ROBUST-H_∞ CONTROL FOR FLEXIBLE STRUCTURES WITH REDUCED DYNAMICS

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Abstract. *A solution for the spillover appearing in the active control of lightly damped flexible structures using the new techniques of robust control H_∞ with feedback is presented. This technique assures the stability of the control system even in the presence of the residual dynamics and results in robust controller that in addition has a good performance. As an example, a control system is synthesized for a flexible structure formed by bar elements and modeled through the finite element method. The design was performed by using Matlab® and Simulink®, and the results show the efficacy of the solution proposed..*

***Keywords**: Flexible structure, Spillover, Robust control*

1. Introduction

Active control of flexible structures does not fit in the classical finite dimensional linear control theory since a flexible structure is infinite dimensional. Hence, to use this theory, one is forced to take a finite dimensional approximation and a good description requires a high order system. Unfortunately, a system of high order also implies in a high order controller whose implementation in practice is not feasible due to limitations in hard as well as in software. The numerical algorithms necessary to solve the Riccati equation in order to implement the control does not function well for high order systems and, also in order to realize the control in hardware, there are limitations in the maximal order of the systems, since there is a limit for the number of equations that can be solved in real time. So, a certain number of modes of the flexible structure cannot be considered in the model. The modes that are not considered (named residual modes) have a dynamics that can destabilize the control system, phenomena which is known in the literature as spillover (Meirovitch, 1990).

The spillover, or the action of the non-considered modes, has two parts, one is the excitation of the residual modes by the control action, and the other is the noise they introduce in the sensors. When this two actions are present the closed loop system may be unstable, and this danger increases when the order of the system decreases and, also when the dissipation of the non-considered modes is taken in account. There are several proposals in the literature to solve this problem but, so far, there is only a partial solution (Chait and Radcliffe,1989), (Meirovitch, 1983).

In this paper it is proposed a solution for the spillover problem appearing in the active control of lightly damped flexible structures by using a modern technique of robust control, synthesizing a controller able to assure the stability of the control system even in presence of residual dynamics.

2. Residual dynamics destabilizing the nominal model

The dynamic model of an undamped flexible structure, obtained by using space discretization techniques (such as the Finite Element Method or Assumed Modes Method), can be expressed by the following relations in the modal form (Junkins and Kim, 1993):

$$\ddot{\mathbf{q}}(t) + \Omega^2\ \mathbf{q}(t) = \mathbf{B}_{ma}\ \mathbf{u}(t) \qquad \textbf{...}(1)$$

where $\mathbf{q}$ is the vector of modal coordinates, Ω the eigenfrequency, and $\mathbf{B}$ the matrix which distributes the control elements of $\mathbf{u}$. The nodal displacement vector corresponding to the nodal points of the structure is given by:

$$\mathbf{v}(t) = \mathbf{U}\ \mathbf{q}(t) \qquad (2)$$

where $\mathbf{U}$ is the modal matrix of the structure. Moreover, in order to actively control the flexible structure, it is necessary to measure the response of the system through any set of variables (for example nodal displacements). This can be represented through equation (3), where $\mathbf{C}_S$ is the measurement matrix:

$$\mathbf{y}(t) = \mathbf{C}_s \ \mathbf{U} \ \mathbf{q}(t) \tag{3}$$

Alternatively, Eq. (1) and Eq. (2) can be written in partitioned form :

$$\begin{bmatrix} \ddot{\mathbf{q}}_N(t) \\ \ddot{\mathbf{q}}_R(t) \end{bmatrix} + \begin{bmatrix} {\Omega_N}^2 & \mathbf{0} \\ \mathbf{0} & {\Omega_R}^2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_N(t) \\ \mathbf{q}_R(t) \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{ma,N} \\ \mathbf{B}_{ma,R} \end{bmatrix} \mathbf{u}(t) \tag{4}$$

$$\mathbf{y}(t) = \mathbf{C}_s \ \begin{bmatrix} \mathbf{U}_N & \mathbf{U}_R \end{bmatrix} \begin{bmatrix} \mathbf{q}_N(t) \\ \mathbf{q}_R(t) \end{bmatrix} \tag{5}$$

