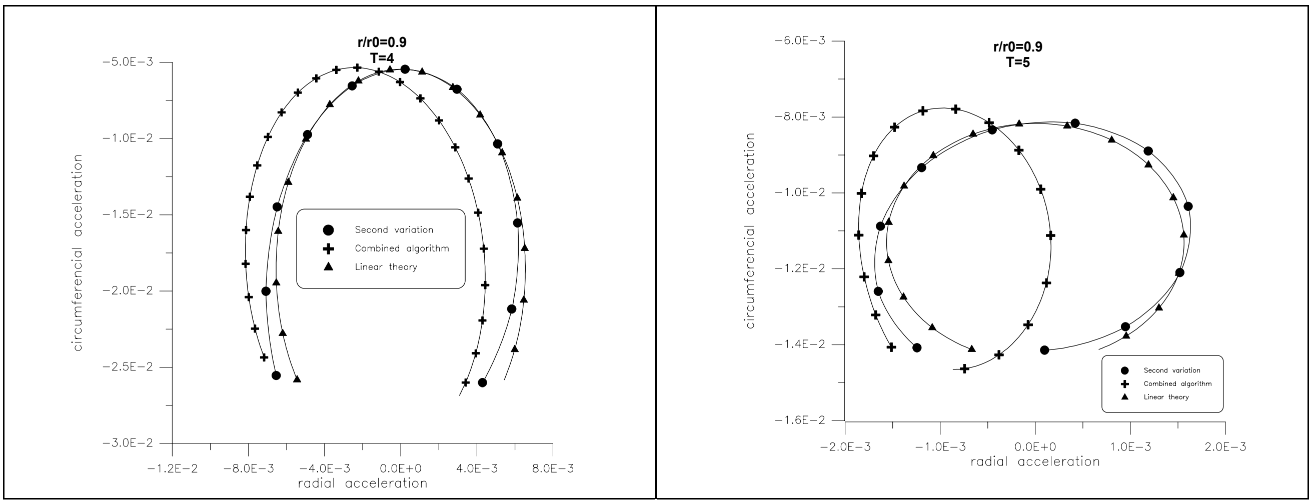

Figure 7 – Acceleration for $\rho = 0.9,\ t_f = 4$ e $t_f = 5$.

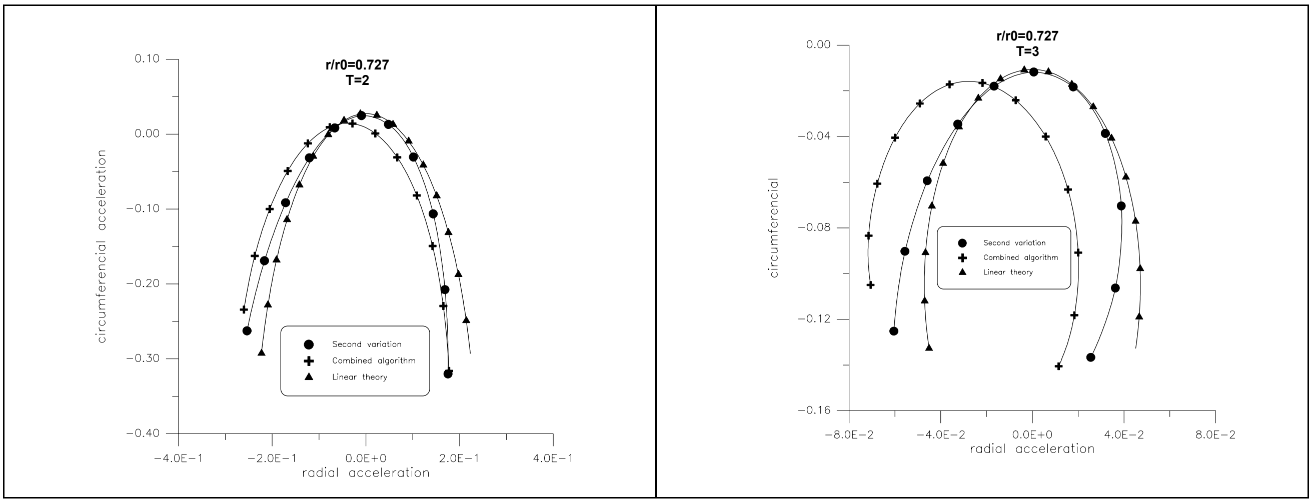

Figure 8 – Acceleration for $\rho = 0.727,\ t_f = 2$ e $t_f = 3$.

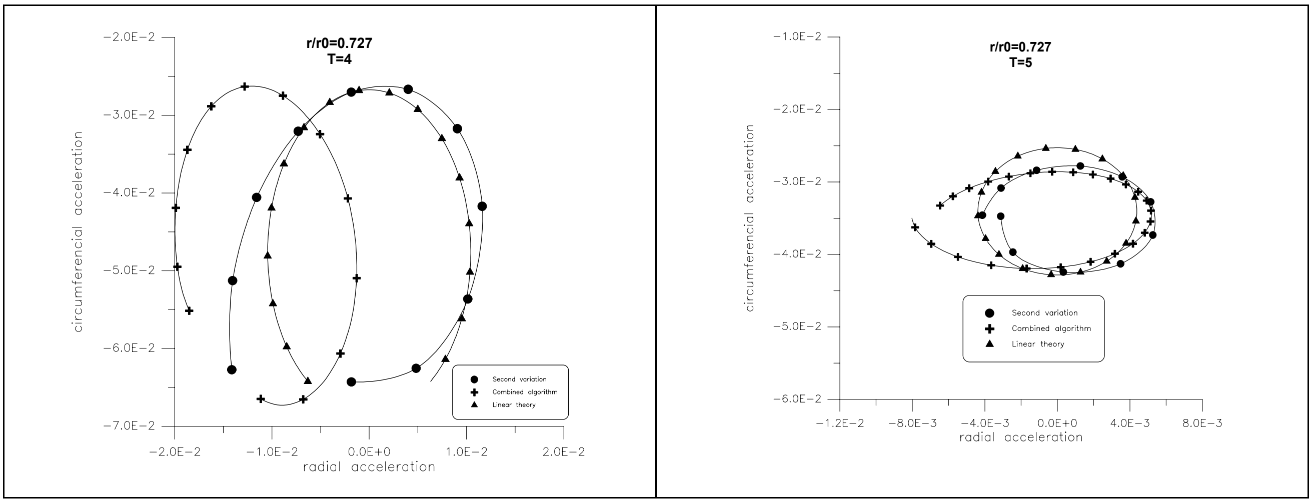

Figure 9 – Acceleration for $\rho = 0.727,\ t_f = 4$ e $t_f = 5$.

5. Conclusions

In this paper two different algorithms based on second variation theory are applied in the analysis of optimal low-thrust limited-power transfer between close circular coplanar orbits in a Newtonian central gravity field. The numerical results provided by the two algorithms have been compared to the analytical ones provided by a linear theory. The great agreement between these results shows that the linear theory provides a good approximation for the solution of the transfer problem and it can be used in mission analysis. On the other hand, the good performance (accuracy of the terminal constraints, number of iterations, …, etc) of the proposed algorithms shows that they are good tools in determining optimal low-thrust limited-power trajectories.

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