GRAVITATIONAL CAPTURE OF A SATELLITE BY THE MAJOR PRIMARY

Antonio Fernando Bertachini de Almeida Prado
Instituto Nacional de Pesquisas Espaciais – INPE
Av. dos Astronautas 1758 – São José dos Campos – SP – Brazil – CEP 12227-010
prado@dem.inpe.br

Abstract. *The objective of the present paper is to study in some detail the ballistic gravitational capture performed by the first primary in a three body system. Analytical equations for the forces involved in this maneuver are derived to estimate their magnitude and to show the best directions of approach for the maneuver.*

Keywords*: Gravitational capture, astrodynamics, restricted three-body problem*

1. Introduction

The ballistic gravitational capture is a characteristic of some dynamical systems in celestial mechanics, as in the restricted three-body problem that is considered in this paper. The basic idea is that a spacecraft (or any particle with negligible mass) can change from a hyperbolic orbit with a small positive energy around a celestial body into an elliptic orbit with a small negative energy without the use of any propulsive system. The force responsible for this modification in the orbit of the spacecraft is the gravitational force of another body involved in the dynamics. In this way, this force is used as a zero cost control, equivalent to a continuous thrust applied in the spacecraft.

The application of this phenomenon in spacecraft trajectories is recent in the literature. The first demonstration of this was in Belbruno, 1987. Further studies include Belbruno (1990); Krish (1991); Miller and Belbruno (1991); Belbruno and Miller (1993). They all studied missions in the Earth-Moon system that use this technique to save fuel during the insertion of the spacecraft in its final orbit around the Moon. Another set of papers that made fundamental contributions in this field, also with the main objective of constructing real trajectories in the Earth-Moon system, are those of Yamakawa, Kawaguchi, Ishii and Matsuo (1992 and 1993) and Yamakawa (1992). The first real application of a ballistic capture transfer was made during an emergency in a Japanese spacecraft, in Belbruno and Miller (1990). After that, some studies that consider the time required for this transfer appeared in the literature. Examples of this approach can be found in the papers by Vieira-Neto and Prado (1995 and 1998). An extension of the dynamical model to consider the effects of the eccentricity of the primaries is also available in the literature (Vieira Neto, 1999; Vieira-Neto and Prado, 1996). A study of this problem, from the perspective of invariant manifolds, was developed by Belbruno (1994).

Examining the literature related to the weak stability boundaries, one finds several definitions of ballistic gravitational capture, depending on the dynamical system considered. Those differences exist to account for the different behavior of the systems. In the restricted three-body problem, the system considered in the present paper, ballistic gravitational capture is assumed to occur when the massless particle stays close to one of the two primaries of the system for some time. A permanent capture is not required, because in this model it does not exist and the phenomenon is always temporary, which means that after some time of the approximation the massless particle escapes from the neighborhood of the primary.

For the practical purposes of studying spacecraft trajectories, the majority of the papers available in the literature study this problem looking at the behavior of the two-body energy of the spacecraft with respect to the primary that performs the gravitational capture. A quantity called C_3 (that is twice the total energy of a two-body system) is defined, with respect to the closer primary, by:

$$C_3 = V^2 - 2(1-\mu)/r \tag{1}$$

where V is the velocity of the spacecraft relative to the primary considered, r is the distance of the spacecraft from this primary and μ is the dimensionless gravitational parameter of that primary. From the value of C_3 it is possible to know if the orbit is elliptical ($C_3 < 0$), parabolic ($C_3 = 0$) or hyperbolic ($C_3 > 0$) with respect to the primary considered. Based upon this definition, it is possible to see that the value of C_3 is related to the velocity variation (ΔV) needed to insert the spacecraft in its final orbit around the primary. In the case of a spacecraft approaching the Earth, it is possible to use the gravitational force of the Moon to lower the value of C_3 with respect to the Earth, so the fuel consumption required to complete this maneuver is reduced.

The present paper has the main goal of studying analytically the potential savings for a maneuver that capture a spacecraft around the first primary. The Earth-Moon system is used for the numerical evaluation.