here the sub-index N denotes nominal vibrational modes (those that are considered in the model used to design the controller) and R denotes the residual modes (those ignored in the nominal model). In the state space form, these equations result:

$$\begin{cases} \dot{\mathbf{x}}_N(t) = \mathbf{A}_N \ \mathbf{x}_N(t) + \mathbf{B}_N \ \mathbf{u}(t) \ , \qquad \mathbf{y}_N(t) = \mathbf{C}_N \ \mathbf{x}_N(t) \\ \dot{\mathbf{x}}_R(t) = \mathbf{A}_R \ \mathbf{x}_R(t) + \mathbf{B}_R \ \mathbf{u}(t) \ , \qquad \mathbf{y}_R(t) = \mathbf{C}_R \ \mathbf{x}_R(t) \end{cases} \tag{6}$$

where the state vectors corresponding to the nominal and residual system are $\mathbf{x}_N(t) = \begin{bmatrix} \mathbf{q}_N(t) & \dot{\mathbf{q}}_N(t) \end{bmatrix}$ and $\mathbf{x}_R(t) = \begin{bmatrix} \mathbf{q}_R(t) & \dot{\mathbf{q}}_R(t) \end{bmatrix}$ respectively. When a state feedback control system is selected, based on the nominal system model $(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N)$ with feedback gain matrix $\mathbf{F}_N$, it is easy to show that the state matrix of the controlled system in presence of residual dynamics is:

$$\mathbf{A}_{re} = \begin{bmatrix} \mathbf{A}_N - \mathbf{B}_N \ \mathbf{F}_N & \mathbf{0} \\ -\mathbf{B}_R \ \mathbf{F}_N & \mathbf{A}_R \end{bmatrix} \tag{7}$$

The gain $\mathbf{F}_N$ is determined such that the matrix $\mathbf{A}_N - \mathbf{B}_N \mathbf{F}_N$ is a Hurwitz matrix (all the eigenvalues have negative real part); from here it is evident that the state matrix $\mathbf{A}_{re}$ is also a Hurwitz matrix. However, if the control system is based on a state observer, the state matrix of the overall system results (Valer, 1999) :

$$\mathbf{A}_{rs} = \begin{bmatrix} \mathbf{A}_N - \mathbf{B}_N \ \mathbf{F}_N & -\mathbf{B}_N \ \mathbf{F}_N & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_N - \mathbf{K}_N \ \mathbf{C}_N & \mathbf{K}_N \ \mathbf{C}_R \\ -\mathbf{B}_R \ \mathbf{F}_N & -\mathbf{B}_R \ \mathbf{F}_N & \mathbf{A}_R \end{bmatrix} \tag{8}$$

where $\mathbf{K}_N$ is the state observer gain and it is determined such that all the eigenvalues of the matrix $\mathbf{A}_N - \mathbf{K}_N \ \mathbf{C}_N$ are placed in the complex left half plane. In this case, since the matrix $\mathbf{A}_{rs}$ is not block diagonal it is not guaranteed to be Hurwitz due to the terms $\mathbf{B}_R \ \mathbf{F}_N$ and $\mathbf{K}_N \ \mathbf{C}_R$. These terms are identified as *control spillover* and *observer spillover* respectively. It is clear that the smaller the damping of the residual modes, or in other words when the eigenvalues of $\mathbf{A}_R$ are closer to the imaginary axis, the larger the possibility of the eigenvalues of the global state matrix $\mathbf{A}_{rs}$ to present a positive real part due to the spillover terms. So, in presence of residual dynamics, the stability of the overall systems is not guaranteed in the case of a observer based control system.

3. H_∞ robust control to stabilize again the nominal model

The H_∞ robust control technique is interpreted from of point of view of the following paradigm: both, the performance specifications and the robustness specifications, can be incorporated in a linear and time invariant generalized plant P, Fig. 1, which contains all the necessary information to be used for the synthesis of the controller C. In this way, the dynamics of the system to be controlled will profit from some frequency dependent weights $\mathbf{W}_i$, which are included in this generalized plant, Fig. 5.

As illustrated in Fig.1, this plant has in general two input vectors and two output vectors, where $\mathbf{u}$ represents the vector of inputs generated by the controller and $\mathbf{y}$ represents the vector of measured variables used in the feedback. The input vector $\mathbf{w}$ represents a vector of exogenous inputs and the vector $\mathbf{z}$ represents variables to be minimized. A time domain realization for the generalized plant is the following:

$$\begin{cases} \dot{\mathbf{x}}_a(t) = \mathbf{A}_a\ \mathbf{x}_a(t) + \mathbf{B}_1\ \mathbf{w}(t) + \mathbf{B}_2\ \mathbf{u}(t) \\ \mathbf{z}(t) = \mathbf{C}_1\ \mathbf{x}_a(t) + \mathbf{D}_{11}\ \mathbf{w}(t) + \mathbf{D}_{12}\ \mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_2\ \mathbf{x}_a(t) + \mathbf{D}_{21}\ \mathbf{w}(t) + \mathbf{D}_{22}\ \mathbf{u}(t) \end{cases} \tag{9}$$

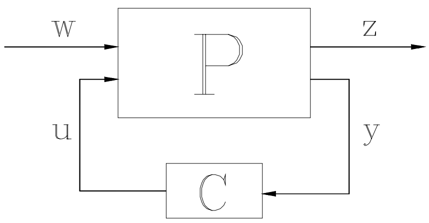

Figure 1 – The modern control paradigm

Now, based on this plant, the general control problem is formulated as the synthesis of a controller C so that the performance variables $\mathbf{z}$ remain small in presence of the exogenous variables $\mathbf{w}$. Mathematically, this means to find a controller such as:

$$\left\| \mathbf{T}_{\mathbf{zw}}(j\omega) \right\|_\infty < 1 \tag{10}$$

where $\mathbf{T}_{\mathbf{zw}}$ is the transfer function between $\mathbf{z}$ and $\mathbf{w}$. Based in a special property of the H_∞ norms, Eq. (10) implies that the H_∞ norm of each sub-matrix H_∞ of the transfer function $\mathbf{T}_{\mathbf{zw}}$ is also less than one, then:

$$\left\| \left(\mathbf{T}_{\mathbf{zw}} \right)_{ij} \right\|_\infty < 1 \tag{11}$$

In this way, the control designer task is reduced to the choice of appropriated weights $\mathbf{W}_i$ artificially included in the generalized plant in order to satisfy the specifications. Solutions for the H_∞ control problem formulation were given by Doyle *et. al.* (6). In general it requires the solution of two coupled Riccati equations .

4. Application

In order to illustrate the method, here there is considered the problem of synthesis of a controller for the truss showed in the Fig. 2. The modeling was performed by using the finite element method and resources of Matlab®. The structural elements are considered as bars with length $l_1 = l_2 = 1\ m$. The material of these elements was assumed to be steel ($E = 207 \times 10^9$ Pa, $\rho = 7806\ kg/m^3$) and the cross section $A = 400\ mm^2$. A small damping, proportional to the modal stiffness matrix, was included in the model.

Since the structure has 41 degrees of freedom, their state space model resulted of order 82. Obviously, due numerical and hardware limitations it is not possible to synthesize directly a controller, and for this reason a reduced model was used as the nominal model where only the $N = 6$ first vibrational modes have been considered. The $R = 41 - 6 = 35$ remaining modes (the residual dynamics) will be interpreted as model uncertainties. Fig. 3-(a) shows clearly the difference between the nominal model and the full model. Moreover, Fig. 3-(b) shows the frequency response function between the disturbance v_d and the output signal y and from here it is evident that structural damping is small and an active control system would be of help.