2. Mathematical model

The model used to study this problem is the planar restricted three-body problem. The system considered for all the simulations shown in this paper is the Earth-Moon system, because this is the system with more likely applications of the ballistic gravitational capture technique. The standard canonical system of units is used, in which the unit of distance is the distance between M_1 (the Earth) and M_2 (Moon); the angular velocity (ω) of the motion of M_1 and M_2 is set to unity; the mass of the smaller primary (M_2) is given by $\mu = m_2/(m_1 + m_2)$ (where m_1 and m_2 are the real masses of M_1 and M_2, respectively) and the mass of M_2 is (1-μ); the unit of time is defined such that the period of the motion of the two primaries is 2π and the gravitational constant is unity.

There are several customary systems of reference for studying this problem (Szebehely, 1967). In this paper the rotating system is used because it is the one that has well-known equations of motion and a simple expression for the Jacobian constant. This system has the following characteristics: origin at the center of mass of the two primaries; horizontal axis lying in the line connecting the two primaries, pointing to M_2; vertical axis perpendicular to the plane of motion of the two primaries. Based upon those conventions, the equations of motion for the spacecraft are (Szebehely, 1967):

$$\ddot{x} - 2\dot{y} = \frac{\partial \Omega}{\partial x} \tag{2}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial \Omega}{\partial y} \tag{3}$$

where Ω is the pseudo-potential given by:

$$\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{(1-\mu)}{r_1} + \frac{\mu}{r_2} \tag{4}$$

The symbols r_1 and r_2 are the distances between the spacecraft and the Earth and the Moon, respectively.

3. Forces involved in the dynamics

Figure 1 shows the gravitational force $\vec{F}_m$ of the Moon acting in a spacecraft M_3 that is approaching the Earth and Fig. 2 shows the centrifugal force acting in the same situation. There is also the Coriolis force, given by $-2\vec{\omega} \times \vec{V}$, where $\vec{\omega}$ is the angular velocity of the reference system and $\vec{V}$ is the velocity of the spacecraft. This force is not analyzed because the main idea of this paper is to explain the ballistic gravitational capture as a result of perturbative forces acting in the direction of motion of the spacecraft and the Coriolis force acts perpendicular to the direction of motion of the spacecraft all the time. In this way, it does not contribute to the phenomenon studied here. The direction $\vec{r}$ points directly to the center of the Earth and the direction $\vec{p}$ is perpendicular to $\vec{r}$, pointing in the counter-clockwise direction. The distance between the spacecraft and the Moon is d, the angle formed by the line connecting the Moon to the spacecraft and the direction $\vec{r}$ is γ. The angle ϕ is used to define instantaneously the direction $\vec{r}$. From geometrical considerations in Figs. 2 and 3, it is possible to write:

$$\left|\vec{F}_m\right| = \frac{\mu}{d^2} \quad \Rightarrow \quad \vec{F}_g = \frac{\mu}{d^2} \cos\gamma \, \vec{r} - \frac{\mu}{d^2} \sin\gamma \, \vec{p} \tag{5}$$

Applying the law of cosines:

$$1 = d^2 + r^2 - 2dr\cos\gamma \qquad \Rightarrow \qquad \cos\gamma = \frac{1 - d^2 - r^2}{-2rd} \tag{6}$$

but

$$d^2 = 1 + r^2 - 2r\cos\phi \quad (7)$$

From Eqs. (6) and (7):

$$\cos\gamma = \frac{1 - 1 - r^2 + 2r\cos\phi - r^2}{-2rd} = \frac{2r^2 - 2r\cos\phi}{+2rd} = \frac{r - \cos\phi}{d} \quad (8)$$

From the law of sines:

$$\frac{d}{\sin\phi} = \frac{1}{\sin\gamma} \Rightarrow \sin\gamma = \frac{\sin\phi}{d} \quad (9)$$

Then, using Eqs. (8) and (9):

$$\vec{F}_g = \frac{\mu(r - \cos\phi)}{d^3}\vec{r} - \frac{\mu\sin\phi}{d^3}\vec{p} = \frac{\mu(r - \cos\phi)}{(1 + r^2 - 2r\cos\phi)^{3/2}}\vec{r} - \frac{\mu\sin\phi}{(1 + r^2 - 2r\cos\phi)^{1/2}}\vec{p} \quad (10)$$