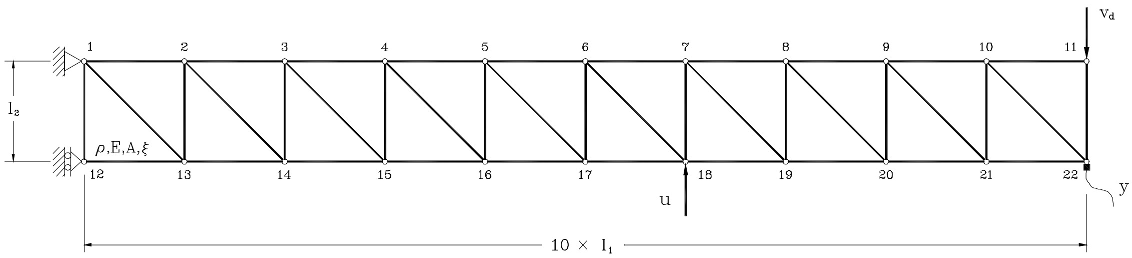

Figure 2 – Flexible structure with many vibrational modes and lightly damped

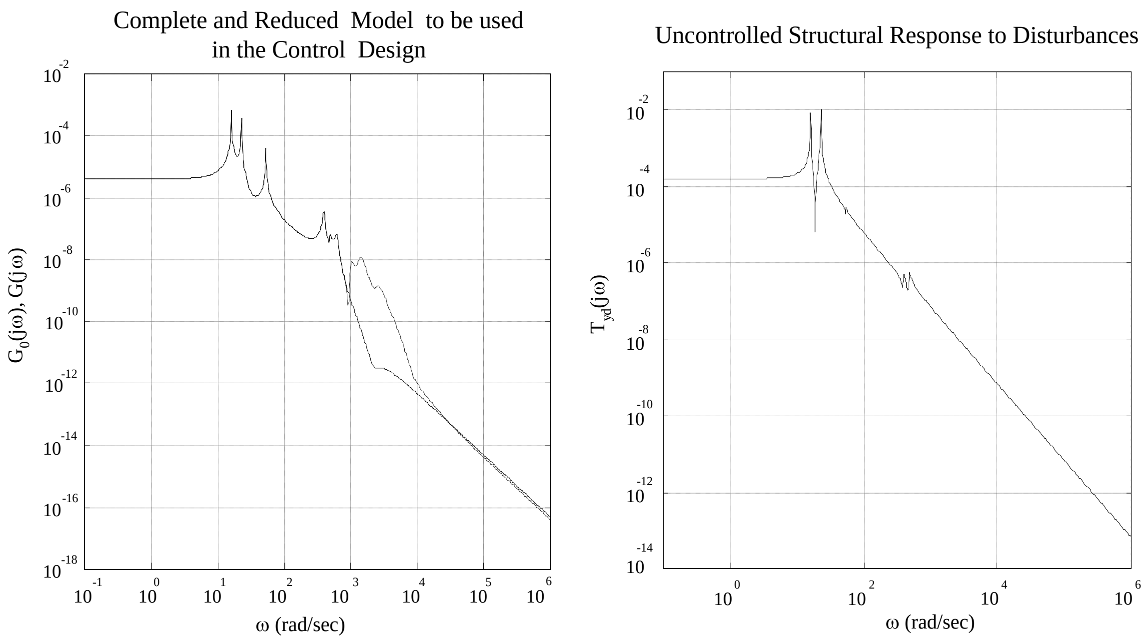

Figure 3 – Open loop frequency response function for the truss

4.1 Representation of the residual dynamics as model uncertainty

In order to synthesize a robust controller, the first step is to represent the residual dynamic as a uncertainty existing in the nominal model due to the model reduction. This uncertainty can be represented through any of the following two alternatives:

$$\mathbf{G}(s) = (\mathbf{I} + \Delta_m(s))\ \mathbf{G}_N(s) \qquad \text{or} \qquad \mathbf{G}(s) = \mathbf{G}_N(s) + \Delta_a(s) \tag{12}$$

where $\mathbf{G}(s) = \mathbf{C}\ (s\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B}$ is the complete model transfer function and $\mathbf{G}_N(s) = \mathbf{C}_N\ (s\mathbf{I} - \mathbf{A}_N)^{-1}\ \mathbf{B}_N$ is the reduced (nominal) model transfer function. The first alternative of representation is referred as multiplicative uncertainty and the second one as additive uncertainty. Fig. 4 shows this two kinds of representations for the flexible structure here considered.

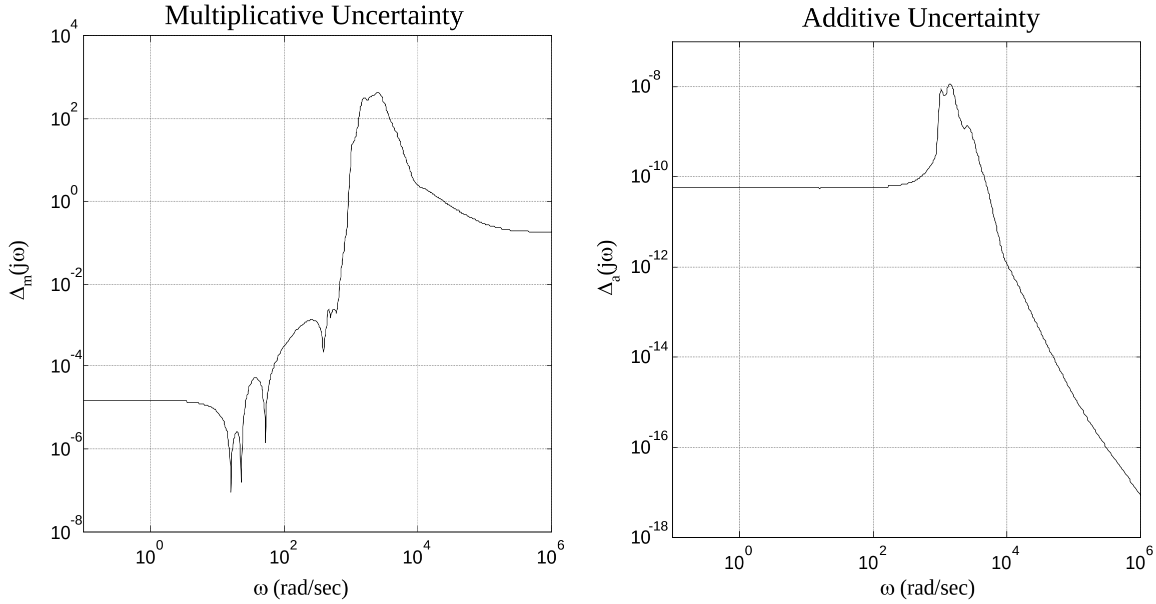

Figure 4 - Uncertainty representation due to the residual dynamics

4.2 Robustness condition

A sufficient condition for robustness in the face of multiplicative uncertainties is given by the following relation (Zhou and. Doyle, 1998):

$$\|\mathbf{T}(j\omega)\|_\infty < 1/l_m(j\omega)\,, \quad \text{with} \qquad l_m(j\omega) > \|\Delta_m(j\omega)\|\,, \qquad \forall\,\omega \in R \tag{13}$$

where the transfer function $\mathbf{T}$ is named complementary sensitivity function of the control system and it's defined as:

$$\mathbf{T}(s) = \mathbf{G}_N(s)\ \mathbf{C}(s)\ [\mathbf{I} - \mathbf{G}_N(s)\ \mathbf{C}(s)]^{-1} \tag{14}$$

The complementary sensitivity function represents the transfer function between the sensor noise and the system output (measured variables).

4.3 Performance specification

The main goal of the control system is to actively damp the flexible structure in order to obtain fast response or equivalently short impulsive response. So, the performance condition can be expressed as a restriction on the closed loop frequency response function between a disturbance v_d and the output y. In the case of the uncontrolled system, this frequency response function was shown in Fig. 3-(b). Mathematically, the performance condition can be expressed by the following relation:

$$\|\mathbf{T}_{yd}(j\omega)\|_\infty < 10^{-3}\,, \qquad \forall\,\omega \in [0, \omega_o] \tag{15}$$

where ω_o is the bandwidth of the closed loop system and represents the limit frequency for the good performance of the system. It is easy to prove that the closed loop transfer function of interest $\mathbf{T}_{yd}$ can be determined by:

$$\mathbf{T}_{yd}(s) = \mathbf{S}(s)\ \mathbf{G}_d(s) \tag{16}$$

where $\mathbf{S}(s) = [\mathbf{I} - \mathbf{G}_N(s)\ \mathbf{C}(s)]^{-1}$ is named sensitivity function and $\mathbf{G}_d(s) = \mathbf{C}_N(s\mathbf{I} - \mathbf{A}_N)^{-1}\mathbf{L}_N$ is the open loop transfer function from the disturbance to the outputs variables. The matrix $\mathbf{L}_N$ is used to indicate the position of the disturbance in the same way that the matrix $\mathbf{B}_N$ is used to indicate the actuators positions.