For the centrifugal force the expression is:

$\vec{F}_{ce} = -F\cos\sigma\,\vec{r} + (F\sin\sigma)\,\vec{p}$, where $F = \omega^2 L$ = L (since ω = 1). (11)

By analogy with the gravitational force:

$$\cos\sigma = \frac{\mu^2 - L^2 - r^2}{-2rL} \quad (12)$$

But, it is also known that $L^2 = \mu^2 + r^2 - 2r\mu\cos\phi$, therefore

$$\cos\sigma = \frac{\mu^2 - \mu^2 - r^2 + 2r\mu\cos\phi - r^2}{-2rL} = \frac{r - \mu\cos\phi}{L} \quad (13)$$

From the law of sines: $\frac{L}{\sin\phi} = \frac{\mu}{\sin\sigma} \Rightarrow \sin\sigma = \frac{\mu\sin\phi}{L}$ (14)

Combining all the results together:

$$\vec{F}_{ce} = (\mu \cos\phi - r)\vec{r} + \mu \sin\phi \; \vec{p} \quad (15)$$

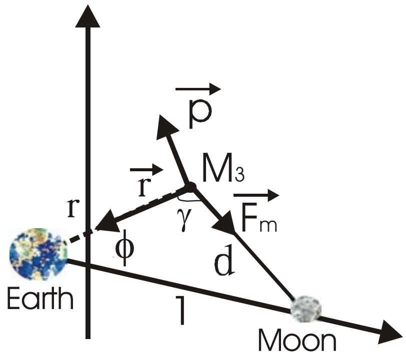

Figure 1. Gravitational force of the Moon.

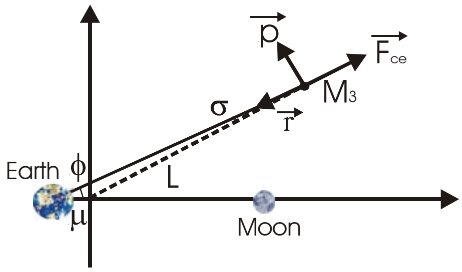

Figure 2. The centrifugal force.

4. Physical explanation of the ballistic gravitational capture

During the approach phase, when the spacecraft is close to the Earth, the force that dominates the dynamics is due to the central body (the Earth). All others forces are perturbations on the motion of the massless particle. In the model considered here, the perturbations are due to the gravitational force of the Moon and the centrifugal force due to the rotation of the system. The Coriolis force also acts in the massless particle, but it does not have any component in the direction of motion, as explained before. In that way, a method to understand the behavior of the perturbing forces is to study the components of each force during the approach phase. The forces involved are divided in the radial and transverse components. The conventions used in this paper imply that in the radial direction the positive sign means that the force is acting in the direction of the Earth and that in the transverse direction the positive sign indicates the force is acting in the counter-clockwise direction.

The relation between the forces and the variation of C_3 can be explained in terms of fundamental physical laws. Suppose that the value of C_3 at the periapsis is called C_{3p} and its value at the crossing point with the sphere of capture of the Earth is called C_{3sc}. From the definition of C_3 (Eq. (1)), the results are:

$$C_{3p} = V_p^2 - \frac{2(1-\mu)}{r_p}, \quad (16)$$

$$C_{3sc} = V_{sc}^2 - \frac{2(1-\mu)}{r_{sc}} \quad (17)$$

where the subscript “sc” stands for values at the sphere of capture of the Earth.

The effects of the three forces studied in the system (gravitational - Earth and Moon, and centrifugal) is to change the velocity of the spacecraft according to the physical law:

$$\int_{t_0}^{t_f} F \, dt = (V_f - V_0), \quad (18)$$

where F is the force per unit mass of the spacecraft, V_0 is the velocity at t_0 and V_f is the velocity at t_f. Then, defining the variation of C_3 (ΔC_3) between the periapsis and the sphere of capture of the Moon as C_{3p} - C_{3sc}, and applying Eq. (18) between the same instants to write V_{sc} in terms of V_p, we have:

$$\Delta C_3 = C_{3p} - C_{3sc} = V_p^2 - \frac{2(1-\mu)}{r_p} - \left(V_p - I_{tot}\right)^2 + \frac{2(1-\mu)}{r_{sc}} = 2(1-\mu)\left(\frac{1}{r_{sc}} - \frac{1}{r_p}\right) + 2V_p I_{tot} - I_{tot}^2 \tag{19}$$

where I_{tot} represents the time integral of the resultant effects of the three forces studied in this system in the direction of the motion of the spacecraft. Eq. (19) gives the variation of C_3 in the rotating frame, because I_{tot} is evaluated in this system.