4. 4 The generalized plant

A convenient internal structure to the generalized plant which is obtained adding frequency dependent weights to the flexible structure nominal model is shown in Fig. 5. This structure is inspired in the LQG control problem definition (5).

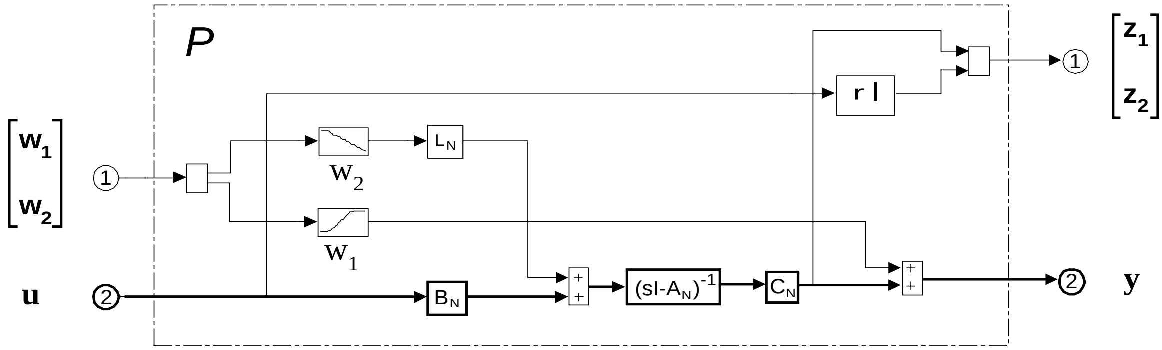

Figure 5 - Generalized plant used to the H$_\infty$ controller synthesis

In this case, the transfer function from the exogenous signals $\mathbf{w}$ to the performance variables $\mathbf{z}$ is given by:

$$\mathbf{T}_{\mathbf{zw}}(s) = \begin{bmatrix} \mathbf{W}_2(s)\ \mathbf{S}(s)\ \mathbf{G}_d(s) & -\mathbf{W}_1(s)\ \mathbf{T}(s) \\ \rho\ \mathbf{W}_2(s)\ \mathbf{K}(s)\ \mathbf{G}_d(s) & \rho\ \mathbf{W}_1(s)\ \mathbf{K}(s) \end{bmatrix} \tag{17}$$

Using the property mentioned in Eq. (11) in the last expression and additionally considering for simplicity that $\mathbf{W}_1 = w_1\ \mathbf{I}$ and $\mathbf{W}_2 = w_2\ \mathbf{I}$, we have:

$$\left\|\mathbf{T}(j\omega)\right\| < 1/w_1(j\omega)\ , \qquad \forall\, \omega \in R \tag{18}$$

$$\left\|\mathbf{S}(j\omega)\ \mathbf{G}_d(j\omega)\right\| = \left\|\mathbf{T}_{yd}(j\omega)\right\| < 1/w_2(j\omega), \qquad \forall\, \omega \in [0, \omega_o] \tag{19}$$

4. 5 Determination of the weights in the generalized plant

Comparing Eq. (18) with the robustness condition given by Eq.(13), and Eq.(19) with the performance specification Eq.(15), we have the following restrictions for the weights:

$$w_1^{-1}(j\omega) < 1/l_m(j\omega),\ \text{com} \qquad l_m(j\omega) > \left\|\Delta_m(j\omega)\right\|, \qquad \forall\, \omega \in R \tag{20}$$

$$w_2^{-1}(j\omega) < 10^{-3}\ , \qquad \forall\, \omega \in [0, \omega_o] \tag{21}$$

Based on these restrictions, a possible convenient choice is:

$$w_1(s) = \left(\frac{s + 10^{1,61}}{10^{-3}s + 10^{2,2}}\right)^8 \tag{22}$$

$$w_2(s) = \frac{10^{-8}s + 10^6}{s + 631} \tag{23}$$

Fig. 6 shows graphically that the restriction equations (20) and (21) are satisfied.

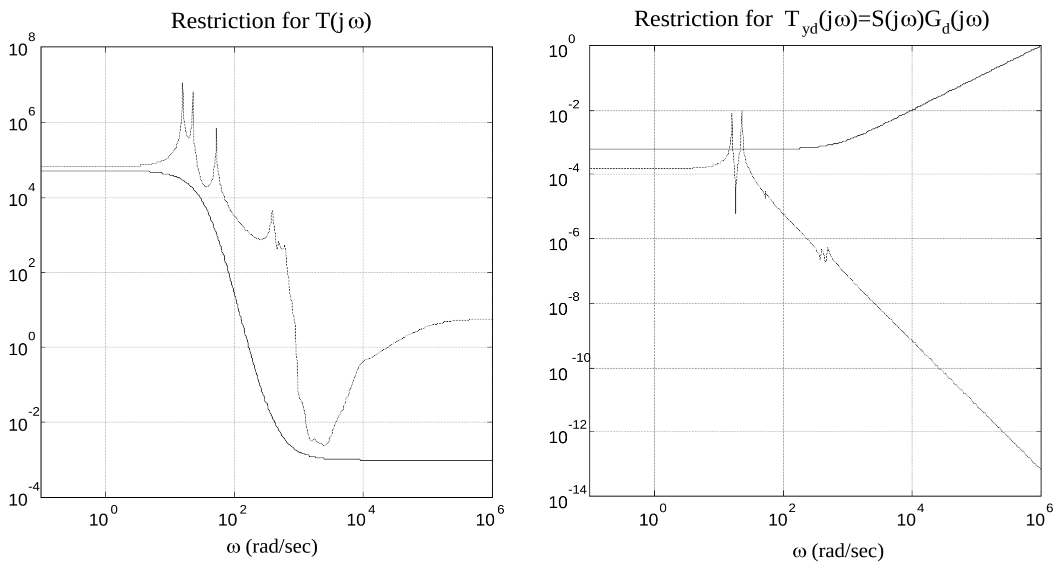

Figure 6 - Determination of the weighting functions used in the H_∞ controller

The parameter ρ in the generalized plant is directly related to the required control effort and it represents the control cost. During the controller design, ρ was changed iteratively until it achieves the H_∞ controller whether or not this controller exists. A value of 10^{-5} was enough for this application.

5. Results

Once the generalized plant has been specified, the H_∞ robust controller was synthesized by solving the Riccati equations using the Matlab® Robust Control Toolbox. Fig. 7 shows the properties of the system with the resulting controller implemented . From here it's clear that both the robustness and performance specifications are satisfied. The controller order resulted to be 21, lightly higher than the nominal model order. This difference is due to the used frequency dependent weights.