5. Analytical analyses of the forces

The next step of this research is to develop analytical expressions for the components of each force, in order to obtain an estimate of their effects. The main idea is to estimate the potential of the field around the Earth to reduce the value of C_3 and not to make predictions for a single trajectory. The analytical equations to measure the effects of this perturbation are derived under the assumption that the trajectory followed by the spacecraft is an idealized trajectory that does not deviate from the radial direction. The real trajectories are not radial, but the equations derived under this assumption can be used to: i) estimate the values of the possible reductions in the value of C_3; ii) show the existence of directions of motion that results in larger reductions of C_3, thereby mapping analytically the decelerating field that exists in the neighborhood of the Earth, and; iii) estimate the effects of the periapsis distance and the size of the sphere of capture, since the equations derived are explicitly functions of those parameters. Another justification for the radial trajectories used to derive the equations is that the reduction of C_3 is a result of the effects of the forces in time during the whole trajectory and, even for trajectories that show several loops before arriving at the periapsis during most of the time, the trajectory can be seen as composed of a set of trajectories close to radial.

For the derivation performed here, only the radial component (the direction of motion under the assumption used here) is derived. Then, assuming that the spacecraft is in free-fall (subject only to the gravitational and centrifugal forces) traveling with zero energy (parabolic trajectory) and that the trajectories do not deviate from a straight line, the result is:

$$\text{Total energy} = E = 0 = \frac{1}{2}V^2 - \frac{(1-\mu)}{r} \Rightarrow V = \sqrt{\frac{2(1-\mu)}{r}} = \frac{ds}{dt} \tag{20}$$

Here ds is the distance traveled by the particle during the time dt. To obtain the integral of the effect of the perturbing forces with respect to time, it is possible to perform the calculations in terms of the radial distance by making the substitution:

$$\int_{t_0}^{t_f} F\,dt = \int_{S_0}^{S_f} (F/V)ds = \int_{r_{min}}^{r_{max}} (F/V)dr \tag{21}$$

The extreme points of the integration change position (S_0 becomes $r_{min} = r_p$ = periapsis distance and S_f becomes $r_{max} = r_{sc}$ = distance for the sphere of capture) here and in all the following integrations to take into account that the positive sense of the radial direction points towards the Earth. Then, for the radial component of the Earth's gravity ,

$$F_g = \frac{\mu(r - \cos\phi)}{(1 + r^2 - 2r\cos\phi)^{3/2}}$$ the integral is:

$$F_1(\alpha) = \int_{r_{min}}^{r_{max}} \frac{\mu(r - \cos\phi)}{(1 + r^2 - 2r\cos\phi)^{3/2}\left(2(1-\mu)/r\right)^{1/2}}\,dr \tag{22}$$

The calculations can be continued now by expanding the equation inside the integral in a power series of r. In this research the expansion was performed up to the first order around a point q, the middle point of the trajectory. The result, after integrating in r, is shown below in the complete form (functions of r_{min}, r_{max}, q, μ and ϕ) because it can be used to compute values for any desirable values of those variables. Since the goal is to obtain only an estimate of the results and because the maximum difference between the first and second-order expansion is about 7%, only the first-order complete equation are show.

$$F_1(\phi) = \left[\frac{\mu(q-\cos\phi)}{(2(1-\mu)/q)^{1/2}(1+q^2-2q\cos\phi)^{3/2}} r + \mu\left(-\frac{3\,(q-\cos\phi)^2}{(2(1-\mu)/q)^{1/2}(1+q^2-2q\cos\phi)^{5/2}} + \right.\right.$$

$$\left.\left. + \frac{(1-\mu)(3q-\cos\phi)}{2q^2(2(1-\mu)/q)^{3/2}(1+q^2-2q\cos\phi)^{3/2}} \right)\left(\frac{r^2}{2} - qr\right) \right]_{r_{min}}^{r_{max}} \qquad (23)$$