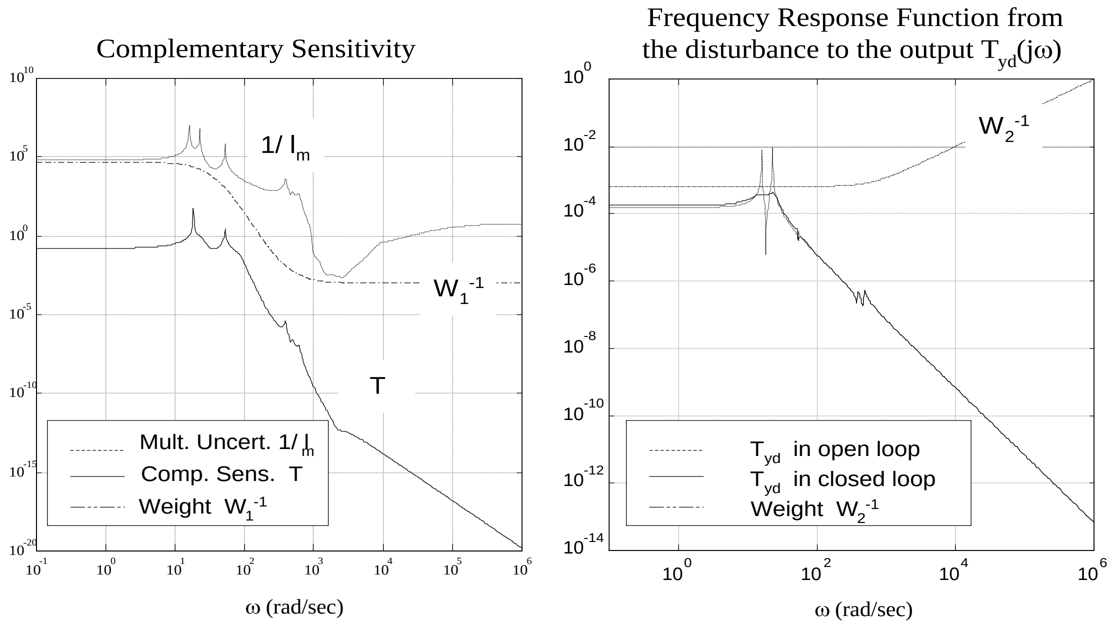

Figure 7 - Resulting system properties with the H_∞ controller implemented

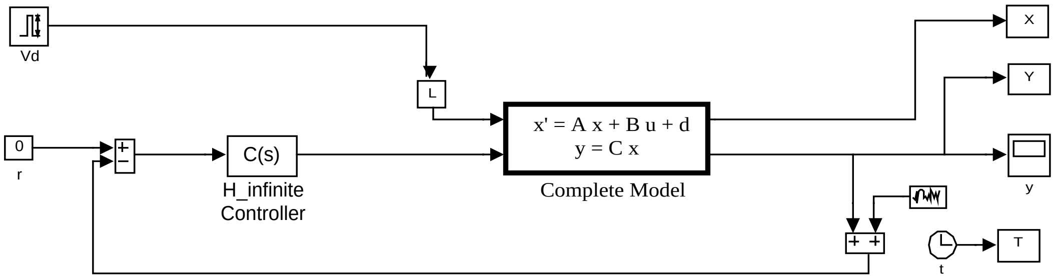

Figure 8 -Implementation of the synthesized robust H_∞ controller

Additionally, in order to verify the closed loop system stability in presence of the residual dynamics, the system eigenvalues were computed resulting all of them with negative real part. Also, it was performed a simulation by using Simulink® with the block diagram shown in Fig. 8. The results are presented in Fig. 9.

6. Conclusions

A methodology based in modern techniques of robust control was applied to a flexible structure with satisfactory results. Through this methodology it was possible to guarantee the stability of the control system in face of residual dynamics, which exist due to the necessary model reduction, and at the same time to ensure a good performance. This technique resulted practical and effective. However, it increased lightly the controller order in relation to the nominal model.

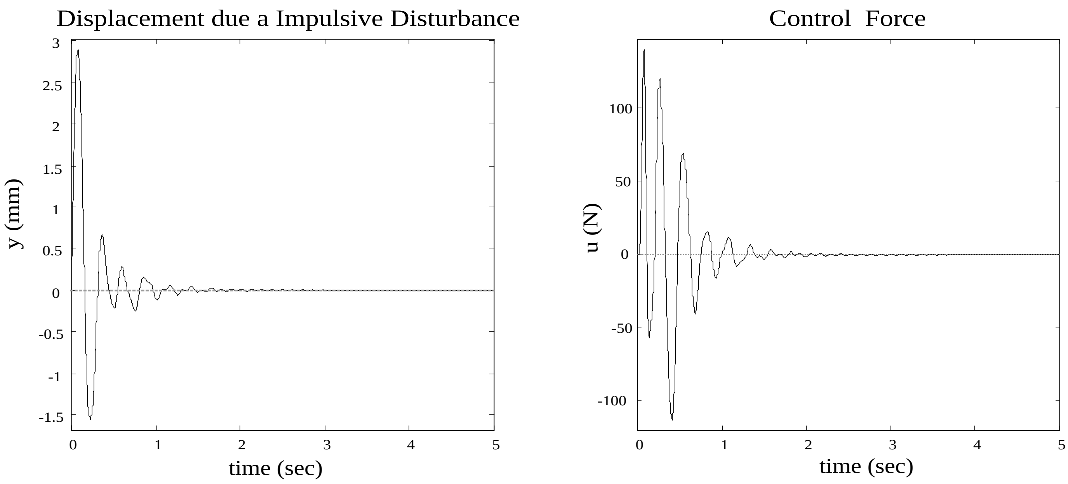

Figure 9 - Impulsive response (controlled displacement in 9a is smaller than the resolution)

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