Using the values r_{min} = 6478/384400 (100 km above the surface of the Earth), r_{max} = 100000/384400 (100000 km above the Earth´s surface), μ = 0.0121 (Earth-Moon system) and $q = (r_{min} + r_{max})/2$ (the medium point of the trajectory) the first-order equation obtained is:

$$F_1(\phi) = (0.000108 + 0.000779\cos(\phi))(1.01918 - 0.2770\cos(\phi))^{-1.5} \qquad (24)$$

This equation is plotted as a function of ϕ in Fig. 3. The curve shows a sinusoidal variation of the integral, with the most favorable angle for the ballistic gravitational capture close to 0° (or the equivalente 360°), where the force has the most negative value. It means that the component of this force applied opposite to the motion of the spacecraft has its maximum effect in reducing the final velocity of the spacecraft, then obtaining a capture with the most negative value for the energy.

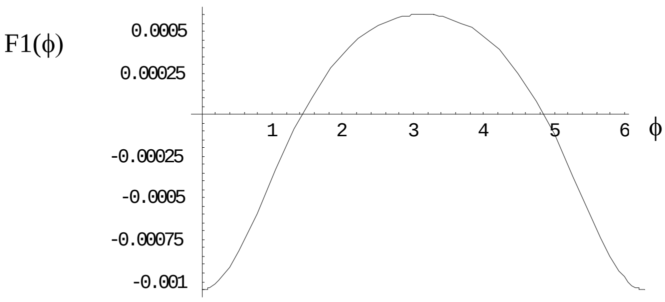

Figure 3. Effects of the gravitational force of the Moon.

For the radial component of the centrifugal force, $[\mu\cos\phi - r]$, the integral is:

$$\int_{r_{min}}^{r_{max}} (F_{ce}/V)ds = \int_{r_{min}}^{r_{max}} (\mu\cos\phi - r)(2(\mu-1)/r)^{-1/2} dr = \left[\left(-0.4\,r^2 + \frac{2}{3} r\mu\cos\phi \right)(2(\mu-1)/r)^{-1/2} \right]_{r_{min}}^{r_{max}} \qquad (25)$$

Using the same values used in the above situation for the variables, this last equation can be reduced to:

$$F_2(\alpha) = -0.009812 + 0.000749\cos\phi \qquad (26)$$

This equation is plotted as a function of ϕ in Fig. 4. It also shows a sinusoidal variation of the integral, with the most favorable angle for the ballistic gravitational capture at ϕ = 180° (the most negative value of the integral). It means that at this point the component of the centrifugal force acting opposite to the motion of the spacecraft has its maximum effect. The sign of the value at this point is opposite to the sign of the effect of the gravity of the Earth, which means that the effects are working against each other. Adding the radial effects of both forces the equation for the resultant force in the radial direction is obtained. This force will be called $F_3(\phi)$ and it is also plotted as a function of ϕ in Fig. 6. From those results, it is clear that the integral of the total effect is always negative, which means that the spacecraft always has its velocity reduced by the perturbation. It is never increased. In this figure it is also possible to obtain the best point to perform the ballistic gravitational capture. This point is at ϕ = 0° (or the equivalente 360°), which has the strongest accumulated effect for the resultant force. There is also a local minimum, at ϕ = 180°.

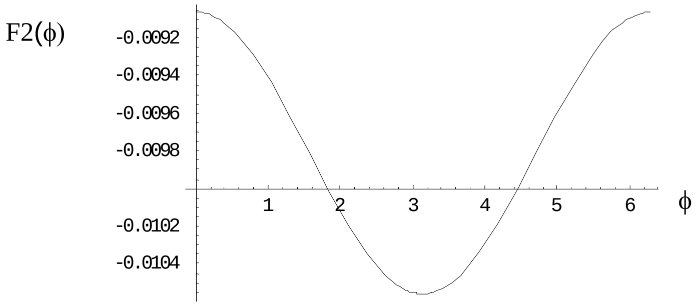

Figure 4. Effects of the centripetal force.

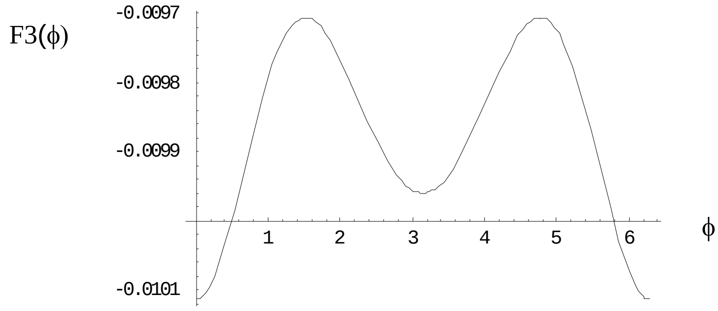

Figure 5. Effects of the resultant radial force.

The next step to be developed here is to obtain an analytical equation to predict the variation of C_3 as a function of the angle ϕ, using Eq. (19). To do that, it is necessary to obtain the value of the integral effect of the gravitational force of the Earth in the direction of motion of the spacecraft under the assumption of radial motion. The gravitational force of the Earth acts only in the radial direction with a magnitude given by $F_M = \frac{(1-\mu)}{r^2}$. So, its integral effect with respect to time is given by (using the same numerical value used before for μ, r_{max} and r_{min}):

$$\int_{t0}^{tf} F dt = \int_{r_{min}}^{r_{max}} (F/V) dr = \sqrt{\frac{(1-\mu)}{2}} \int_{r_{min}}^{r_{max}} r^{-3/2} dr = 8.07196 \quad (27)$$

Then, the total effect I_{tot} is given by $F_3(\phi)$ + 8.07196, where $F_3(\phi)$ is given by $F_1(\phi) + F_2(\phi)$ (Eqs. (25) and (27)). Equation (19) becomes (using the numerical values available): $2\mu\left(\frac{1}{r_{sc}} - \frac{1}{r_p}\right) + 2V_p(F_3(\phi) + 8.07196) - (F_3(\phi) + 8.07196)^2$, which completes the required derivation. Fig. 6 shows a plot with the numerical results.

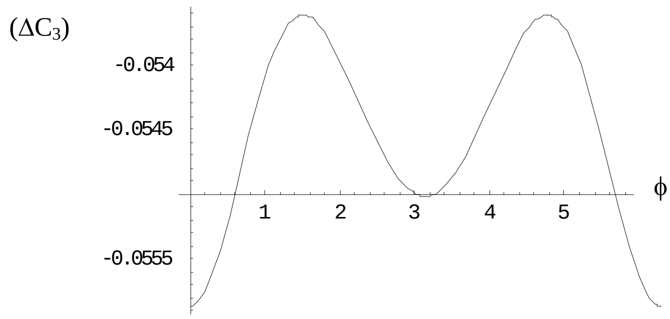

Figure 6. Variation of C_3 as a function of ϕ.

6. Conclusions

This paper had the main goal of studying the ballistic gravitational capture problem by the main primary of a three-body system. It showed an explanation of the phenomenon based in the calculation of the forces involved in the dynamics as a function of time and in their integration with respect to time. Analytical equations are derived to study this problem under the assumption of radial motion, which leads to the derivation of an equation that estimates the reduction of C_3. Then, the forces acting on the ballistic gravitational capture problem are obtained in closed forms. There are two forces that act as disturbing forces in the direction of motion: the gravitational force due to the Moon and the centrifugal force. These forces can decelerate the spacecraft, working opposite to its motion. This is equivalent to applying a continuous propulsion force against the motion of the spacecraft. In the radial direction the gravitational force due to the Moon and the centrifugal force work in opposite directions, but the resultant force always works against the motion of the spacecraft.

7. Acknowledgements

The author is grateful to CNPq (National Council for Scientific and Technological Development) - Brazil for the contract 300828/2003-9 and to FAPESP (Foundation to Support Research in São Paulo State) for the contract 03/03262-4.

4. References

Belbruno, E.A., "Examples of the Nonlinear Dynamics of Ballistic Capture and Escape in the Earth-Moon System", *AIAA-90-2896. In: AIAA Astrodynamics Conference*, Portland, Oregon, Aug. 1990.

Belbruno, E.A., "Lunar Capture Orbits, a Method of Constructing Earth Moon Trajectories and the Lunar Gas Mission", *AIAA-87-1054. In: 19th AIAA/DGLR/JSASS International Electric Propulsion Conference*, Colorado Springs, Colorado, May 1987.

Belbruno, E.A., Humble, R., Coil, J., Ballistic Capture Lunar Transfer Determination for the U.S. Air Force Academy Blue Moon Mission, Advances in Astronautical Science, Spaceflight Mechanics, Vol. 95, Paper AAS 97-171, pp. 869-880, 1997.

Belbruno, E.A., Miller, J.K., "A Ballistic Lunar Capture Trajectory for Japanese Spacecraft Hiten", *Jet Propulsion Lab., JPL IOM 312/90.4-1731, Internal Document*, Pasadena, CA, Jun. 1990.

Belbruno, E.A., Miller, J.K., "Sun-Perturbed Earth-to-Moon Transfers With Ballistic Capture, *Journal of Guidance, Control and Dynamics,* Vol. 16, nº 4, 1993, pp. 770-775.

Belbruno, E.A., The Dynamical Mechanism of Ballistic Lunar Capture Transfers in the Four-Body Problem from the Perspective of Invariant Manifolds and Hill´s Regions, Institut D´Estudis Catalans, CRM Research Report Nº 270, (November 1994).

Krish, V., "An Investigation Into Critical Aspects of a New Form of Low Energy Lunar Transfer, the Belbruno-Miller Trajectories", *Master's Dissertation, Massachusetts Institute of Technology,* Cambridge, MA, Dec1991.

Miller, J.K., Belbruno, E.A., "A Method for the Construction of a Lunar Transfer Trajectory Using Ballistic Capture", *AAS-91-100. In: AAS/AIAA Space Flight Mechanics Meeting*, Houston, Texas, Feb. 1991.

Prado, A.F.B.A., "Optimal Transfer and Swing-By Orbits in the Two- and Three-Body Problems," Ph.D. Dissertation, University of Texas at Austin, TX, Dec. 1993.

Szebehely, V.G., "Theory of orbits", *Academic Press*, New York, 1967., pp. 7-41.

Vieira Neto, E. and Prado, A.F.B.A., A Study of the Gravitational Capture in the Restricted-Problem. *Proceedings of the "International Symposium on Space Dynamics"* pg. 613-622. Toulouse, France, 19-23/June/1995.

Vieira Neto, E. and Prado, A.F.B.A., Study of the Gravitational Capture in the Elliptical Restricted Three-Body Problem. *Proceedings of the "International Symposium on Space Dynamics"* pg. 202-207. Gifu, Japan, 19-25/May/1996.

Vieira Neto, E. and Prado, A.F.B.A., Time-of-Flight Analyses for the Gravitational Capture Maneuver. *Journal of Guidance, Control and Dynamics,* Vol. 21, No. 1, pp. 122-126, 1998.

Vieira Neto, E., Estudo Numérico da Captura Gravitacional Temporária Utilizando o Problema Restrito de Três Corpos. *Ph.D. Dissertation*, Instituto Nacional de Pesquisas Espaciais, Brazil, 1999.

Yamakawa, H., Kawaguchi, J., Ishii, N., Matsuo, H., "A Numerical Study of Gravitational Capture Orbit in Earth-Moon System", *AAS paper 92-186, AAS/AIAA Spaceflight Mechanics Meeting,* Colorado Springs, Colorado, 1992.

Yamakawa, H., Kawaguchi, J., Ishii, N., Matsuo, H., "On Earth-Moon transfer trajectory with gravitational capture", *AAS paper 93-633, AAS/AIAA Astrodynamics Specialist Conference*, Victoria, Canada, 1993.

Yamakawa, H., On Earth-Moon Transfer Trajectory with Gravitational Capture. *Ph.D. Dissertation*, University of Tokyo, 1992.